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AN ELEMENTARY TEXT-BOOK

ON THE

DIFFERENTIAL AND INTEGRAL CALCULUS

BY

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PREFACE.

This text-book is designed with special reference to the needs of the undergraduate work in mathematics in American Colleges.

The preparation for it consists in fairly good elementary courses in Algebra, Geometry, Trigonometry, and Analytical Geometry.

The course is intended to cover about one year's work. Experience has taught that it is best to confine the attention at first to functions of only one variable, and to subsequently introduce those of two or more. For this reason the text has been divided into two books. Great pains have been taken to develop the subject continuously, and to make clear the transition from functions of one variable to those of more than one. The ideas which lie about the fundamental elements of the calculus have been dwelt upon with much care and frequent repetition.

The change of intellectual climate which a student experiences in passing from the finite and discrete algebraic notions of his previous studies to the transcendental ideas of analysis in which are involved the concepts of infinites, infinitesimals, and limits is so marked that it is best to ignore, as far as possible on first reading, the abstruse features of those philosophical refinements on which repose the foundations of the transcendental analysis.

The Calculus is essentially the science of numbers and is but an extension of Arithmetic. The inherent difficulties which lie about its beginning are not those of the Calculus, but those of Arithmetic and the fundamental notions of number. Our elementary algebras are beginning now to define more clearly the number system and the meaning of the number continuum. This permits a clearer presentation of the Calculus, than heretofore, to elementary students.

As an introduction and a connecting link between Algebra and the Calculus, an Introduction has been given, presenting in review those essential features of Arithmetic and Algebra without which it is hopeless to undertake to teach the Calculus, and which are unfortunately too often omitted from elementary algebras.

The introduction of a new symbolism is always objectionable.

Nevertheless, the use of the "English pound" mark for the symbol of "passing to the limit" is so suggestive and characteristic that its convenience has induced me to employ it in the text, particularly as it has been frequently used for this purpose here and there in the mathematical journals.

The use of the "parenthetical equality" sign $(=)$ to mean "converging to" has appeared more convenient in writing and printing, more legible in board work, and more suggestive in meaning than the dotted equality, $\doteq$, which has sometimes been used in American texts.

An equation must express a relation between finite numbers. The differentials are defined in finite numbers according to the best modern treatment. In order to make clear the distinction between the derivative and the differential-quotient, I have at first employed the symbol Df, after Arbogast, or the equivalent notation f' of Lagrange exclusively, until the differential has been defined, and then only has Leibnitz's notation been introduced. After this, the symbols are used indifferently according to convenience without confusion.

The word *quantity* is never used in this text where number is *meant*. True, numbers are quantities, but a special kind of quantity. Quantity does not necessarily mean number.

The word ratio is not used as a relation between numbers. It is taken to mean what Euclid defined it to be, a certain relation between quantities. The corresponding relation between numbers is in this book called a quotient. The quotient of a by b is that *number* whose product by b is equal to a.

In preparing this text I have read a number of books on the subject in English, French, German, and Italian. The matter presented is the common property now of all mankind. The subject has been worked up afresh, and the attempt been made to present it to American students after the best modern methods of continental writers.

I am especially indebted to the following authors from whose books the examples and exercises have been chiefly selected: Todhunter, Williamson, Price, Courtenay, Osborne, Johnson, Murray, Boole, Laurent, Serret, and Frost.

My thanks are due Dr. John E. Williams for great assistance in reading the proof and for working out all of the exercises.

W. H. E.

UNIVERSITY OF VIRGINIA, October, 1902.

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INTRODUCTION TO THE CALCULUS.

SECTION I.

ON THE VARIABLE.

1. **Calculus,** like Arithmetic and Algebra, has for its object the investigation of the relations of Numbers. It is necessary to understand that the symbols employed in Analysis either represent numbers or operations performed on numbers.

2. **The Symbols.**

$$1,\ 2,\ 3,\ 4,\ \ldots \qquad \text{(i)}$$

are symbols used to represent the *groups of marks* which we call *integers*. Thus*

$$\begin{aligned} 1 &\equiv \mathrm{I}, \\ 2 &\equiv \mathrm{I} + \mathrm{I}, \\ 3 &\equiv \mathrm{I} + \mathrm{I} + \mathrm{I}, \\ &\cdots\cdots\cdots \end{aligned}$$

The system of integers (i) extends indefinitely toward the right, as indicated by the sign of continuation. This system is called the *table of integers*. Each integer has its *assigned* place, once and for all, in the table. Any integer in the table is, conventionally, said to be greater than any other integer to the left of it, and less than any integer to the right of it in the table (i).

3. **Definition of Infinite Integer.**—When an integer is so great that its place in the table of integers cannot be assigned in such a manner that it can be uniquely distinguished from each and every other integer, that integer is said to be *unassignably great* or *infinite*. Mathematical infinity has no further or deeper meaning than this.

4. **The Inverse Integer.**—The reciprocals of the integers

$$\ldots,\ \tfrac{1}{4},\ \tfrac{1}{3},\ \tfrac{1}{2},\ 1 \qquad \text{(ii)}$$

constitute an extension of the table (i) to the left of the integer 1, which number is its own reciprocal. As before, any number in this table is said to be greater than any number to the left of it, and less than any number to the right of it.

Corresponding to each number in (i) there is a number in (ii), and conversely. Those numbers in (ii) which are the reciprocals of the infinite or unassignably great integers, are said to be *infinitesimals* or unassignably small.*

---

* The symbol $\equiv$ is to be read, "*is identical with*," or "*is the same as*."

5. The Absolute Number. The Absolute-Number Continuum. When in the table of numbers

$$\ldots, \tfrac{1}{3}, \tfrac{1}{2}, 1, 2, 3, \ldots \qquad \text{(iii)}$$

the gap between each pair of consecutive numbers is filled in with all the rational (fractional) and irrational numbers that are greater than the lesser and less than the greater of the pair, we construct a table of numbers which is called the *absolute-number continuum*. Each number in this system has its assigned place. It is said to be greater than any number to the left of it and less than any number to the right of it. Each number in the absolute-number continuum is called an *absolute number*.

Any and all numbers in the table that are greater than any integer that can be uniquely assigned, as in § 3, are said to be *infinite* or *unassignably great*. In like manner any number in the table that is less than any reciprocal-integer that can be uniquely assigned a place in the table is *infinitesimal* or *unassignably small*.

The absolute continuum is thus divided into two classes of numbers: the uniquely assigned or simply the *assigned* numbers, which we call the *finite* numbers; and the numbers which cannot be uniquely assigned or *transfinite* numbers.

The transfinite numbers greater than 1 are called infinite, those less than 1 infinitesimal numbers.

6. Zero and Omega.—The absolute-number system, as constructed in § 5, extends indefinitely both ways, in the direction of the indefinitely great and in that of the indefinitely small. In this system there is no number greater than all other numbers in the system, nor is there any number that is less than all others in the system.

The system is conventionally *closed* on the left by assigning in the table a number *zero* whose symbol is 0, which shall be less than any number in the absolute system. Since now there is no number greater than 1 to correspond to the reciprocal of this number 0, we design arbitrarily a number *omega* whose symbol is Ω as the reciprocal of 0, and which is greater than any number in the absolute system.

The number 0 is the familiar *naught* of Arithmetic. The number Ω is the *ultimate* number of the Theory of Functions, and with which we shall not be further concerned in this book.

The number 0 is not an absolute number, but is the inferior boundary number of that system. In like manner the number Ω is not an absolute number, but is the superior boundary number of the absolute system.

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* The words 'great' and 'small' have in no sense whatever a magnitude meaning when applied to numbers. They are mere conventional phrases and the words 'right' and 'left' or 'in' and 'out,' might just as well be employed.

7. The conventional symbol for the whole class of unassignably great or infinite numbers is ∞. There has been adopted no conventional symbol for the class of infinitesimals; the symbol most commonly used is the Greek letter iota, ι.

8. The Real-Number System.—When in the algebraic system of numbers

$$-\Omega, \ldots, -3, -2, -1, 0, +1, +2, +3, \ldots, +\Omega,$$

the gap between each consecutive pair is filled in with all the rational and irrational numbers that are greater than the lesser and less than the greater of the pair, the system thus constructed is called the *real-number continuum.*

It is understood that any number in this table is greater than any number to the left of it and less than any number to the right of it.

The *modulus* of any real number is its *arithmetical* or *absolute* value. Thus, the modulus or absolute value of $+3$ or -3 is the absolute number 3. If we employ the symbol x to represent any number in the real continuum, then its modulus or absolute value is represented by $|x|$ or mod x.

In this book we shall be directly concerned only with real numbers and their absolute values. Hereafter when we speak of a *number*, we mean a real number unless otherwise specially mentioned.*

9. Geometrical Picture of the Real-Number System.—We assume a correspondence between the points on a straight line and the numbers in the real continuum.

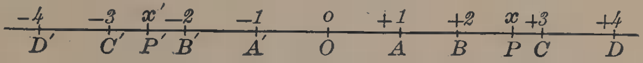

FIG. 1.

Select any point O on a straight line. Choose arbitrarily any unit length; with which construct a scale of equal parts, A, B, C, . . . starting at O proceeding

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* The real-number continuum is a closed system of numbers to all operations save that of the extraction of roots. When we consider the square root of a negative number we introduce a new number. The *complex* or *complete* number of analysis is

$$x + iy,$$

where x and y are any two real numbers, and i is a conventional symbol representing $+\sqrt{-1}$. Corresponding to any real number y there are as many complex numbers as there are real numbers x; and corresponding to any real value x there are as many complex numbers as there are real numbers y. The complex system is a double system. In the theory of functions of complex numbers, which includes that of real numbers as a special case, the ultimate number Ω is conventionally a number common to all systems in the same way as is 0.

The student is already familiar with the impossibility of solving all questions in analysis with real numbers only. For example, in the theory of equations when seeking the roots of equations. All the more so is this true in the Calculus, for we cannot solve the fundamental problem of expanding functions in series without the use of complex numbers, except in a very few particular cases.

If z is any complex number $x + iy$, its modulus or absolute value is

$$|z| \equiv +\sqrt{x^2 + y^2}.$$

toward the right, and A', B', C', . . . toward the left. Mark the points of division, 0 at the origin O, and $+1$, $+2$, etc., toward the right; -1, -2, etc., toward the left. Then, corresponding to any real number x there is a point P on the line to the right of O if x is positive; and P' to the left of O if x is negative. The number x is the measure of the length OP with respect to the unit length chosen. Conversely, corresponding to each point P on the line there is a number in the real-number system.

10. Variable and Constant.—In the continuous number system, as designed in § 8, it is convenient to use letters as general symbols to represent temporarily the numbers in that system. Thus, we can think of a symbol x as representing any particular number in that system. Further, we can think of a symbol x as representing any particular number, say $+3$, and then representing continuously in succession every number between $+3$ and any other number, say $+5$, and finally attaining the value $+5$. We speak of such a symbol x, representing successively different numbers, as a number, and we speak of any particular number which it represents, as its value.

Definition.—A number x is said to be *variable* or *constant* according as it *does* or *does not* change its value during an investigation concerning it.

We shall frequently be concerned with symbols of numbers which are variable during part of an investigation and are constant during another part.

Generally, variables are represented by the terminal letters u, v, w, x, y, z, etc., and constants by the initial letters a, b, c, etc., of the alphabet. This is not always the case, however, as the context will show.

11. Interval of a Variable.—We shall sometimes confine our attention to a portion of the number system. For example, we may wish to consider only those numbers between a and b. We shall employ the symbol (a, b), a being less than b, to represent the numbers a, b and all numbers between them. If we wish to exclude from this system b only, we write $(a, b($; if a only, we write $)a, b)$; when we wish to exclude a and b and consider only those numbers greater than a and less than b, we represent the system by $)a, b($.

If x is a general symbol representing any number in such a portion of the number system, or interval, defined by a and b, we have the equivalent notations,

$$\begin{aligned}
(a, b) &\equiv a \leq x \leq b, \\
(a, b(&\equiv a \leq x < b, \\
)a, b) &\equiv a < x \leq b, \\
)a, b(&\equiv a < x < b.
\end{aligned}$$

A variable x is said to vary continuously through an interval (a, b), when x starts with the value a and increases to b in such a manner as to pass through the value of each and every number in (a, b). Or, x

passes in like manner from b to a. The number x is said to increase continuously from a to b, or decrease continuously from b to a.

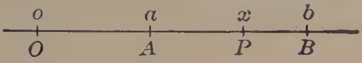

FIG. 2.

In the geometrical picture, of § 9, illustrating the number system, if the points A, P, B, correspond to the numbers a, x, b, the segment AB represents the interval (a, b). As x varies continuously from a to b, or through the interval (a, b), the point P corresponding to the number x generates the segment AB.

12. The Limit of a Variable.

Definition.—When the successive values of a variable x approach nearer and nearer to the value of an assigned constant number a in such a manner that the absolute value of the difference $x - a$ becomes and remains less than any *given* assigned *constant* absolute number ϵ whatever, we say that the number x has a for its *limit*.

In symbols, under the above conditions we write

$$\pounds(x) = a,$$

which is read, "the limit of x is a." The variable is said to converge to its limit.

EXAMPLES.

Arithmetic furnishes examples of a limit .

1. In the extraction of roots of numbers. Whenever a number has no rational number for a root, its root, if real, is an *irrational* number called a *surd*, which is the limit of a sequence of rational numbers constructed according to a certain law.

2. In general, the definition of a number is:[*]

Any sequence of rational numbers

$$a_1, a_2, \dots, a_n, \dots$$

defines and assigns a number, when it is constructed according to any law which requires each number in the sequence to be finite and such that, whatever assigned number ϵ be given (however small), we can always assign an integer n for which

$$|a_n - a_{n+p}| < \epsilon,$$

for any assigned value of the integer p (however great).

The very definition of a number, on which all analysis is founded, is a limit. The number assigned by the above *regular* sequence of numbers is but the limit to which converges the element a_r of that sequence, as r increases indefinitely. If α be the symbol of the number thus defined, then in symbols we write,

$$\alpha = \underset{r=\infty}{\pounds}(a_r).$$

It should be observed that irrational numbers having been thus defined, the numbers a_r in a regular sequence can be any numbers rational or irrational. The regular sequence defines and assigns a number in its place in the table of numbers.

Algebra furnishes a useful and an interesting example of a limit in the evaluation of the infinite geometrical progression.

3. The identity

$$1 - x^{n+1} \equiv (1 - x)(1 + x + x^2 + \dots + x^n)$$

is established by multiplying the two factors on the right.

---

[*] Due to Cantor and Weierstrass.

Therefore, in compact symbolism, which we shall frequently employ,

$$\sum_{r=0}^{r=n} x^r = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}.$$

If x is any number such that $|x| < 1$, we can make and keep x^{n+1}, and therefore also the second term of the member on the right, less than any assigned number ϵ, by making n sufficiently great. Therefore, the limit of the sum of the series on the left is $1/(1-x)$, or in symbols

$$\underset{n=\infty}{\pounds} \sum_{r=0}^{n} x^r \equiv 1 + x + x^2 + \ldots$$
$$= \frac{1}{1-x}.$$

In this example the variable is

$$S_r \equiv 1 + x + \ldots + x^r.$$

If now x is any assigned number in $)0, 1($, x is positive, and S_r continuously increases as r increases. The variable S_r is always less than the limit. If x is in $)-1, 0($, it is negative, say $x \equiv -a$; then

$$\sum_{r=0}^{n} (-1)^r a^r \equiv 1 - a + a^2 - \ldots + (-1)^n a^n,$$
$$= \frac{1}{1+a} + \frac{(-1)^n a^{n+1}}{1+a}.$$

When n is even the variable S_n is greater than its limit; when n is odd the variable S_n is less than its limit. Therefore, as n increases through integral values, the variable converges to its limit, changing from greater than the limit to less as n changes from even to odd and *vice versa*.

It is to be observed that if $|x| > 1$, the sum of the series and the equivalent member on the right increase indefinitely with n, in absolute value, and can be made greater than any assigned number and therefore become infinite. Under these circumstances the series has no limit; its value becomes indeterminately great.

Geometry furnishes numerous illustrations of the limit. The most notable being:

4. The evaluation of the area of the circle as the limit to which converge the areas of the circumscribed and inscribed regular polygons as the number of sides is indefinitely increased.

5. The evaluation of the irrational and transcendental number π representing the ratio of the circumference of a circle to its diameter.

Trigonometry furnishes an illustration of a limit which will be found useful later:

6. To evaluate the limit of the quotient $\sin x \div x$ as x diminishes indefinitely in absolute value.

Draw a circle with radius 1. Draw $MA = MB$ perpendicular to OT. Then

Area quadrilateral $OATB = \tan x$,
Area triangle $OAMB = \sin x$,
Area sector $OANB = x$,

where x is, of course, the circular measure of $\angle AOT$.

Then, obviously, from geometrical considerations,

$$\sin x < x < \tan x,$$

or

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x},$$

$$\therefore \quad 1 > \frac{\sin x}{x} > \cos x.$$

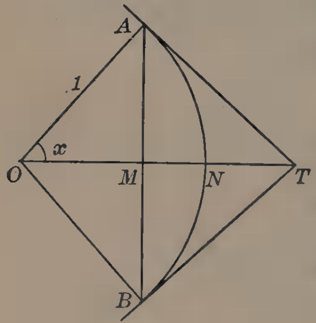

FIG. 3.

When x diminishes indefinitely in absolute value, $\cos x$ becomes more and more nearly equal to 1, and has the limit 1 as x converges to 0. Consequently the quotient $(\sin x)/x$ converges to the limit 1 as x converges to 0. In our symbolism,

$$\mathcal{L}_{x(=)0}\left(\frac{\sin x}{x}\right) = 1$$

13. Definition.—When a symbol x, representing a variable number, has become and subsequently remains always less, in absolute value, than any arbitrarily small assigned absolute number, x is said to be *infinitesimal.*

When a variable becomes and remains greater, in absolute value, than any arbitrarily great assigned number, the variable is said to be *infinite.*

When a variable x is infinitesimal, we write* $x(=)0$. It follows from the definition that when a variable becomes infinitesimal it has the limit 0, or assigns the number 0.

When x has the limit a, or $\mathcal{L}x = a$, then by definition

$$\mathcal{L}(x - a) = 0.$$

When $x - a$ is infinitesimal, we write

$$x - a(=)0.$$

This same relation we shall frequently express by the symbol

$$x(=)a,$$

meaning that the absolute value of the difference between x and a is infinitesimal. When a is the limit of x, the symbol $x(=)a$ is to be read, "as x converges to a," or "x converges to a."

We shall frequently use the symbol ϵ (*epsilon*) to represent an arbitrarily small assigned absolute number. We then speak of the interval $(a - \epsilon,\ a + \epsilon)$ as the *neighborhood* of an assigned number a. The symbol $x(=)a$ means that "x is in the neighborhood of a." All numbers that are in the *neighborhood* of an assigned number are said to be *consecutive* numbers.

When a variable x becomes infinite we write $x = \infty$. Such a variable has no limit, it simply becomes indeterminately great. The symbol $x = \infty$ merely means that x is some number in the class of unassignably great numbers.

14. The Principle of Limits.

I. A variable cannot simultaneously converge to two different limits.

---

* The equality sign in parenthesis $(=)$ may be read "parenthetically equal to," the word 'parenthetically' carrying with it the explanation of the nature of the approximate equality. It is simply another way of saying that the difference between two numbers is infinitesimal.

$$|x - a| = \iota \qquad \text{and} \qquad x - a(=)0$$

mean the same thing. The symbol $\doteq$ has been used for $(=)$, but appears less convenient, expressive, and explicit.

It is impossible for a (one-valued) variable x to converge to two unequal limits a and b. For, the differences $|x - a|$ and $|x - b|$ cannot each be less than the assigned constant number $\frac{1}{2}|b - a|$ for the same value of x.

The direct proof of this statement rests on this:

The number x must be either greater than, equal to, or less than the number $\frac{1}{2}(a + b)$, where say $a < b$.

$$\text{If } x = \tfrac{1}{2}(a + b), \quad \therefore \quad x - a = \tfrac{1}{2}(b - a).$$
$$\text{If } x > \tfrac{1}{2}(a + b), \quad \therefore \quad x - a > \tfrac{1}{2}(b - a).$$
$$\text{If } x < \tfrac{1}{2}(a + b), \quad \therefore \quad b - x > \tfrac{1}{2}(b - a).$$

II. If two variables x and y are always equal and *each converges to a limit*, then the limits are equal.

If $\pounds x = a$, and $\pounds y = b$, and $x = y = z$, then, by I, the variable z cannot converge to two unequal limits simultaneously. Therefore $a = b$.

15. Theorems on the Limit.*

I. If the limit of x is 0, then also the limit of cx is 0, where c is finite and constant.

For, whatever be the assigned constant absolute number ϵ, we can by definition of a limit make and keep $|x|$ less than the constant $|\epsilon/c|$, and therefore cx less than ϵ in absolute value. Consequently, by definition

$$\underset{x(=)0}{\pounds}(cx) = 0 = c\pounds(x).$$

II. If each of a *finite* † number of variables $x_1, x_2, \ldots, x_n$, has the limit 0, then the algebraic sum of these variables has the limit 0.

Let x be the greatest, in absolute value, of the n variables. Then

$$|x_1 + x_2 + \ldots + x_n| \leqq nx.$$

Since n is finite, the limit of this sum is 0, by I.

III. If $\pounds x_1 = a_1$, $\pounds x_2 = a_2$, ..., $\pounds x_n = a_n$, then when n is a finite integer

$$\pounds(x_1 + x_2 + \ldots + x_n) = \pounds x_1 + \pounds x_2 + \ldots + \pounds x_n.$$

For, put $x_1 = a_1 + \alpha_1$, ..., $x_n = a_n + \alpha_n$. By definition, the limits of $\alpha_1, \ldots, \alpha_n$ are 0. Hence

$$x_1 + x_2 + \ldots + x_n = (a_1 + \ldots + a_n) + (\alpha_1 + \ldots + \alpha_n),$$

by II, gives

$$\pounds(x_1 + \ldots + x_n) = a_1 + \ldots + a_n.$$

Therefore the limit of the sum of a *finite* number of variables is equal to the sum of their limits.

---

* The theorems of this article are of such fundamental importance and so absolutely necessary for the foundation of the Calculus that it will, in general, be assumed hereafter that they are so well known as to require no further reference to them.

† If the number of variables is *not finite*, this theorem does not hold in general.

IV. The limit of the product of two variables x_1 and x_2 which have assigned limits a_1 and a_2, is equal to the product of their limits.

Let, as in III, $x_1 = a_1 + \alpha_1$, $x_2 = a_2 + \alpha_2$.

$$\therefore \quad x_1x_2 = a_1a_2 + a_1\alpha_2 + a_2\alpha_1 + \alpha_1\alpha_2.$$

By III, we have

$$\pounds(x_1x_2) = a_1a_2 + a_1\pounds\alpha_2 + a_2\pounds\alpha_1 + \pounds(\alpha_1\alpha_2).$$

But, $\pounds\alpha_1 = 0$, $\pounds\alpha_2 = 0$, and *a fortiori* $\pounds(\alpha_1\alpha_2) = 0$. Therefore

$$\pounds(x_1x_2) = a_1a_2 = (\pounds x_1)(\pounds x_2).$$

Cor. The limit of the product of a *finite* number of variables having *assigned* limits, is equal to the product of their limits. In symbols *

$$\pounds \prod_{r=1}^{n}(x_r) = \prod_{r=1}^{n}\pounds(x_r).$$

V. The limit of the quotient, x_1/x_2, of two variables is equal to the quotient of their limits, *provided* the limit of the denominator is not 0.

With the same symbolism as in IV,

$$\frac{x_1}{x_2} = \frac{a_1 + \alpha_1}{a_2 + \alpha_2} = \frac{a_1}{a_2} + \frac{a_1 + \alpha_1}{a_2 + \alpha_2} - \frac{a_1}{a_2},$$

$$= \frac{a_1}{a_2} + \frac{a_2\alpha_1 - a_1\alpha_2}{a_2(a_2 + \alpha_2)}.$$

By hypothesis, $\pounds\alpha_1 = 0$, $\pounds\alpha_2 = 0$, and $a_2 \neq 0$. Therefore the denominator of the second term on the right is always finite, while, by III, the limit of the numerator is 0. The limit of this term is 0, by I.†

$$\therefore \quad \pounds\left(\frac{x_1}{x_2}\right) = \frac{a_1}{a_2} = \frac{\pounds x_1}{\pounds x_2}.$$

VI. If x and y are two variables and a is a constant, such that y always lies between x and a, then if $\pounds x = a$, also $\pounds y = a$.

---

* As the symbol Σ is used to indicate the sum, so Π is used to indicate the product of a set of numbers. Thus,

$$\sum_1^n x_r \equiv x_1 + x_2 + \ldots + x_n,$$

$$\prod_1^n x_r \equiv x_1 \times x_2 \times \ldots \times x_n.$$

The advantage of such symbolism is in compactness of the formulæ.

† Notice particularly the provision that $\pounds x_2 \neq 0$. For, when $\pounds x_2 = 0$ and $\pounds x_1 \neq 0$, the quotient x_1/x_2 increases beyond all limit or becomes infinite as x_1 and x_2 converge to their limits. An infinite number cannot be a limit under the definition.

Again, if $\pounds x_2 = 0$ and also $\pounds x_1 = 0$, the quotient of the limits 0/0 is completely indeterminate, while the quotient $x_1/x_2 = q$ may or may not converge to a determinate limit. The value of this quotient as x_1 and x_2 converge to 0 depends on the law connecting the variables x_1 and x_2 as they converge to 0. This case is one of profound importance and is the foundation of the Differential Calculus.

The truth of this is obvious, since $|x-a|>|y-a|$, and $x-a$ has the limit 0.

In like manner, it follows that if x and z have the common limit a, and y is a third variable between x and z, then also must $\pounds y = a$. For, $|y-a|$ must at all times be less than one or the other of the differences $|x-a|$ and $|z-a|$, and each of these differences has the limit 0.

VII. If one of two variables is always positive and the other is always negative, and they have a common limit, that limit is 0.

Let a be the common limit of x and y, where x is always positive and y is always negative. Then

$$+|x| = a + \alpha, \quad \text{and} \quad -|y| = a + \beta,$$

where $\pounds \alpha = 0$, $\pounds \beta = 0$. Subtracting,

$$|x| + |y| = \alpha - \beta.$$

Since $\pounds(\alpha - \beta) = 0$, $\therefore$ $a + a = 2a = 0$, and a, the common limit of x and y, is 0.

VIII. If a variable x continually increases and assumes a value a but is never greater than a given constant A, then there must exist a superior limit of x equal to or less than A.

(1). No number such as a which x once attains can be a limit of x. For, since x continually increases, it must subsequently take some value $a' > a$, and it is never possible thereafter for $x - a$ to be less than the constant $a' - a$.

(2). The variable x cannot attain the number A, since if it did, x continually increasing must become greater than A, which is contrary to hypothesis.

(3). Divide the interval $A - a = h$ into 10 equal parts. The variable x after attaining a must either attain $a + \frac{1}{10}h$ or remain always less than $a + \frac{1}{10}h$. If x attains $a + \frac{1}{10}h$, it must either attain $a + \frac{2}{10}h$ or remain always less than $a + \frac{2}{10}h$. We continue to reason thus until we find a digit p_1 such that x must attain $a + \frac{p_1}{10}h$ and remain always less than $a + \frac{p_1 + 1}{10}h$. That is, x must enter and always remain in one of the 10 intervals.

In like manner, divide the interval

$$\left(a + \frac{p_1}{10}h, \quad a + \frac{p_1 + 1}{10}h\right)$$

into 10 equal parts. In the same way we find that x must enter and always remain in one of these intervals, and that there is a digit p_2 such that

$$a + \frac{p_1}{10}h + \frac{p_2}{10^2}h < x < a + \frac{p_1}{10}h + \frac{p_2}{10^2}h + \frac{h}{10^2}.$$

In like manner, continue this process n times. Then

$$a + h\sum_{1}^{n}\frac{p_r}{10^r} < x < a + h\sum_{1}^{n}\frac{p_r}{10^r} + \frac{h}{10^n}.$$

This process can be carried on indefinitely. Consequently the construction leads to the constant number

$$\alpha = a + h\sum_{1}^{\infty}\frac{p_r}{10^r},$$

from which x can be made to differ by a number less than $h/10^n$ which can be made and kept less than any given number ϵ, for all values of n greater than m, where $h/10^m < \epsilon$.

Therefore the constant α is the limit of x, and is either equal to or less than A.

In the same way, we prove the theorem: If a variable x always diminishes and attains a value a, but is never less than an assigned constant number A, then the variable x has an inferior limit that is equal to or greater than A.

IX. If there be two variables x and y, such that y is always greater than x, and if x continually increases and y continually decreases, and the difference $y - x$ becomes less in absolute value than any assigned absolute number ϵ, then there is a constant number greater than x and less than y which is the common limit of x and y.

By Theorem VIII, x has a superior limit a, and y has an inferior limit b. For, any particular value y_1 of y fixes a constant than which x cannot be greater, and any particular value x_1 of x fixes a constant than which y cannot be less. Hence, if we put

$$x = a - \alpha, \text{ and } y = b + \beta,$$

we have

$$y - x = (a - b) - (\alpha + \beta).$$

But, $\pounds(y - x) = 0$, $\pounds(\alpha + \beta) = 0$; $\therefore\ a - b = 0$. This defines the equality of a and b. Therefore x and y converge to a common limit.

EXERCISES.

1. The successive powers of any assigned number greater than 1 increase indefinitely and become infinite as the exponent becomes infinite.

Let α be any absolute number, and m any integer.

Then $$(1+\alpha)^m > 1 + m\alpha. \qquad (1)$$

In fact, $$(1+\alpha)^2 = 1 + 2\alpha + \alpha^2 > 1 + 2\alpha.$$

The formula (1) is true when $m = 2$. Assume it to be true when $m = n$. Then

$$(1+\alpha)^n > 1 + n\alpha.$$

Multiply both sides by $1 + \alpha$.

$$\therefore \quad (1+\alpha)^{n+1} > 1 + (n+1)\alpha + n\alpha^2,$$
$$> 1 + (n+1)\alpha.$$

(1) is true also for $n + 1$. But, being true for $m = 2$, it is also true for $m = 3$, and therefore for $m = 4$, etc., and generally. Therefore, since $m\alpha$ and consequently $(1+\alpha)^m$ can be made greater than any assigned number, the proposition is demonstrated.

2. The successive powers of any assigned absolute number less than 1 diminish indefinitely and have 0 for limit.

Any number less than 1 can be written as the quotient $1/(1+\alpha)$. By Ex. 1,

$$\frac{1}{(1+\alpha)^m} < \frac{1}{1+m\alpha} < \frac{1}{m\alpha}.$$

This can be made less than any assigned number ϵ, by sufficiently increasing m.

3. The successive roots of an *absolute* number greater than 1 continually diminish; those of an absolute number less than 1 continually increase; and in either case have the limit 1.

Whatever be the absolute number a,

$$a^{\frac{1}{n}} = a^{(n+1)\frac{1}{n(n+1)}} = \left\{a^{\frac{1}{n(n+1)}}\right\}^{n+1},$$
$$a^{\frac{1}{n+1}} = a^{n\frac{1}{n(n+1)}} = \left\{a^{\frac{1}{n(n+1)}}\right\}^{n}.$$

Therefore, by Exs. 1, 2, whatever be the integer n,

$$a^{\frac{1}{n}} > a^{\frac{1}{n+1}}, \text{ if } a > 1;$$
$$a^{\frac{1}{n}} < a^{\frac{1}{n+1}}, \text{ is } a < 1.$$

If $a > 1$, then $$a^{\frac{1}{n}} > 1.$$

Let $$a = 1 + \alpha, \quad \text{and} \quad a^{\frac{1}{n}} = 1 + \beta;$$

then $$(1+\alpha)^{\frac{1}{n}} = 1 + \beta,$$

or $$(1+\alpha) = (1+\beta)^n > 1 + n\beta.$$

$$\therefore \quad \beta < \alpha/n, \quad \text{and we have}$$

$$a^{\frac{1}{n}} < 1 + \frac{\alpha}{n}.$$

Hence $$\underset{n=\infty}{\pounds}\, a^{\frac{1}{n}} = 1.$$

Let $a < 1$, say $a = 1/(1 + \alpha)$.

Also, $a^{\frac{1}{n}} < 1$, say $a^{\frac{1}{n}} = 1/(1 + \beta)$.
Then, as before, $\beta < \alpha/n$, and

$$1 > a^{\frac{1}{n}} > \frac{1}{1 + \alpha/n},$$

which shows again that

$$\underset{n=\infty}{\pounds}\, a^{\frac{1}{n}} = 1.$$

4. Show that when a is any assigned positive number,

$$\underset{x(=)0}{\pounds}\, a^{x} = 1,$$

whatever be the way in which x converges to 0.

(1). Let m, n, p, q be any positive integers. Then

$$a^{\frac{m}{n}} a^{\frac{p}{q}} = a^{\frac{m}{n} + \frac{p}{q}}.$$

If $a > 1$, then $a^{\frac{p}{q}} > 1$.

$$\therefore\quad a^{\frac{m}{n} + \frac{p}{q}} > a^{\frac{m}{n}}, \qquad \text{and} \qquad a^{-\left(\frac{m}{n} + \frac{p}{q}\right)} < a^{-\frac{m}{n}}.$$

Therefore a^x continually increases as x increases by rational numbers.

If $a < 1$, then $a^{\frac{p}{q}} < 1$.

$$\therefore\quad a^{\frac{m}{n} + \frac{p}{q}} < a^{\frac{m}{n}}, \qquad \text{and} \qquad a^{-\left(\frac{m}{n} + \frac{p}{q}\right)} > a^{-\frac{m}{n}}.$$

Therefore a^x continually diminishes as x increases by rational numbers.

When $|x|$ is rational and less than 1, there can always be assigned two consecutive integers m and $m + 1$ such that

$$\frac{1}{m + 1} < |x| < \frac{1}{m}.$$

The above results show that whether a be greater or less than 1, a^x lies between $a^{\frac{1}{m+1}}$ and $a^{\frac{1}{m}}$. When $m = \infty$, $a^{\frac{1}{m+1}}$ and $a^{\frac{1}{m}}$ converge to 1, Ex. 3, and therefore also does a^x; and $\pounds a^x = 1$, when $x(=)0$.

(2). When x is irrational there can always be assigned two rational numbers α and β differing from each other as little as we choose, such that $\alpha < x < \beta$. The number a^x is defined by its lying between a^α and a^β. Since $x(=)0$ when $\alpha(=)\beta(=)0$, we have, as before, a^x converging to 1 along with a^α and a^β.

5. Show that $\pounds a^x = a^{\pounds x} = a^\beta$, if $\pounds x = \beta$.

We have $$a^\beta - a^x = a^\beta(1 - a^{x-\beta}).$$

Passing to limits, we have, by Ex. 4,

$$a^\beta - \pounds a^x = 0.$$

6. If a and β are positive numbers, and $\pounds x = \beta$, show that

$$\pounds \log_a x = \log_a \pounds(x) = \log_a \beta.$$

We have $$\log_a \beta - \log_a x = \log_a \frac{\beta}{x}.$$

The above exercises show that however x converges to β, $\pounds \log_a (\beta/x) = 0$. Therefore

$$\log_a \beta - \pounds \log_a x = 0.$$

7. Utilize Ex. 6, to prove IV, V, from III, § 15.

8. Use Ex. 6, to show that

$$\mathcal{L}(y^x) = (\mathcal{L}y)^{\mathcal{L}x},$$

where y has a positive limit, and the limit of x is determinate.

9. A set of numbers $a_1, a_2, \ldots, a_r, \ldots$, arranged in order is called a *sequence*. Any number of the sequence, a_r, is called an element of the sequence; the number r is called the order of the element a_r. Any sequence is said to be known when each element is finite and known when its order is known.

If $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of numbers such that a_r is finite when r is finite, then will $\mathcal{L}a_n$, when $n = \infty$, be 0 or ∞ according as

$$\mathcal{L}_{n=\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

is less or greater than 1, respectively.

Let, when $n = \infty$, $\mathcal{L}(a_{n+1}/a_n) = k$, and $k > 1$. Then, by the definition of a limit, we can always assign a number k' such that $1 < k' < k$, whence corresponding to k' we can find an integer m for which we have, for all values of n,

$$\frac{a_{n+m+1}}{a_{n+m}} > k'.$$

$$\therefore \quad a_{m+1} > k' a_m.$$

$$a_{m+2} > k' a_{m+1} > k'^2 a_m,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a_{m+n} > k'^n a_m.$$

By hypothesis, a_m is finite. Since we can make k'^n greater than any assigned number by sufficiently increasing n, we have $\mathcal{L}a_n = \infty$.

In like manner, if $\mathcal{L}(a_{n+1}/a_n) = k < 1$,

$$a_{m+n} < k'^n a_m,$$

which can be made less than any assigned number by increasing n, when as before $1 > k' < k$. $\quad \therefore \quad \mathcal{L}a_n = 0$, when $n = \infty$.

In order that the element a_n may have a finite-limit different from 0, it is necessary that *

$$\mathcal{L}\frac{a_{n+1}}{a_n} \,|{=}|\, 1.$$

The quotient, a_{n+1}/a_n, of each element by the preceding one will hereafter be called the *convergency quotient* of the sequence. This theorem is of importance and will be used later.

10. The series of numbers

$$a_1 + a_2 + \ldots + a_n + \ldots \qquad \text{(i)}$$

is said to be absolutely convergent when the corresponding series of the absolute values of the terms is convergent.

That is, when

$$S_n \equiv |a_1| + |a_2| + \ldots + |a_n|$$

has a determinate limit when $n = \infty$.

Show that (i) is absolutely convergent if

$$\mathcal{L}\frac{a_{n+1}}{a_n} \,|{<}|\, 1;$$

and if this limit is greater than 1, the sum of the series is ∞.

---

* When the symbols $|{=}|$, $|{>}|$, $|{<}|$ are used, they mean that the equality or inequality of the *absolute* values of the two members of the equation is asserted.

Let the letters in (i) represent absolute numbers, and let

$$\pounds \frac{a_{n+1}}{a_n} = k < 1.$$

Then there can always be assigned an integer m corresponding to any number k' such that $k < k' < 1$, for which

$$\frac{a_{m+n+1}}{a_{m+n}} < k',$$

for all values of n. As in Ex. 9, we have

$$a_{m+n} < k'^n a_m.$$

Hence the sum of the series after a_m is less than

$$k'a_m + \ldots + k'^n a_m + \ldots = a_m(k' + \ldots + k'^n + \ldots),$$
$$= a_m \frac{k'}{1 - k'}.$$

This is finite, since $k' \neq 1$. Therefore S_∞ must be finite. Also, by Ex. 9, $\pounds\, a_m = 0$, when $m = \infty$. Consequently we can always assign an integer n such that

$$S_{n+m} - S_n < \epsilon,$$

for all values of m, where ϵ is any assigned number. Hence S_n has a determinate limit. Otherwise, the existence of the limit of S_n follows at once from VIII, § 15. For S_n continually increases, but can never exceed

$$a_1 + a_2 + \ldots + a_m + a_m \frac{k'}{1 - k'}.$$

Again, if $\pounds(a_{n+1}/a_n) > 1$, say equal to $k > 1$. Then, as before, we can assign k' between k and 1, and have the sum of the series after a_m greater than

$$a_m(k' + \ldots + k'^n + \ldots),$$

which is ∞.

The number $\pounds(a_{n+1}/a_n)$ is called the convergency quotient of the series.

11. The arithmetical average, or mean value of a sequence of n numbers,

$$a_1, a_2, \ldots, a_n,$$

is one nth of their sum, or

$$\alpha_n = \frac{1}{n}\sum_1^n a_r.$$

Show that when the number of elements in a sequence increases indefinitely according to any given law, the mean value has a determinate limit, if all the elements are finite.

Since

$$L < \alpha_n < M,$$

where L and M are the least and greatest elements respectively, the mean value must remain finite. Also,

$$\alpha_{n+p} - \alpha_n = \frac{1}{n+p}\sum_1^{n+p} a_r - \frac{1}{n}\sum_1^n a_r,$$
$$= \frac{1}{n+p}\sum_{n+1}^{n+p} a_r - \frac{p}{n(n+p)}\sum_1^n a_r.$$

But

$$\frac{p}{n(n+p)}\sum_{1}^{n} a_r \,|<|\, \frac{pnG}{n(n+p)} = \frac{pG}{n+p};$$

$$\frac{1}{n+p}\sum_{n+1}^{n+p} a_r \,|<|\, \frac{pG}{n+p}.$$

G being an assigned number, than which no element can be greater in absolute value. Whatever be the assigned integer p, we can always assign an integer n that will make $\alpha_{n+p} - \alpha_n$ less than any assigned number ϵ. The mean value therefore converges to a determinate limit. The value of this limit depends on the law by which the sequence is formed.

12. Find the limit of

$$\left(1+\frac{1}{z}\right)^{z}$$

when z becomes infinite in any way whatever.

Divide both numerator and denominator in

$$\frac{x^{\frac{m+1}{m}}-1}{x-1},$$

where m is a positive integer, by $x^{\frac{1}{m}}-1$. Whence results

$$\frac{1+x^{\frac{1}{m}}+\ldots+\left(x^{\frac{1}{m}}\right)^{m}}{1+x^{\frac{1}{m}}+\ldots+\left(x^{\frac{1}{m}}\right)^{m-1}} = 1+\frac{1}{x^{-1}+x^{\frac{1}{m}-1}+\ldots+x^{-\frac{1}{m}}}.$$

(1). If $x = 1+\frac{1}{m+1}$, then each of the m terms in the denominator of the fraction on the right is less than 1.

$$\therefore \quad \frac{x^{\frac{m+1}{m}}-1}{x-1} > 1+\frac{1}{m} = \frac{m+1}{m}.$$

Hence

$$x^{\frac{m+1}{m}}-1 > \frac{1}{m},$$

or

$$\left(1+\frac{1}{m+1}\right)^{m+1} > \left(1+\frac{1}{m}\right)^{m}.$$

Therefore, the value of the expression continually increases with m, and is always greater than 2, by Ex. 1.

(2). If $x = 1-\frac{1}{m+1}$, each of the m terms in the denominator of the same fraction is greater than 1.

$$\therefore \quad \frac{1-x^{\frac{m+1}{m}}}{1-x} < 1+\frac{1}{m} = \frac{m+1}{m}.$$

Hence

$$1-x^{\frac{m+1}{m}} < \frac{1}{m},$$

or

$$x^{\frac{m+1}{m}} > 1-\frac{1}{m},$$

or
$$\left(1 - \frac{1}{m+1}\right)^{m+1} > \left(1 - \frac{1}{m}\right)^{m}.$$
$$\therefore \quad \left(1 - \frac{1}{m+1}\right)^{-(m+1)} < \left(1 - \frac{1}{m}\right)^{-m}.$$

Therefore the expression continually diminishes as the positive integer m increases.

(3). Whatever be the positive number x, we have
$$x^2 > x^2 - 1.$$
$$\therefore \quad \frac{x}{x-1} > \frac{x+1}{x}.$$

Hence
$$\left(1 - \frac{1}{x}\right)^{-x} > \left(1 + \frac{1}{x}\right)^{x},$$
whatever positive value x may have.

(4). Also,
$$\left(1 - \frac{1}{x}\right)^{-x} = \left(\frac{x-1}{x}\right)^{-x} = \left(\frac{x}{x-1}\right)^{x},$$
$$= \left(1 + \frac{1}{y}\right)^{y} + \frac{1}{y}\left(1 + \frac{1}{y}\right)^{y},$$
if we put $x = y + 1$.

These results (1), . . . , (4), show that
$$\left(1 + \frac{1}{z}\right)^{z}$$
continually increases as z increases by positive integers, and continually decreases as z decreases by negative integers, and that the latter set of numbers is always greater than the former, by (3). Also, these ascending and descending sequences have a common limit,* by (4).

The value of this limit lies somewhere between
$$(1 + 1/6)^6 = 2.521 \ldots \quad \text{and} \quad (1 - 1/6)^{-6} = 2.985 \ldots$$

We represent it, conventionally, by the symbol e. More accurately computed, its value is
$$e = 2{\cdot}7182818285 \ldots$$

A more convenient method of computing e will be given later. It only remains now to show that the limit is the same, whether z increases by rational or irrational values, or continuously.

If z is any positive number, rational or irrational, we can always find two consecutive integers m and $m + 1$, such that
$$m < z < m + 1,$$
and
$$\left(1 + \frac{1}{m+1}\right)^{m} < \left(1 + \frac{1}{z}\right)^{z} < \left(1 + \frac{1}{m}\right)^{m+1},$$
or
$$\left(1 + \frac{1}{m+1}\right)^{-1}\left(1 + \frac{1}{m+1}\right)^{m+1} < \left(1 + \frac{1}{z}\right)^{z} < \left(1 + \frac{1}{m}\right)^{m}\left(1 + \frac{1}{m}\right).$$

This shows that when $m = \infty$, then $z = \infty$, and
$$\mathop{\mathcal{L}}_{z=\infty}\left(1 + \frac{1}{z}\right)^{z} = e.$$

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* Put $(1 - m^{-1})^{-m} = a_m$, $[1 + (m-1)^{-1}]^{m-1} = b_m$. Then assigning to m the values 1, 2, 3, . . . , we have two sequences of positive numbers. The sequence a_m always diminishes, the sequence b_m always increases. The difference $a_m - b_m$ is a positive number converging to 0 when $n = \infty$. The two sequences therefore define a common limit e.

The result in (4) shows this is true whether z be positive or negative.* This limit is the most important one in analysis.

13. Show that $\underset{x(=)0}{\pounds}(1+x)^{\frac{1}{x}} = e.$

14. Show that

$$\underset{x=\infty}{\pounds}\log_a\left(1+\frac{1}{x}\right)^x = \underset{x(=)0}{\pounds}\log_a(1+x)^{\frac{1}{x}} = \log_a e,$$

and is 1, if $a \equiv e$. Use Ex. 6.

15. Show that $\underset{x=\infty}{\pounds}\left(1+\frac{y}{x}\right)^x = \underset{x(=)0}{\pounds}(1+xy)^{\frac{1}{x}} = e^y.$

16. Show that $\underset{x(=)0}{\pounds}(a^x-1)/x = \log_e a.$

Hint. Put $a^x = 1 + z$.

17. If m is a positive integer, show that

$$\underset{x(=)a}{\pounds}\frac{x^m-a^m}{x-a} = ma^{m-1}.$$

18. Show that Ex. 17 also holds true when m is a negative integer, also if m is any positive or negative rational number.

Hint. Put $m = p/q$. Divide the numerator and denominator by $x^{\frac{1}{q}} - a^{\frac{1}{q}}$, to obtain the quotient in determinate form for evaluation.

19. Show that $\underset{x(=)a}{\pounds}\dfrac{\sin x - \sin a}{x-a} = \cos a.$ Use Ex. 6, § 12.

20. Let p_r represent some particular one of the digits 0, 1, . . . , 9, for a particular value of r. Show that the periodic decimal

$$a \cdot p_1 \ldots p_l p_{l+1} \ldots p_{l+m} p_{l+1} \ldots p_{l+m} \ldots$$

has for its limit the rational number

$$M + \frac{N}{10^l(10^m-1)},$$

where $M \equiv a \cdot p_1 \ldots p_l$, and $N \equiv p_l p_{l+1} \ldots p_{l+m}$, and $p_{l+r} = p_{l+qm+r}$, q being any integer, and r any integer less than or equal to q.

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* The evaluation here given is a modification of one due to Fort, *Zeitschrift für Mathematik*, vii, p. 46 (1862). See also Chrystal's *Algebra*, Part II, p. 77.

SECTION II.

ON THE FUNCTION OF A VARIABLE.

16. Definition.—When two variables x and y are so related that corresponding to each value of one there is a value of the other they are said to be *functions* of each other.

If we fix the attention on y as the function, then x is called the variable; if on x as the function, then y is called the variable.

Such functions as x and y defined above are not amenable to mathematical analysis until the *law of connectivity* between them can be expressed in mathematical language.

Classification of Functions.

Functions are classed as *explicit* or *implicit* functions according as the law of connectivity between the function and the variable is direct, explicit, or indirect, implied, implicit.

17. Explicit Functions.—The simplest form of a function of a variable x is any mathematical expression containing x. Such a function is called an *explicit* function of x, because it is expressed explicitly in terms of the variable.

Our attention will be confined in Book I principally to explicit functions of one variable.

The three standard or elementary functions,

$$x^a, \qquad \sin x, \qquad \log_a x,$$

and their inverse functions,

$$x^{-a}, \qquad \sin^{-1}x, \qquad a^x,$$

represent the three fundamental classes of functions called algebraic, circular, and logarithmic or exponential. All the elementary explicit functions of analysis are formed by combining these standard functions by repetitions of the three fundamental laws of algebra,

Addition, Multiplication, Involution,

and their inverses,

Subtraction, Division, Evolution.

Explicit functions are classified as *algebraic* or *transcendental* according as the number of operations (including only

addition, multiplication, involution,
subtraction, division, evolution,

by which the function is constructed from the variable), is *finite* or *infinite*.

18. The Explicit Rational Functions.

I. The Explicit Integral Rational Function.

The function of the variable x,

$$a_0 + a_1x + a_2x^2 + \ . \ . \ . + a_nx^n,$$

where the numbers a_0, . . . , a_n are independent of x, and n is a *finite* integer, is called an *explicit integral rational* function of x, or briefly a *polynomial* in x.

This is the familiar function which is the subject of inquiry in the Theory of Equations. Its place and properties in the system of functions correspond in many respects to the place and properties of the integer in the system of numbers. It can advantageously be expressed by the compact symbolism

$$\sum_{r=0}^{n} a_r x^r,$$

meaning the sum of terms of type a_rx^r from $r = 0$ to $r = n$.

II. The Explicit Rational Function.

The quotient of two explicit integral rational functions of a variable x,

$$\frac{a_0 + a_1x + \ . \ . \ . + a_nx^n}{b_0 + b_1x + \ . \ . \ . + b_mx^m},$$

is called an *explicit rational* function of x, or simply a *rational* function of x.

Its place in the system of functions corresponds to that of the rational or fractional number in the number system.

III. The Explicit Irrational Algebraic Function.

Any expression involving a variable x, or an integral or rational function of x, in which evolution a finite number of times (fractional exponents) is the only irrational part of the construction, is said to be an *explicit irrational algebraic* function of x.

Such a function in the function system corresponds to those irrational numbers in the number system called *surds*.

For example,

$$\sqrt{a^2 - x^2}, \qquad a + bx^{\frac{1}{3}}, \qquad \sqrt{1 + x}/\sqrt{1 - x},$$

are irrational algebraic functions.

19. Explicit Transcendental Functions.—Any expression which is constructed by an *infinite* (and cannot be constructed by a finite) number of algebraic operations on a variable x is said to be an explicit transcendental function of x.

Examples of such functions are $\sin x$, e^x, $\log x$, $\tan^{-1} x$, etc., which can only be constructed from x by an infinite number of operations, such as infinite series or products, or continued fractions.

20. Implicit Functions.—Whenever we have *any* equation involving two variables, x and y, this equation is an expression of the law of connectivity between the two variables and defines one of them as a function of the other. The functional relation is *implied* by the equation and is not explicit until the equation is solved with respect to one or the other of the variables.

For example, the equation

$$ax^2 + by^2 - c = 0$$

defines x as a function of y, and, just as much so, y as a function of x. These functions can be expressed explicitly by solving for x and y. Thus we have

$$x = \sqrt{\frac{c - by^2}{a}}, \quad \text{and} \quad y = \sqrt{\frac{c - ax^2}{b}},$$

or x and y are expressed as explicit irrational algebraic functions of each other.

In general, any algebraic polynomial in two variables x and y when equated to zero defines y as an algebraic function of x, and x as an algebraic function of y. The explicit algebraic functions of § 18 are but particular cases of this more generally defined algebraic function.

21. Conventional Symbolism for Functions.—We frequently have to deal with a class of functions having a common property or common properties, and with functions of complicated form, which makes it convenient to adopt abbreviated symbols for functions. Thus, we frequently represent a function of the variable x by the symbol $f(x)$, or $F(x)$, $\phi(x)$, $\psi(x)$, etc., when it is necessary or advisable to indicate the variable and the function in one compact symbol. When the variable is clearly understood, the parenthesis and the variable are frequently omitted and the function symbol written f, F, ϕ or ψ, etc.

We frequently employ the symbols y, z, u, v, etc., as functions of x.

In like manner we write a function of two variables x, y as $\phi(x, y)$ or $f(x, y)$, etc., meaning a mathematical expression containing x and y. The equation

$$\phi(x, y) = 0$$

implies, as said before, a functional relation between x and y, and defines y as an implicit function of x, or x as an implicit function of y.

If $f(x)$ is a function of x, and if a is any particular assigned value of x, we write $f(a)$ as the value of the function when $x = a$, or, as we say, the value of $f(x)$ at a.

For the present, when we use the word function we mean an explicit function of one variable.

A function, $f(x)$, is said to be *uniform* or one-valued at a when the function has one determinate value at a.

For example,

$$ax^2 + bx + c, \qquad e^x, \qquad \sin x,$$

are one-valued functions for any value of x.

If $f(x)$ has two, three, etc., distinct values corresponding to a value of the variable, it is said to be a two-, three- valued, etc., function.

For example, $ax^{\frac{1}{2}}$, $\sqrt{a^2 - x^2}$, are two-valued functions of x.

Frequently a function does not exist (in real values or finite values) for certain values of the variable. Then, it is necessary to define the interval of the variable in which the function does exist and in which the investigation is confined.

For example, the function $\sqrt{a^2 - x^2}$ exists as a real function only in the interval $(-a, +a)$; the function represented by the series

$$1 + x + x^2 + \dots$$

exists as a determinate finite function only in the interval $)-1, +1($.

22. Continuity of a Function.

Definition: $f(x)$ is said to be a continuous function of x at $x = a$, when $f(x)$ converges to $f(a)$ as a limit, at the same time that x converges continuously to a as a limit.

The definition and condition of continuity of $f(x)$ at a are compactly expressed in symbols by

$$\pounds f(x) = f(\pounds x).$$

The function $f(x)$ is said to be continuous in an interval (α, β) when it is continuous for all values of x in (α, β).

The definition of continuity of $f(x)$ at x asserts that whatever absolute number δ is assigned, we can always assign a corresponding absolute number h such that for all values of x_1 satisfying the inequality

$$|x_1 - x| < h,$$

we have

$$|f(x_1) - f(x)| < \delta.$$

Since, by definition, the limit of $f(x_1)$ is $f(x)$, we can make and keep

$$|f(x_1) - f(x)|$$

less than any assigned absolute number δ for all values of $f(x_1)$ subsequent to an assigned value $f(x')$.

If $x + h$ is the value of the variable corresponding to $f(x')$, then all the values of the function corresponding to the values of the variable in $(x, x + h)$ satisfy the inequality above.

An important corollary to the above and a principle which will constantly be employed later is: If $f(a)$, the value of $f(x)$ at a, is different from 0 and is finite, then we can always assign a finite number h such that for all values of x in the interval $(a - h, a + h)$ the function $f(x)$ has the same sign as $f(a)$.

The above definition shows that a continuous function must change its value gradually as the variable changes gradually, and that the difference of the values of the function

$$f(x_1) - f(x_2),$$

must be arbitrarily small in absolute value when the difference of the corresponding values of the variable, $x_1 - x_2$, is arbitrarily small.

It also shows that a function cannot be infinite at a value of the variable for which the function is continuous, and *vice versa.*

In symbols, when $f(x)$ is continuous at a, we must have simultaneously

$$\pounds(x-a)=0 \qquad \text{and} \qquad \pounds[f(x)-f(a)]=0.$$

The definition and condition of continuity at a can be expressed in the compact symbol

$$\pounds_{x(=)a} f(x)=f(a).$$

23. **Fundamental Theorem of Continuity.**—If $f(x)$ is a uniform (§ 21) and continuous function of x in an interval (a, b), then whatever number N be assigned between the numbers $f(a)$ and $f(b)$, there is a value ξ of x in (a, b) such that at ξ we have

$$f(\xi)=N.$$

The proof of this theorem falls under two heads.

I. If a function $f(x)$ is one-valued and continuous throughout an interval (a, b), and $f(a)$ and $f(b)$ have contrary signs, then there is a number ξ in (a, b), at which we have

$$f(\xi)=0.$$

Suppose $f(a)$ is negative and $f(b)$ positive. Then $f(x)$ cannot be 0 or $+$ arbitrarily near to $x=a$, nor can $f(x)$ be 0 or $-$ arbitrarily near to $x=b$, by definition of continuity of $f(x)$ at a and b.

Let $b-a=h$. Divide this interval into 10 equal parts by the numbers

$$a,\ a+\tfrac{1}{10}h,\ \ldots,\ a+\tfrac{9}{10}h,\ b.$$

Either $f(x)$ is 0 for x equal to one of these numbers, in which case the theorem is proved, or it is not. In the latter case let a_1 be the last of these numbers, proceeding from the left, at which $f(x)$ is negative, and b_1 the first at which it is positive.

Proceed in exactly the same way, subdividing the interval (a_1, b_1) into 10 equal parts. Then if $f(x)$ is not 0 at one of the new division numbers, let a_2 be the last at which it is negative and b_2 the first at which it is positive.

Continuing this process n times, we find that either $f(x)$ is 0 at one of the interpolated numbers, or that $f(a_n)$ is negative and $f(b_n)$ is positive (see § 15, VII, 3), and

$$a_n=a+h\sum_{1}^{n}\frac{p_r}{10^r}, \qquad b_n=a+h\sum_{1}^{n}\frac{p_r}{10^r}+\frac{h}{10^n},$$

where each $p_r (r=1, 2, \ldots, n)$ represents some one of the digits $0, 1, \ldots, 9$. If $f(x)$ is 0 for some one of the interpolated numbers obtained by continually subdividing (a, b), the theorem is proved; if not, then the two numbers a_n and b_n, the former always increasing, the latter always diminishing, converge to the common limit

$$\xi = a + h \sum_{1}^{\infty} \frac{p_r}{10^r}.$$

Meanwhile $f(a_n)$ and $f(b_n)$ converge to the common limit $f(\xi)$, by the definition of continuity. The first of these $f(a_n)$ is always negative, the second $f(b_n)$ is always positive. Also, since $b_n - a_n = h/10^n$, we must have

$$\begin{aligned} \pounds_{n=\infty}\{f(b_n) - f(a_n)\} &= \pounds\{|f(b_n)| + |f(a_n)|\}, \\ &= 2|f(\xi)|. \end{aligned}$$

But this limit is 0, by definition of continuity.

$$\therefore \quad f(\xi) = 0.$$

In like manner we prove the theorem when $f(a)$ is positive and $f(b)$ is negative.

II. The general theorem now follows immediately. For, whatever be the numbers $f(a)$ and $f(b)$, if N lies between them, then

$$f(x) - N$$

must have contrary signs when $x = a$, $x = b$. Therefore, by I, there must be a number ξ in (a, b) at which

$$f(\xi) - N = 0.$$

The important fact demonstrated by this theorem is this: If a function $f(x)$ is uniform and continuous in an interval (a, b) of the variable, then as the variable x varies continuously through the interval (a, b), the function must vary continuously through the interval determined by the numbers $f(a)$ and $f(b)$. That is, the function $f(x)$ must pass through every number between $f(a)$ and $f(b)$ at least once.

24. General Theorems.—The following general theorems result immediately from the theorems on the limit, § 15, and the definition of a continuous function.

I. The sum of a finite number of continuous functions is a continuous function throughout any common interval of continuity of these functions.

If $f_1(x)$, $f_2(x)$, . . . , $f_n(x)$, are continuous at x, then

$$\phi(x) \equiv f_1(x) + \, . \, . \, . \, + f_n(x)$$

is a continuous function at x. For we have

$$\begin{aligned} \pounds_{x'(=)x} \phi(x') &= \pounds[f_1(x') + \, . \, . \, . \, + f_n(x')], \\ &= \pounds f_1(x') + \, . \, . \, . \, \pounds f_n(x'), \\ &= f_1(\pounds x') + \, . \, . \, . \, f_n(\pounds x'), \\ &= f_1(x) + \, . \, . \, . \, + f_n(x) = \phi(x). \end{aligned}$$

II. The product of a finite number of continuous functions is a continuous function in any common interval of continuity of these functions.

If $$\phi(x) \equiv f_1(x) \cdot f_2(x) \ . \ . \ . \ f_n(x),$$

then
$$\begin{aligned}\underset{x'(=)x}{\pounds}\phi(x') &= \pounds[f_1(x') \ . \ . \ . \ f_n(x')],\\ &= \pounds f_1(x') \ . \ . \ . \ \pounds f_n(x'),\\ &= f_1(\pounds x') \ . \ . \ . \ f_n(\pounds x'),\\ &= f_1(x) \ . \ . \ . \ f_n(x) = \phi(x).\end{aligned}$$

Corollary. Any finite integral power of a continuous function is a continuous function in the same interval of continuity.

III. The quotient of two continuous functions is a continuous function in their common interval of continuity, except at the values of the variable for which the denominator is zero.

If $f(x) \equiv \phi(x)/\psi(x)$, then we can consider $f(x)$ as the product of $\phi(x)$ and $1/\psi(x)$. The theorem is then true by the reasoning of the preceding theorem.

Otherwise,
$$\begin{aligned}\underset{x'(=)x}{\pounds} f(x') &= \pounds\frac{\phi(x')}{\psi(x')} = \frac{\pounds\phi(x')}{\pounds\psi(x')},\\ &= \frac{\phi(x)}{\psi(x)},\end{aligned}$$

provided $\psi(x) \neq 0$. If $\psi(x) = 0$ and $\phi(x) \neq 0$, then $f(x) = \infty$ and is not continuous at x. If $\psi(x) = 0$ and also $\phi(x) = 0$, then $f(x)$ may or may not have a determinate value as the limit of $f(x')$, a case which we shall investigate later.

IV. It has been shown, Exercises, Sec. I, Ex. 5, that
$$\pounds a^{f(x)} = a^{\pounds f(x)} = a^{f(\pounds x)},$$
when a is positive. Therefore $a^{f(x)}$ is continuous when $f(x)$ is continuous.

V. In like manner, Ex. 6, Exercises, Sec. I,
$$\pounds \log_a f(x) = \log_a \pounds f(x) = \log_a f(\pounds x),$$
$f(x)$ being positive. Therefore, $\log_a f(x)$ is continuous.

VI. Again, if $f(x)$ and $\phi(x)$ are continuous and $f(x)$ is positive, we have
$$\begin{aligned} y &\equiv [f(x)]^{(\phi x)}.\\ \therefore\quad \log y &= \phi(x)\log f(x),\\ \pounds \log y &= \pounds[\phi(x)\log f(x)],\\ &= \pounds\phi(x)\cdot\pounds\log f(x).\\ \therefore\quad \log \pounds y &= \phi(\pounds x)\log f(\pounds x),\\ &= \log\,[f(\pounds x)]^{\phi(\pounds x)}.\\ \therefore\quad \pounds[f(x)]^{\phi(x)} &= [f(\pounds x)]^{\phi(\pounds x)},\end{aligned}$$

and the function y is continuous when $\phi(x)$ is continuous and $f(x)$ is continuous and positive.

SPECIAL THEOREMS.

Since $y \equiv x$ is a continuous function of x, the product $a_r x^r$, where a_r is independent of x, and r is any finite integer, is continuous for all finite values of x. Also the sum of any finite number of terms of this type is continuous. Therefore the algebraic polynomial in x is a continuous function for all finite values of x.

By the theorem for the quotient, it follows that the algebraic fraction or rational function is continuous everywhere, except at the roots of the denominator.

By Trigonometry, since

$$\sin x' = \sin x + 2 \cos \tfrac{1}{2}(x' + x) \sin \tfrac{1}{2}(x' - x),$$

and $$\pounds \sin \tfrac{1}{2}(x' - x) = 0, \quad \text{when} \quad x' (=) x,$$

we have $$\pounds \sin x' = \sin x = \sin \pounds x.$$

Therefore $\sin x$ is everywhere continuous.

In like manner we show that $\cos x$ is everywhere continuous, and by § 24, III, $\tan x$, $\cot x$, $\sec x$ and $\csc x$ are continuous functions everywhere except at the roots of their denominators, $\cos x$, $\sin x$.

The continuity of any algebraic function of

$$x^a, \qquad a^x, \qquad \sin x, \qquad \log x,$$

can now be easily determined.

25. Geometrical Illustration of Functions.—If we adopt the method of representing the variable, in §§ 8, 11, by points on a straight line, such as Ox, then at any point M on Ox corresponding to $x = a$ we can represent the corresponding value of a uniform function $f(x)$ by a point P in a plane xOy. The point P is constructed by laying off a perpendicular MP to Ox, such that the distance MP is equal to the number $f(a)$, and is measured upward if $f(a)$ is positive, and downward if $f(a)$ is negative.

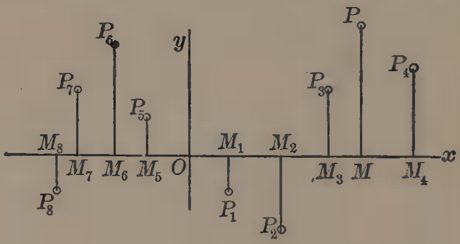

FIG. 4.

For each and every value of x for which $f(x)$ is a defined function, such as a_1, a_2, . . ., we can construct corresponding points P_1, P_2, . . ., representing $f(a_1)$, $f(a_2)$, . . .

This is the familiar method of Analytical Geometry, invented by Descartes. If we put $y = f(x)$, then Oy perpendicular to Ox can be called the axis of the function, corresponding to Ox, the axis of the variable; and x, y are the Cartesian coordinates of the point P representing the functional form $f(x)$.

If the function $f(x)$ is continuous in any interval (a_7, a_5), then corresponding to any point M_6 in M_7M_5 there is a point P_6 in the plane xOy, at a finite distance from Ox, representing the function. Moreover, any two such points P', P'' corresponding to M', M'' can be brought as near together as we choose by bringing M' and M'' sufficiently near together. Can we say that the assemblage of *all* the points, P, representing a continuous function in a given interval (a, b) of x, is a line?

To answer this question it is necessary to consider the question: What constitutes a line, or in general a curve?

Geometrically speaking, the older definitions, now antiquated, required a line to have in the first place a determinate *length* corresponding to any two arbitrarily chosen points on the line, and also to have *direction* at any point. This requires a definition of *direction* and of *length*, concepts themselves abstruse. The old definition, "a line has length without breadth or thickness," is now taken to mean that a line is simply extension in one dimension.

In order that the assemblage of points in the plane xOy representing a continuous function $f(x)$ can be defined as a line, this assemblage must have some analytical property at each point that will define a determinate direction, and corresponding to any two points some analytical property that will define a determinate length. These properties must be inherent in the function $f(x)$ of which the assemblage of points is the geometric picture.

To define the first of these properties, i.e., a determinate direction, is the province of the Differential Calculus; the second, which gives meaning to a definite length, is furnished by the Integral Calculus.

At our present stage of knowledge, then, we cannot say that the assemblage of points which represents a continuous function is a line. But it will be demonstrated in what follows that such continuous functions as those with which we shall be concerned can be represented by curves, and we shall in the course of our work develop an analytical definition of a line, and find means of measuring its direction, length, and curvature, and many other properties that are unattainable save through the Calculus.

In order to take advantage of the intuitive suggestiveness of geometrical pictures as illustrations of the text, we shall assume for the present that the assemblage of points $P_1, \ldots, P_n$ representing values of a continuous function in the interval M_1M_n, has the following properties:

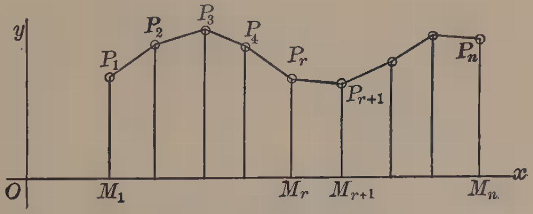

FIG. 5.

Join the consecutive points by straight lines. Consider the broken polygonal line $P_1P_2 \ldots P_n$. Then, if M_1 and M_n correspond to two fixed values a, b of x, and we increase the number of points, M, between M_1 and M_n indefinitely, in such a manner that the distance between any two consecutive points M_r and M_{r+1} converges to zero, we shall have:

First. The distance between the corresponding points P_r and P_{r+1} converges to 0. For, P_rP_{r+1} is the hypothenuse of a right-angled triangle, P_rNP_{r+1}, whose sides P_rN and $NP_{r+1} = M_rM_{r+1}$ converge to 0 together, when $M_r(=)M_{r+1}$, since the function $f(x)$ is continuous.*

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* The point N, not shown in the figure, is the point in which a straight line through P_{r+1} parallel to Ox cuts M_rP_r.

Second. We assume that the angle $P_{r-1} P_r P_{r+1}$ between any pair of consecutive sides of the polygonal line, such as $P_{r-1} P_r$ and $P_r P_{r+1}$, converges to two right angles as a limit.

Third. We assume that the sum of the lengths of the sides of this polygonal line $P_1 P_n$ converges to a determinate limit length.

The first consideration secures continuity, the second determinate direction, and the third determinate length.

The three together constitute the necessary conditions that the assemblage of points shall be a curve.

The analytical equivalents of the second and third conditions will be developed later. That for the first has already been established in the definition of a continuous function.

EXERCISES.

1. If $f(x) \equiv 2x^3 - x^2 - 12x + 1$, show that the function has a root in each of the intervals $(0, 1)$, $(2, 3)$, $(-3, -2)$.

2. If $\phi(x) \equiv (x - 1)/(x + 1)$, show that

$$\frac{\phi a) - \phi(b)}{1 + \phi(a)\phi(b)} = \frac{a - b}{1 + ab}.$$

3. If $\psi(t) \equiv e^t + e^{-t}$, show that

$$\psi(3t) = [\psi(t)]^3 - 3\psi(t),$$
$$\psi(x + y) \times \psi(x - y) = \psi(2x) + \psi(2y).$$

4. If $F(x) \equiv \log \frac{1 - x}{1 + x}$, show that

$$F(p) + F(q) = F\left(\frac{p + q}{1 + pq}\right).$$

5. What functions satisfy the functional equations

$$f(x + y) = f(x) \cdot f(y),$$
$$\phi(x) + \phi(y) = \phi(xy),$$
$$\psi(x) - \psi(y) = \psi(x/y),$$
$$F(x - y) = F(x)/F(y).$$

6. If $f(x) \equiv ax^2 - bx + c$, write $f(\sin x)$.

7. If $y = x^2 + x - 5$, write x as a function of y.

8. Show that $e^{\frac{1}{x}}$ is discontinuous at $x = 0$. Examine the behavior of this function as x increases through 0.

9. If $y = \log(x + \sqrt{x^2 + 1})$, show that

$$x = \tfrac{1}{2}(e^y - e^{-y}).$$

This last function is called the hyperbolic sine of y and is written

$$\sinh y \equiv \tfrac{1}{2}(e^y - e^{-y}).$$

10. If $y = \log(x + \sqrt{x^2 - 1})$, x is called the hyperbolic cosine of y and written $\cosh y$. Find this as a function of y.

11. Show that $e^y = \sinh y + \cosh y$.

12. Let x be any assigned real number. Consider the function

$$F(n) \equiv \frac{x^n}{n!},$$

where n takes only positive integral values. Show that $F(n)$ has the limit 0 when $n = \infty$, whatever may be the finite value of x.

13. Show that

$$f(r) \equiv \frac{a(a - 1) \ldots (a - r + 1)}{r!} x^r,$$

in which a and x are assigned real numbers, has the limit 0 when $r = \infty$, provided $|x| < 1$. What is the value of $f(\infty)$ when $|x| > 1$?

14. Investigate

$$\underset{n=\infty}{\mathcal{L}} \frac{x^n}{n},$$

for $|x| \gtrless 1$.

15. The identity

$$ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

shows that the geometrical mean, $\sqrt{ab}$, of two unequal numbers lies between them and is less than their arithmetical mean $\frac{1}{2}(a+b)$.

Finding the square root of any absolute number β can be reduced to finding the square root of a number between 1 and 100. For, we can always assign an integer n such that $10^{2n}\beta = \alpha$, where $1 < \alpha < 100$; n being $+$ or $-$ according as β is less or greater than 1. Then

$$\sqrt{\beta} = 10^{-n}\sqrt{\alpha}.$$

Consider any given number between 1 and 100. Choose x_1 from one of the integers 2, . . ., 10, such that

$$(x_1 - 1)^2 < \alpha < x_1^2.$$

Then
$$\frac{\alpha}{x_1} < \sqrt{\alpha} < x_1,$$
$$\frac{\alpha}{x_2} < \sqrt{\alpha} < \frac{1}{2}\left(x_1 + \frac{\alpha}{x_1}\right) = x_2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Show that if this construction be continued to x_m, then

$$x_m - \sqrt{\alpha} < \frac{1}{2^{2m-1}},$$

and therefore the sequence of numbers $x_1, x_2, \ldots$ defines the square root of α, and

$$\underset{m=\infty}{\mathcal{L}} x_m = \sqrt{\alpha}.$$

16. Apply 15 to show that $\sqrt{5}$, to six decimal places, is

$$x_4 = \frac{2207}{987} = 2.2360689.$$

17. Show that the cubic function of x,

$$\begin{vmatrix} a-x, & h\ , & g \\ h\ , & b-x, & f \\ g\ , & f\ , & c-x \end{vmatrix} \equiv f(x)$$

always has three real roots.

Expanding with respect to the first row,

$$f(x) \equiv (a-x)[(b-x)(c-x) - f^2] - [h^2(c-x) - 2fgh + g^2(b-x)].$$

Let p, q, of which p is not less than q, be the two roots of the quadratic function

$$(b-x)(c-x) - f^2 \equiv x^2 - (b+c)x + bc - f^2.$$

Then
$$p + q = b + c, \quad \text{and} \quad pq = bc - f^2.$$

Therefore neither p nor q can be between b and c or equal to b or c, and p is greater and q is less than either b or c. But

$$\begin{aligned} f(+\infty) &= -\infty, \\ f(p) &= +[h\sqrt{p-c} + g\sqrt{p-b}]^2, \\ f(q) &= -[h\sqrt{c-q} - g\sqrt{b-q}]^2, \\ f(-\infty) &= +\infty. \end{aligned}$$

Hence, by § 23, I, $f(x)$ vanishes between $+\infty$ and p, between p and q, also between q and $-\infty$, and the three roots are real. This exercise will be needed in subsequent work.

18. Determine the condition that the function

$$ax^2 + 2bx + c$$

shall retain its sign unchanged for all values of the variable x.

The function can be written

$$a\left(x^2 + 2\frac{b}{a}x + \frac{c}{a}\right) \equiv a\left\{\left(x + \frac{b}{a}\right)^2 + \frac{c}{a} - \frac{b^2}{a^2}\right\},$$

$$\equiv \frac{(ax + b)^2 + (ac - b^2)}{a}.$$

In order that this shall retain its sign unchanged for all values of x, it is necessary and sufficient that $ac - b^2$ shall be positive. This condition being satisfied, the function has the same sign as a for all values of x.

19. Determine the condition that the function

$$ax^2 + 2hxy + by^2$$

shall retain its sign unchanged for all values of the variables x and y.

By completing the square, the function can be written

$$\frac{(ax + hy)^2 + y^2(ab - h^2)}{a},$$

which, when $ab - h^2$ is positive, has the same sign as a for all values of x and y.

20. Determine the condition that the function

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy$$

shall keep its sign unchanged for all values x, y, z.

By completing the square, the function can be written

$$\frac{1}{a}\left\{(ax + gz + hy)^2 + (ab - h^2)y^2 + 2(fa - hg)yz + (ac - g^2)z^2\right\}.$$

The function will keep its sign unchanged and have the same sign as a whatever be the values of x, y, z, provided the quadratic function

$$(ab - h^2)y^2 + 2(fa - hg)yz + (ac - g^2)z^2$$

is always positive. This will be the case, by Ex. 19, when

$$ab - h^2 \quad \text{and} \quad (ab - h^2)(ac - g^2) - (fa - hg)^2$$

are both positive. Or, what is the same thing, $ab - h^2$ and

$$a(abc + 2fgh - af^2 - bg^2 - ch^2) \equiv a\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

must be positive.

Exercises 18, 19, 20 will be drawn on in the sequel.

BOOK I.

FUNCTIONS OF ONE VARIABLE.

PART I.

PRINCIPLES OF THE DIFFERENTIAL CALCULUS.

CHAPTER I.

ON THE DERIVATIVE OF A FUNCTION.

26. The Difference of the Variable.—The *difference* of a variable x is a technical term, which means the result obtained by subtracting a *particular* value of the variable, say x, from an *arbitrarily* assigned value of the variable, say x_1.

Or, in symbols,

$$x_1 - x.$$

We use the characteristic letter Δ to represent the symbol of this operation, and write

$$\Delta x \equiv x_1 - x.$$

This difference, Δx, is of course positive or negative according as x_1 is greater or less than x.

We sometimes for convenience write

$$\Delta x \equiv x_1 - x = h,$$

so that

$$x_1 = x + h,$$

and call h the increment of the variable x.

27. The Difference of the Function.—The *difference* of the function is a corresponding technical term, which means the result obtained by subtracting the value of a function at a *particular* value of the variable, say x, from the value of the function at an *arbitrarily* chosen value of the variable, x_1. In symbols

$$f(x_1) - f(x)$$

is the *difference* of the function $f(x)$ at x.

As in the case of the difference of the variable, we use the letter Δ as the symbol of this operation, and write

$$\Delta f(x) \equiv f(x_1) - f(x).$$

28. The Difference-Quotient of a Function.

A difference of a function and a difference of the variable are said to "correspond" when the same values of the variable occur in the same way in these differences.

Definition.—The quotient obtained by dividing a difference of the function by the *corresponding* difference of the variable is called the *difference-quotient* of the function at the *particular* value of the variable.

Thus, in symbols,

$$\frac{\Delta f(x)}{\Delta x} \equiv \frac{f(x_1) - f(x)}{x_1 - x} = q_1$$

is the *difference-quotient* of the function $f(x)$ at x.

For an assigned particular value x, the number q_1 depends on the value assigned to the arbitrary number x_1.

29. The Derivative of a Function.

Definition.—Whenever the function $f(x)$ is such that when we assign to the arbitrary value of the variable successive arbitrarily chosen values

$$x_1,\ x_2,\ \ldots,\ x_n,\ \ldots$$

in such a manner that this sequence converges to the particular value x as a limit, and the corresponding sequence of difference-quotients,

$$\frac{f(x_1) - f(x)}{x_1 - x} = q_1,\ \frac{f(x_2) - f(x)}{x_2 - x} = q_2,\ \ldots,\ \frac{f(x_n) - f(x)}{x_n - x} = q_n,\ \ldots$$

has a determinate number as a limit, this *limit* is called the *derivative* of the function $f(x)$ at x.

In other words, the function $f(x)$ is said to be *differentiable* at x when the difference-quotient

$$\frac{\Delta f(x)}{\Delta x} \equiv \frac{f(x') - f(x)}{x' - x}$$

converges to a determinate limit as x' converges to x as a limit in any arbitrary manner whatever.

In symbols,

$$\mathop{\mathcal{L}}_{x'(=)x} \frac{f(x') - f(x)}{x' - x}$$

is called the *derivative* of $f(x)$ at x.

This derivative is, in general, a function of x, and we shall represent it, after Lagrange, by the symbol $f'(x)$, a convenient and characteristic symbolism because it shows the association of the derivative $f'(x)$ with the *primitive* function $f(x)$ from which it has been derived.

We shall also use sometimes another symbolism, to represent the operation by which this limit is derived, instead of the cumbersome one employed above representing the limit of the difference-quotient.

We use the characteristic letter D as a symbol to represent the operation gone through of dividing the difference of the function by the corresponding difference of the variable, and determining the limit of this difference-quotient when the arbitrary value of the variable converges to the particular value of the variable as a limit.

In compact symbols, we write

$$Df(x) \equiv \underset{x'(=)x}{\pounds} \frac{f(x') - f(x)}{x' - x}.$$

But we have already agreed that this limit, the derivative, shall be represented by $f'(x)$. Hence we have the equivalent symbolism

$$Df(x) \equiv f'(x).$$

Or, the operation D performed on the function $f(x)$ results in the derivative $f'(x)$.

This operation is called *differentiation.*

30. Observations on the Derivative.—We observe that in order that a function $f(x)$ may be differentiable (have a derivative), it must be continuous. For, unless we have

$$\underset{x'(=)x}{\pounds} \Delta f(x) \equiv \underset{x'(=)x}{\pounds} [f(x') - f(x)] = 0,$$

as is required by the definition of a continuous function, then, since we do have

$$\underset{x'(=)x}{\pounds} (x' - x) = 0,$$

the value of the corresponding difference-quotient would be ∞, or no limit exists.

Hence the Differential Calculus deals directly with none but continuous functions.

The converse of the above statement is not true, i.e., a function that is uniform and continuous is not always differentiable. There exist functions that are uniform and continuous and yet the limit of the difference-quotient is completely indeterminate for all values of the variable in certain finite intervals.* We shall not have occasion to meet any of these highly transcendental functions in this book, and the functions with which we deal will, in general, be differentiable. Only for isolated values of the variable will the derivatives of these functions be found indeterminate. Such values are singular values and receive treatment in their appropriate places.

The evaluation of the derivative of a function falls under the case specially excepted in § 15, V. Here, the limit of the numerator

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* See Appendix, note 1.

(the difference of the function, Δf), and the limit of the denominator, Δx, are each o.

The quotient of the limits o/o is always indeterminate.

We are not concerned, in evaluating the derivative, with the quotient of the limits, but only with the limit of the quotient. We are not concerned or interested in the *difference-ratio* but with the *difference-quotient.*

This is a variable number which does or does not have a limit according as the function is or is not differentiable at the particular value of the variable considered.

The derivative of any *constant* is necessarily o by the definition. For, the quotient of differences is constantly o and remains o for $\Delta x (=) 0$.

EXAMPLES.

1. Differentiate the function x^2.

We have the difference-quotient,

$$\frac{x'^2 - x^2}{x' - x} = x' + x.$$

The limit of this number when $x'(=)x$ is $2x$.

$$\therefore \quad Dx^2 = 2x.$$

2. Differentiate the function $x^{\frac{1}{2}}$.

We have $$\frac{x'^{\frac{1}{2}} - x^{\frac{1}{2}}}{x' - x} = \frac{1}{x'^{\frac{1}{2}} + x^{\frac{1}{2}}},$$

the limit of which is $x^{-\frac{1}{2}}/2$ when $x'(=)x$.

$$\therefore \quad Dx^{\frac{1}{2}} = \tfrac{1}{2}x^{-\frac{1}{2}}.$$

3. If $f(x) = \sin x$, show that $D \sin x = \cos x$.

We have, by Trigonometry,

$$\sin x' - \sin x = 2 \cos \tfrac{1}{2}(x' + x) \sin \tfrac{1}{2}(x' - x).$$

$$\therefore \quad \frac{\sin x' - \sin x}{x' - x} = \cos \tfrac{1}{2}(x' + x) \frac{\sin \frac{1}{2}(x' - x)}{\frac{1}{2}(x' - x)}.$$

But, by § 12, Ex. ~~4~~ 6, $$\mathop{\mathcal{L}}_{x'(=)x} \frac{\sin \frac{1}{2}(x' - x)}{\frac{1}{2}(x' - x)} = 1,$$

$$\therefore \quad f'(x) = \cos x.$$

4. Show that the derivative of any constant is zero.

If A is any constant, it keeps its value unchanged whatever be the value of x. Therefore the difference-quotient is

$$\frac{A - A}{x_1 - x} = 0$$

for all values of $x_1 \neq x$ and when $x_1(=)x$. Consequently $DA = 0$.

5. Show that the derivative of the product of a constant and any function of x is equal to the product of the constant and the derivative of the function.

Let a be constant and y a function of x. Let y take the value y_1 when x takes the value x_1. The difference-quotient of ay is

$$\frac{ay_1 - ay}{x_1 - x} = a\frac{y_1 - y}{x_1 - x},$$

the limit of which is aDy. $\therefore \quad Day = aDy.$

31. Geometrical Picture.—We have seen that a differentiable function is necessarily continuous. We shall now see that the assemblage of points taken to represent it possesses the characteristic property of a determinate direction at each point and can be considered as a curve.

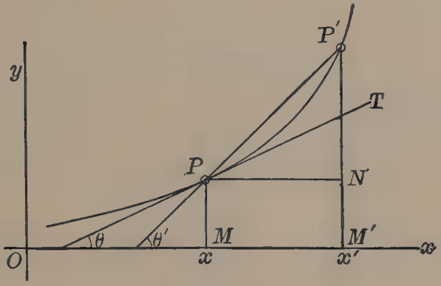

FIG. 6.

In the figure, if P, P' represent $f(x)$, $f(x')$, then

$$\Delta x = x' - x = MM',$$
$$\Delta f(x) = f(x') - f(x) = NP',$$
$$\frac{\Delta f(x)}{\Delta x} = \frac{NP'}{PN} = \tan\theta',$$

where $\theta' = \angle NPP'$ is the angle which the secant PP' makes with Ox. By the definition of a tangent to a curve, the limiting position of the secant PP' as the point P' moves along the curve and converges to P as a limit is the tangent PT to the curve at P. At the same time θ' converges to θ as a limit, θ being the angle which the tangent PT makes with Ox. But $\tan\theta$ is the limit of $\tan\theta'$, and is therefore the limit of the difference-quotient, or is the derivative of $f(x)$ at x.

Therefore we have

$$Df(x) = f'(x) = \tan\theta.$$

Hence the derivative of a function is represented by the *slope* of the tangent to the curve which represents the function. The direction of a curve at any point on it is the direction of the tangent there, and the slope or declivity of the curve is that of its tangent at the point.

32. Ab Initio Differentiation.—The process of differentiating a given function directly from the definition by evaluating the limit of the difference-quotient is, in general, a complicated and tedious process. We shall in the next chapter deduce certain rules of differentiation by which, when once we have differentiated log x and sin x by the *ab initio* process, we can write down directly the derivatives of all the elementary functions in terms of these derivatives and those which follow. Meanwhile, in order not to lose sight of the *ab initio* process and the rationale of differentiation which is at bottom always the evaluation of a limit, the limit of the difference-quotient, the following exercises are set for solution by this method.

EXERCISES.

Differentiate the following functions :

1. $3x^2 - 6x$.	$6(x - 1)$.
2. $7x^4 - 13$.	$28x^3$.
3. $(x - 1)(2x + 3)$.	$4x + 1$.
4. x^{-1}.	$- x^{-2}$.
5. ax^{-3}.	$- 3ax^{-4}$.
6. $(x - a)/(x + a)$.	$2a(x + a)^{-2}$.
7. $x^{\frac{3}{2}}$.	$\frac{3}{2}x^{\frac{1}{2}}$.
8. $(x^2 - 2)^{\frac{1}{2}}$.	$x(x^2 - 2)^{-\frac{1}{2}}$.
9. $2(x + 1)^{-\frac{1}{2}}$.	$-(x + 1)^{-\frac{3}{2}}$.
10. $x^{\frac{1}{3}}$.	$\frac{1}{3}x^{-\frac{2}{3}}$.
11. x^n. (n any finite integer.*)	nx^{n-1}.
12. $x^{\frac{p}{q}}$. (p and q positive integers.*)	$\frac{p}{q}x^{\frac{p}{q}-1}$.
13. cos x.	− sin x.
14. tan x.	sec² x.
15. log x. (See § 15, Ex. 11.)	x^{-1}.
16. sec x.	sec x tan x.
17. a^x. (Use Ex. 15, § 15.)	$a^x \log_e a$.
18. x^a. (x positive, a rational.)	ax^{a-1}.

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* Divide numerator and denominator of the diff.-quot. by $x_1 - x$ in Ex. 11, and by $x_1^{\frac{1}{q}} - x^{\frac{1}{q}}$ in Ex. 12.

CHAPTER II.

RULES FOR ELEMENTARY DIFFERENTIATION.

33. As was stated in Chapter I, when we have once differentiated x^a, $\sin x$, $\log x$, by the *ab initio* process, we can differentiate directly any elementary function of these functions by certain rules for differentiation, without recourse to the *ab initio* process directly.* These rules are themselves deduced by that process, and their application to differentiation is but a short method of evaluating the limits which we call derivatives. We shall see that the direct differentiation of only two, $\sin x$ and $\log x$, are necessary, for x^a can be differentiated by means of $\log x$. Independent proofs, however, are given in each case.

34. Derivative of $\log_a x$.—We have for the difference-quotient, writing $x_1 - x = h$,

$$\begin{aligned}\frac{\log_a(x+h) - \log_a x}{h} &= \frac{1}{h}\log_a\left(1+\frac{h}{x}\right),\\ &= \frac{1}{x}\log_a\left(1+\frac{h}{x}\right)^{\frac{x}{h}},\\ &= \frac{1}{x}\log_a\left(1+\frac{1}{z}\right)^{z},\end{aligned}$$

writing $x/h = z$. When $h(=)0$, $z = \infty$.

$$\begin{aligned}\therefore D\log_a x &= \mathop{\mathcal{L}}_{z=\infty}\frac{1}{x}\log_a\left(1+\frac{1}{z}\right)^{z},\\ &= \frac{1}{x}\log_a \mathop{\mathcal{L}}_{z=\infty}\left(1+\frac{1}{z}\right)^{z}. \qquad \S\,15,\ \text{Ex. } 6.\end{aligned}$$

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* As a matter of fact, the evaluation of only one of these functions, $\log x$, by the *ab initio* process is necessary. That is, the differentiation of all functions can be reduced to the evaluation of the single limit, $(1 + 1/x)^x$, when $x = \infty$, § 15, Ex. 12. For, the differentiation of $\log x$ gives that of e^x, and we have $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$, where $i \equiv +\sqrt{-1}$. We do not, however, recognize complex numbers in this book directly, which necessitates an independent differentiation of $\sin x$, and restricts us to a geometrical definition and differentiation of that function.

The evaluation of this limit is effected in § 15, Ex. 12, and is the number

$$\mathop{\mathcal{L}}_{z=\infty}\left(1+\frac{1}{z}\right)^{z} = e = 2\cdot7182\ .\ .\ .$$

$$\therefore\ D \log_a x = \frac{1}{x}\log_a e.$$

In particular, if $a \equiv e$, then $\log_e e = 1$, and

$$D \log x = \frac{1}{x}.$$

According to common usage, when the base of the logarithm employed is e we omit writing the base and put $\log x$ for $\log_e x$.

35. Derivative of x^a.—Let $a = p/q$, where p and q are positive integers.

Dividing the numerator and denominator of the difference-quotient

$$\frac{x_1^{\frac{p}{q}} - x^{\frac{p}{q}}}{x_1 - x}$$

by $x_1^{\frac{1}{q}} - x^{\frac{1}{q}}$, the difference quotient-becomes

$$\frac{\left(x_1^{\frac{1}{q}}\right)^{p-1} + \left(x_1^{\frac{1}{q}}\right)^{p-2} x^{\frac{1}{q}} + \ .\ .\ .\ + x_1^{\frac{1}{q}}\left(x^{\frac{1}{q}}\right)^{p-2} + \left(x^{\frac{1}{q}}\right)^{p-1}}{\left(x_1^{\frac{1}{q}}\right)^{q-1} + \left(x_1^{\frac{1}{q}}\right)^{q-2} x^{\frac{1}{q}} + \ .\ .\ .\ + x_1^{\frac{1}{q}}\left(x^{\frac{1}{q}}\right)^{q-2} + \left(x^{\frac{1}{q}}\right)^{q-1}}.$$

In the numerator there are p terms each of which has the limit $\left(x^{\frac{1}{q}}\right)^{p-1}$, and in the denominator there are q terms each of which has the limit $\left(x^{\frac{1}{q}}\right)^{q-1}$, when $x_1(=)x$. Therefore the limit of the difference-quotient is

$$Dx^{\frac{p}{q}} = \frac{p}{q}\frac{\left(x^{\frac{1}{q}}\right)^{p-1}}{\left(x^{\frac{1}{q}}\right)^{q-1}} = \frac{p}{q}\left(x^{\frac{1}{q}}\right)^{p-q},$$

$$= \frac{p}{q}x^{\frac{p}{q}-1}.$$

If $a = -p/q$, then the difference-quotient is

$$\frac{x_1^{-\frac{p}{q}} - x^{-\frac{p}{q}}}{x_1 - x} = \frac{-1}{x_1^{\frac{p}{q}}x^{\frac{p}{q}}}\ \frac{x_1^{\frac{p}{q}} - x^{\frac{p}{q}}}{x_1 - x},$$

the limit of which for $x_1(=)x$ is, by the above,

$$-\frac{1}{x^{\frac{2p}{q}}}\ \frac{p}{q}x^{\frac{p}{q}-1} = -\frac{p}{q}x^{-\frac{p}{q}-1}.$$

Therefore, whatever be the rational number a,

$$Dx^a = ax^{a-1}.$$

Rule: Multiply by the exponent and diminish the exponent by 1.

36. Derivative of sin x, cos x.—It has been shown in Chapter I, § 30, Ex. 3, that

$$D \sin x = \cos x.$$

The derivatives of all the other circular functions can and should be deduced in like manner. They can, however, as we shall see, all be deduced from that of the sin x.

For immediate use we have, from Trigonometry,

$$\cos x' - \cos x = -2 \sin \tfrac{1}{2}(x' + x) \sin \tfrac{1}{2}(x' - x).$$

$$\therefore \quad \frac{\cos x' - \cos x}{x' - x} = -\sin \tfrac{1}{2}(x' + x) \frac{\sin \frac{1}{2}(x' - x)}{\frac{1}{2}(x' - x)}.$$

Hence, on passing to the limit,

$$D \cos x = -\sin x.$$

RULES FOR DIFFERENTIATION.

37. We proceed to establish rules for the derivative of the (1) **sum,** (2) **product,** (3) **quotient,** (4) **inverse function, and** (5) **function of a function,** in terms of the derivatives of the functions involved.

These are the general rules for the differentiation of all functions with which we shall be concerned. It is necessary to know them perfectly, for they are the tools with which the Differential Calculus works.

38. Derivative of an Algebraic Sum.

Let $$y = u + v + w,$$

where u, v, w are differentiable functions of x. Let the differences of these functions be Δy, Δu, Δv, Δw, respectively, corresponding to the difference Δx of the variable x. Then, if y, u, v, w take the values y_1, u_1, v_1, w_1 when x takes the value x_1, we have

$$y_1 - y = \Delta y, \qquad \therefore \; y_1 = y + \Delta y,$$

and so for u, v, w.

$$y_1 = u_1 + v_1 + w_1,$$

$$y_1 - y = (u_1 - u) + (v_1 - v) + (w_1 - w),$$

or

$$\Delta y = \Delta u + \Delta v + \Delta w.$$

$$\therefore \; \frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} + \frac{\Delta w}{\Delta x}.$$

The student should observe the detail with which the difference-quotient is worked out here, as this detail will be omitted hereafter and he will be expected to supply it.

Since the limit of a sum is equal to the sum of the limits, we have for $\Delta x (=) 0$, on passing to limits,

$$Dy = Du + Dv + Dw, \tag{I}$$

or

$$D(u + v + w) = Du + Dv + Dw.$$

Corollary. What has been proved for three functions here is equally true for any finite number of functions $u_1, \ldots u_n$, and it can be proved in the same way that

$$D\sum_1^n u_r = \sum_1^n Du_r;$$

hence the rule:

The derivative of the algebraic sum of a finite number of differentiable functions is equal to the sum of their derivatives.

In all cases in which we pass from an equation in difference-quotients to one in derivatives, the student is required to quote the corresponding theorem of limits, § 15, which justifies the equality.

EXAMPLES.

1. The derivative of any polynomial in x,

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n,$$

is

$$a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1}.$$

This can be expressed in the following rule:

Strike out every term independent of x, since its derivative is zero, and multiply each remaining term by the exponent of that term and diminish that exponent by 1.

2. If

$$y = 2x^{\frac{5}{2}} + \log x^5 - 3 \sin x,$$

show that

$$Dy = 5x^{\frac{3}{2}} + 5/x - 3 \cos x.$$

3. If

$$f(x) \equiv \frac{cx^2 + bx + a}{x}, \quad \text{show that}$$

$$f'(x) = c - ax^{-2}.$$

4. Make use of the identity

$$\sin (a + x) = \sin a \cos x + \cos a \sin x,$$

to show that

$$D \sin (a + x) = \cos (a + x).$$

39. Derivative of a Product of Functions.

Let

$$y = uv.$$

Then, with notation as in § 40, we have

$$\begin{aligned} \Delta y &= (u + \Delta u)(v + \Delta v) - uv, \\ &= v\,\Delta u + u\,\Delta v + \Delta u \cdot \Delta v. \end{aligned}$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = v\frac{\Delta u}{\Delta x} + u\frac{\Delta v}{\Delta x} + \frac{\Delta u \cdot \Delta v}{\Delta x}. \tag{i}$$

Since, by hypothesis,

$$Du = \mathcal{L}\frac{\Delta u}{\Delta x}, \quad \text{and} \quad Dv = \mathcal{L}\frac{\Delta v}{\Delta x}$$

are finite, the last term on the right of (i) has the limit 0 when $\Delta x (=) 0$; for it can be written either

$$\Delta u \left(\frac{\Delta v}{\Delta x}\right) \quad \text{or} \quad \left(\frac{\Delta u}{\Delta x}\right) \Delta v,$$

and $\Delta u (=) 0$, $\Delta v (=) 0$, when $\Delta x (=) 0$, the functions being continuous.

Therefore, in the limit, (i) becomes

$$D(uv) = v\,Du + u\,Dv. \tag{II}$$

In particular, if v is constant, $v = a$, then $Da = 0$, and

$$D(au) = a\,Du.$$

Corollary. Show that

$$D(uvw) = uv\,Dw + uw\,Dv + vw\,Du,$$

and, in general, that the derivative of the product of a finite number of functions is equal to the sum of the derivative of each function multiplied by the product of the others.

EXAMPLES.

1. Show that $D(x^n \sin x) = nx^{n-1} \sin x + x^n \cos x$.

2. $D(x^a \log x) = x^{a-1} (\log x^a + 1)$.

3. Show that $\mathcal{L}_{x(=)0} (D \log x^{\sin x} - \cos x \log x) = 1$.

4. Show that $D \sin^2 x = \sin 2x$.

5. If $y = (\log x)^2$, show that $Dy = \log (x)^{\frac{2}{x}}$.

6. If $f(x) \equiv \log x^2$, then $f'(x) = 2/x$.

7. Show that $D \sin 2x = 2 \cos 2x$.

8. Show that $D \cos 2x = -2 \sin 2x$.
Use $\cos 2x = (\cos x + \sin x)(\cos x - \sin x)$.

9. Show that $D (\log x^x) = \log x + 1$.

40. Derivative of a Quotient.

Let
$$y = \frac{u}{v}.$$

Then y, u, v, become $y + \Delta y$, $u + \Delta u$, $v + \Delta v$, when x becomes $x + \Delta x$, and we have

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v},$$

$$= \frac{v\Delta u - u\Delta v}{v(v + \Delta v)}.$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}.$$

Since $\Delta v (=) 0$ when $\Delta x (=) 0$, we have, provided $v \neq 0$, on passing to limits

$$Dy = D\left(\frac{u}{v}\right) = \frac{v\,Du - u\,Dv}{v^2}. \tag{III}$$

In particular, if $u \equiv a$, any constant, then $Du = 0$, and

$$D\left(\frac{a}{v}\right) = -\,a\frac{Dv}{v^2}. \tag{IV}$$

EXAMPLES.

1. Show that $D \tan x = \sec^2 x.$

We have
$$D \tan x = D\frac{\sin x}{\cos x}, \quad = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}, \quad = \sec^2 x.$$

2. Show that $D \cot x = -\csc^2 x$, using both
$\cot x = \cos x/\sin x$ and $\cot x = 1/\tan x$.

3. Show that $D \sec x = \sec x \tan x$,
using both $\sec x = 1/\cos x$ and $\sec x = \tan x/\sin x$.

4. Show that $D \csc x = -\csc x \cot x$.

5. Show that $D \operatorname{vers} x = \sin x$.

6. Show that $D\dfrac{a+x}{b+x} = \dfrac{b-a}{(b+x)^2}$.

7. Show that $D\left(\dfrac{a+x}{a-x}\right)^2 = 4a\dfrac{a+x}{(a-x)^3}$.

8. Show that $D\dfrac{\log x}{\log ax} = \dfrac{\log a^{\frac{1}{x}}}{(\log ax)^2}$.

41. Derivative of the Inverse Function.—If y is a continuous function of x, we must have $\Delta y (=) 0$ when $\Delta x (=) 0$, by the definition of continuity. Therefore for any particular value of x at which y is a continuous function of x we can always make Δy converge to 0 continuously in any manner we choose, such that simultaneously we have $\Delta x = 0$. Also, for corresponding differences Δy and Δx, we have

$$\frac{\Delta y}{\Delta x}\frac{\Delta x}{\Delta y} = 1.$$

If we represent the derivative of y with respect to x, by $D_x y$, and the derivative of x with respect to y, by $D_y x$, then whenever y is a differentiable function of x and $D_x y \neq 0$, we shall have x a differentiable function of y, and the relation

$$D_x y \cdot D_y x = 1$$

always exists.

Therefore, if y and x are functions of each other and the derivative of the first with respect to the second can be found, then the derivative of the second with respect to the first is the reciprocal or inverse of the first derivative.

If $y = f(x)$, then $x = \phi(y)$, obtained by solving $y = f(x)$ for x, is the inverse function of $f(x)$.

GEOMETRICAL ILLUSTRATION.

If the curve AB represents the function $y = f(x)$, and we consider x as the function and y as the variable, we have $x = \phi(y)$ represented by the same curve, except that now Oy is the axis of the variable and Ox the axis of the function. For a particular x, the point X represents $f(x)$ and $\phi(y)$, and we have

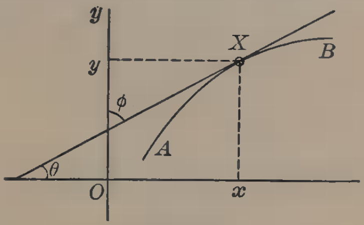

FIG. 7.

$$xX = f(x); \quad yX = \phi(y).$$

Again, if θ, ϕ are the angles made by the tangent to AB at X, with Ox, Oy respectively, measured according to the conventions of Cartesian Geometry, we have

$$D_xy = f'(x) = \tan\theta,$$
$$D_yx = \phi'(y) = \tan\phi.$$

But, since we always have $\tan\theta \tan\phi = 1$,

$$\therefore\ D_xy \cdot D_yx = 1.$$

EXAMPLES.

1. If $y = x^2 + 2ax + b$, then

$$D_xy = 2(x + a).$$

$$\therefore\ D_yx = \frac{1}{2(x + a)}.$$

If we solve for x, we get the inverse function

$$x = -a \pm \sqrt{a^2 + y - b},$$

a function which we do not yet know how to differentiate, but we know its derivative must be the value D_yx obtained above.

2. If $y = x^{\frac{3}{5}}$, find D_xy; D_yx, and verify $Dy\,Dx = 1$.

3. Differentiate $\sin^{-1}x$.

If $y = \sin^{-1}x$, then $x = \sin y$.

$$\therefore \quad D_y x = \cos y = \sqrt{1 - x^2}.$$

Hence $$D_x y = \frac{1}{\sqrt{1 - x^2}}.$$

We know from Trigonometry that the angle whose sine is x, $\sin^{-1}x$, is multiple-valued and that

$$\sin [n\pi + (-1)^n\theta] = \sin \theta,$$

where n is any integer. In the derivative of $\sin^{-1}x$ above, the radical shows its value is ambiguous as to sign. But if we agree to take $\sin^{-1}x$ to mean that angle between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$ whose sine is x, there is no ambiguity, since then $\cos y$ is positive.

Then we have

$$D \sin^{-1}x = \frac{1}{+\sqrt{1 - x^2}}.$$

4. Show in like manner that

$$D \cos^{-1}x = \frac{1}{-\sqrt{1 - x^2}},$$

where $$0 < \cos^{-1}x < \pi.$$

This can also be shown immediately by differentiating the identity

$$\sin^{-1}x + \cos^{-1}x = \tfrac{1}{2}\pi.$$

5. Show that $$D \tan^{-1}x = \frac{+1}{1 + x^2}.$$

Put $y = \tan^{-1}x$, then $x = \tan y$, and

$$D_y x = \sec^2 y = 1 + x^2. \qquad \text{Ex. 1, § 41.}$$

$$\therefore \quad D \tan^{-1}x = \frac{+1}{1 + x^2},$$

where we take $\tan^{-1}x$ to be that number such that

$$-\tfrac{1}{2}\pi < \tan^{-1}x < +\tfrac{1}{2}\pi.$$

6. Show in like manner that

$$D \cot^{-1}x = \frac{-1}{1 + x^2},$$

where $$0 < \cot^{-1}x < \pi.$$

Also, by § 38, from

$$\tan^{-1}x + \cot^{-1}x = \tfrac{1}{2}\pi.$$

7. Show that $$D \sec^{-1}x = \frac{+1}{x\sqrt{x^2 - 1}}.$$

If $y = \sec^{-1}x$, then $x = \sec y$, and

$$D_y x = \sec y \tan y = x\sqrt{x^2 - 1}. \qquad \text{Ex. 3, § 41.}$$

$$\therefore \quad D_x y = \frac{+1}{x\sqrt{x^2 - 1}}.$$

8. Show, as in Ex. 7, that

$$D \csc^{-1}x = \frac{-1}{x\sqrt{x^2 - 1}}.$$

Also, by § 38, using the identity

$$\csc^{-1}x + \sec^{-1}x = \tfrac{1}{2}\pi.$$

9. Differentiate a^x.

Put $y = a^x$, then $x = \log_a y$.

Therefore, by § 34, we have

$$D_y x = \frac{1}{y} \log_a e.$$

$$\therefore \quad D_x y = \frac{y}{\log_a e} = a^x \log_e a.$$

In particular, if $a \equiv e$, then

$$Da^x = a^x \log a$$

becomes

$$De^x = e^x,$$

or the function e^x is not changed by differentiation.

42. Differentiation of a Function of a Function.—We come now to consider one of the most powerful methods of differentiating certain classes of functions.*

Let z be a function of the variable y, say $z = f(y)$, and let y be a function of x, say $y = \phi(x)$. We require the derivative of z with respect to the variable x.

If z is a differentiable function of the variable y, and y is a differentiable function of the variable x, for corresponding values of z, y and x, then we shall have

$$D_x z = D_y z \cdot D_x y, \tag{VI}$$

or

$$f'_x(y) = f'_y(y) \cdot D_x y.$$

For, we have

$$\frac{\Delta z}{\Delta x} \equiv \frac{\Delta z}{\Delta y} \frac{\Delta y}{\Delta x},$$

and since by hypothesis $D_y z$ and $D_x y$ are determinate limits, $D_x z$ is a determinate limit equal to their product, and (VI) is true.

Corollary. If u is a function of v, v a function of w, w a function of z, z a function of y, and finally y a function of x, then the difference-relation

$$\frac{\Delta u}{\Delta x} \equiv \frac{\Delta u}{\Delta v} \frac{\Delta v}{\Delta w} \frac{\Delta w}{\Delta z} \frac{\Delta z}{\Delta y} \frac{\Delta y}{\Delta x}$$

leads to the derivative

$$D_x u = D_v u \cdot D_w v \cdot D_z w \cdot D_y z \cdot D_x y$$

whenever the derivatives on the right are determinate. Hence the following rule: The derivative of a function of a function, etc., is equal to the product of the derivatives of the functions, each derivative taken with respect to its particular variable.

EXAMPLES.

1. Differentiate x^α, when x is positive and α irrational.

Put $y = x^\alpha$, then taking the logarithm or, as we shall say, "logarating,"† we have

$$\log y = \alpha \log x.$$

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* For a geometrical picture of a function of a function, see Appendix, Note 2.

† The term "taking the logarithm" is the meaning of an operation so frequently used that it seems to deserve a verb "to logarate."

Differentiate with respect to x. We have

$$\frac{1}{y}Dy = \frac{\alpha}{x}.$$

$$\therefore \quad Dy = \alpha\frac{y}{x} = \alpha x^{\alpha-1},$$

the same formula as when α is rational.

2. Differentiate $(a + bx)^{\alpha}$.

Put $f(y) = y^{\alpha}$, where $y = a + bx$.

$$\therefore \quad f'_x(y) = \alpha y^{\alpha-1} Dy, \quad \text{and} \quad Dy = b.$$

$$\therefore \quad D(a+bx)^{\alpha} = b\alpha(a+bx)^{\alpha-1}.$$

3. To find $D\cos x$ from $D\sin x = \cos x$.

We have $\cos x = \sin(\frac{1}{2}\pi - x)$.

$$\begin{aligned}\therefore \quad D\cos x &= D\sin(\tfrac{1}{2}\pi - x),\\ &= \cos(\tfrac{1}{2}\pi - x)\, D(\tfrac{1}{2}\pi - x),\\ &= -\sin x.\end{aligned}$$

4. Deduce in like manner $D\cot x$, $D\csc x$, given the derivatives of $\tan x$ and $\sec x$.

5. If $y = \cos^{-1}x$, then $x = \cos y$.

Differentiate both sides with respect to x.

$$\therefore \quad 1 = -\sin y\, Dy.$$

$$\therefore \quad D\cos^{-1}x, \text{ as before.}$$

6. Find in like manner $D\cot^{-1}x$, $D\csc^{-1}x$, from $D\tan x$, $D\sec x$.

7. If $y = a^x$, then $\log y = x \log a$.

Differentiating with respect to x, we have

$$D_y \log y \cdot D_x y = \log a,$$

or

$$\frac{1}{y} D_x y = \log a.$$

$$\therefore \quad D_x y = a^x \log a, \text{ as before}$$

8. Differentiate $\sqrt{a^2 - x^2}$. Put $u = a^2 - x^2$.

$$\begin{aligned}\therefore \quad D_x u^{\frac{1}{2}} &= D_u u^{\frac{1}{2}} D_x u,\\ &= \tfrac{1}{2}u^{-\frac{1}{2}}(-2x),\\ &= \frac{-x}{\sqrt{a^2 - x^2}}.\end{aligned}$$

9. As an example of the differentiation of a complicated function of functions, differentiate

$$\log \sin e^{\cos(a-bx)^3}.$$

Let

$$\begin{aligned}
y &= a - bx. & \therefore \quad D_x y &= -b.\\
z &= (a - bx)^3 = y^3, & \therefore \quad D_y z &= 3y^2.\\
u &= \cos(a - bx)^3 = \cos z, & \therefore \quad D_z u &= -\sin z.\\
v &= e^{\cos(a-bx)^3} = e^u, & \therefore \quad D_u v &= e^u.\\
w &= \sin e^{\cos(a-bx)^3} = \sin v, & \therefore \quad D_v w &= \cos v.
\end{aligned}$$

Therefore the required derivative is the function

$$\frac{3b}{w} y^2 e^u \sin z \cos v,$$

which can be expressed as a function of x directly.

43. Examples of Logarithmic Differentiation.—The differentiation of products, quotients, and exponential functions are frequently simplified by taking the logarithm before differentiation.

EXAMPLES.

1. Show that

$$\frac{D(uv^{\pm 1})}{uv^{\pm 1}} = \frac{Du}{u} \pm \frac{Dv}{v},$$

the upper signs going together and lower signs going together. Put $y = uv^{\pm 1}$, then taking the logarithm,

$$\log y = \log u \pm \log v.$$

$$\therefore \quad \frac{Dy}{y} = \frac{Du}{u} \pm \frac{Dv}{v}.$$

This expresses compactly the formulæ for differentiating the product and the quotient of two functions.

2. Show that if $u_1 u_2 \ldots u_n$ is the product of n functions of x, the derivative of the product is given by

$$\frac{D\Pi_1^n u_r}{\Pi_1^n u_r} = \sum_1^n \frac{Du_r}{u_r}.$$

3. Differentiate u^v, where u and v are functions of x.

Put $y = uv$ and take the logarithm.

$$\therefore \quad \log y = v \log u.$$

Differentiating,

$$\frac{Dy}{y} = Dv \cdot \log u + v\frac{Du}{u}.$$

$$\therefore \quad Du^v = u^v\left(\log u \, Dv + \frac{v}{u}Du\right).$$

4. Differentiate $\log_v u$.

Put $y = \log_v u$, then $v^y = u$. Logarate this with respect to the base e, and we have

$$y \log v = \log u.$$

Differentiating with respect to x,

$$\log v \, Dy + \frac{y}{v}Dv = \frac{Du}{u}.$$

$$\therefore \quad D \log_v u = \left(\frac{Du}{u} - \frac{\log u}{\log v}\frac{Dv}{v}\right)\frac{1}{\log v}.$$

44. For general reference in differentiation a table or catechism of the standard rules and elementary derivatives is compiled and should be memorized.

In this table u or v is any differentiable function of a variable with respect to which the differentiation is performed.

The Derivative Catechism.

1. $D(cu) = c\,Du.$
2. $D(u + v) = Du + Dv.$
3. $D(uv) = u\,Dv + v\,Du.$
4. $D\left(\dfrac{u}{v}\right) = \dfrac{v\,Du - u\,Dv}{v^2}.$
5. $D\left(\dfrac{1}{v}\right) = -\dfrac{Dv}{v^2}.$
6. $Du^a = au^{a-1}\,Du.$
7. $D \log_a u = \dfrac{Du}{u} \log_a e.$
8. $D \log u = \dfrac{Du}{u}.$
9. $Da^u = a^u \log a\,Du.$
10. $De^u = e^u\,Du.$
11. $Du^v = u^v \log u\,Dv + vu^{v-1}\,Du.$
12. $D \sin u = + \cos u\,Du.$
13. $D \cos u = - \sin u\,Du.$
14. $D \tan u = + \sec^2 u\,Du.$
15. $D \cot u = - \csc^2 u\,Du.$
16. $D \sec u = + \sec u \tan u\,Du.$
17. $D \csc u = - \csc u \cot u\,Du.$
18. $D \sin^{-1}u = + \dfrac{Du}{\sqrt{1 - u^2}}.$
19. $D \cos^{-1}u = - \dfrac{Du}{\sqrt{1 - u^2}}.$
20. $D \tan^{-1}u = + \dfrac{Du}{1 + u^2}.$
21. $D \cot^{-1}u = - \dfrac{Du}{1 + u^2}.$
22. $D \sec^{-1}u = + \dfrac{Du}{u\sqrt{u^2 - 1}}.$
23. $D \csc^{-1}u = - \dfrac{Du}{u\sqrt{u^2 - 1}}.$
24. $D \operatorname{vers}^{-1}u = \dfrac{Du}{\sqrt{2u - u^2}}.$

EXERCISES.

1. Differentiate by the *ab initio* process, and check by the catechism, the following functions :

(1), x. (2), cx. (3), $2x^3$. (4), cx^4. (5), x^{-1}. (6), ax^{-4}. (7), $x^2 - 2x$. (8), $5x^3 - 4x + 7$. (9), $1/(ax + b)$. (10), $x^4 - 3x - 2x^{-2}$. (11), $(x - 1)(3x + 2)$. (12), $(x - 3)/(x + 5)$. (13), $x^{\frac{1}{2}}$. (14), $x^{\frac{2}{3}}$. (15), $x^{-\frac{1}{3}}$.

The solution in each case depends on the fact that $a^n - b^n$ is divisible by $a - b$ when n is an integer.

(16), $\cos\dfrac{x}{a}$. (17), $\sin ax$. (18), $\tan ax$. (19), $\csc ax$.

2. Draw the curve $y = 3x^2$ and find the slope of the tangent where $x = 2$.

3. Draw the curve $y = x^2 + 2x - 3$, and find the angle at which it crosses the Ox axis.

4. Use the relation of the derivatives of inverse functions to find the derivatives of $x^{\frac{1}{2}}$, $x^{\frac{1}{5}}$, $x^{-\frac{1}{3}}$, $x^{-\frac{1}{n}}$, and check the results by the rule for differentiating a function of a function.

5. Show that the equation to the tangent to any curve $y = f(x)$ is

$$y = f(a) + (x - a)f'(a),$$

the point representing $f(a)$ being the point of contact.

6. Differentiate $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, $\sqrt{a^2 + 2bx}$.

Ans. $-x(a^2 - x^2)^{-\frac{1}{2}}$, $x(x^2 - a^2)^{-\frac{1}{2}}$, $b(a^2 + 2bx)^{-\frac{1}{2}}$.

(1) $D(a + x)^c = c(a + x)^{c-1}$.

(2) $D(a + x^2)^3 = 6x(a + x^2)^2$.

(3) $D(c + bx^3)^4 = 12bx^2(c + bx^3)^3$.

(4) $D(ax^2 + bx + c)^5 = 5(ax^2 + bx + c)^4(2ax + b)$.

(5) $D(a^2 - x^2)^5 = -10x(a^2 - x^2)^4$.

(6) $D(a^2x + bx^2)^7 = 7(a^2x + bx^2)^6(a^2 + 2bx)$.

(7) $D(b + cx^m)^n = mncx^{m-1}(b + cx^m)^{n-1}$.

(8) $D(1 + ax^2)^{-\frac{1}{2}} = -ax(1 + ax^2)^{-\frac{3}{2}}$.

(9) $D(a^2 - x^3)^{\frac{1}{2}} = -\frac{3}{2}x^2(a^2 - x^3)^{-\frac{1}{2}}$.

(10) $D\sin^2 ax = -D\cos^2 ax = a\sin 2ax$.

(11) $D\sin^n ax = na\sin^{n-1}ax\cos ax$.

(12) $D\sin(\sin x) = \cos x\cos(\sin x)$.

7. Show that the equation to the tangent at $x = \alpha$, $y = \beta$, for the curve

(1) $x^2 + y^2 = a^2$ is $x\alpha + y\beta = a^2$.

(2) $\dfrac{x^2}{a^2} \pm \dfrac{y^2}{b^2} = 1$ is $\dfrac{x\alpha}{a^2} \pm \dfrac{y\beta}{b^2} = 1$.

(3) $y^2 = 4px$ is $y\beta = 2p(x + \alpha)$.

8. Given $\sin 3x = 3\sin x - 4\sin^3 x$, find $\cos 3x$.

9. Given $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$, find $\sin 5x$.

10. Verify $\cos x = 1 - 2\sin^2 \frac{1}{2}x$, by differentiating.

11. Obtain new identities by differentiating

$$\sin 3a + \sin 2a - \sin a = 4\sin a\cos\tfrac{1}{2}a\cos\tfrac{3}{2}a,$$
$$\sin b\sin(\tfrac{1}{3}\pi - b)\sin(\tfrac{1}{3}\pi + b) = \tfrac{1}{4}\sin 3b,$$

a and b being variables.

12. Differentiate the identity

$$\cos^3 2x - 3\cos 2x = 4(\cos^6 x - \sin^6 x).$$

13. Differentiate $x^3 \sin x$, $x^2 \sqrt{a + bx}$, $(ax + b)^{\frac{2}{3}}$.

14. $D[(x + 1)^5(2x - 1)^3] = (16x + 1)(x + 1)^4(2x - 1)^2$.

15. $D[(x^2 + 1)(x^3 - x)^{\frac{1}{2}}] = \dfrac{7x^4 - 2x^2 - 1}{2\sqrt{x^3 - x}}$.

16. $D\{(1 - 2x + 3x^2 - 4x^3)(1 + x)^2\} = -20x^3(1 + x)$.

17. $D\{(1 - 3x^2 + 6x^4)(1 + x^2)^3\} = 60x^5(1 + x^2)^2$.

18. Show that

$$D\sqrt{\frac{1 + x}{1 - x}} = \frac{1}{(1 - x)\sqrt{1 - x^2}}; \qquad D\frac{x + 3}{x^2 + 3} = \frac{3 - 6x - x^2}{(x^2 + 3)^2};$$

$$D\frac{x^n}{(1 - x)^n} = \frac{nx^{n-1}}{(1 - x)^{n+1}}; \qquad D\frac{x}{\sqrt{1 - x^2}} = (1 - x^2)^{-\frac{3}{2}};$$

$$D\frac{1 - x}{\sqrt{1 + x^2}} = -\frac{1 + x}{(1 + x^2)^{\frac{3}{2}}}; \qquad D\frac{2\sqrt{x}}{3 + x^2} = \frac{3(1 - x^2)}{(3 + x^2)^2\sqrt{x}};$$

$$D\frac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{a + x} - \sqrt{a - x}} = -\frac{a^2 + a\sqrt{a^2 - x^2}}{x^2\sqrt{a^2 - x^2}}.$$

19. Show that

$$D\tan^{-1}\frac{x}{\sqrt{1 - x^2}} = \frac{1}{\sqrt{1 - x^2}}; \qquad D\sin^{-1}\frac{3 + 2x}{\sqrt{13}} = \frac{1}{\sqrt{1 - 3x - x^2}}.$$

20. Differentiate $\sin^{-1}(x/a)$, $\tan^{-1}(ax + b)$, $\cos^{-1}\dfrac{x}{\sqrt{a^2 + x^2}}$, $\sec^{-1}(a/x)$, $\sec^{-1}(x + cx^2)$.

21. $D \log \sin x = \cot x$; $D \log \cos x = ?$

22. Differentiate e^{2x}, e^{-x}, e^{nx}, $e^{\sin x}$, $e^{\log x}$.

23. Differentiate a^{cx}, $a^{\sin ax}$, $a^{\log x}$, $a^{\tan x}$.

24. Differentiate $\log x^{\frac{1}{2}}$, $\log(a + x)$, $\log(ax + b)$, $x^n e^x$, $a^x e^x$, 2^x, $e^x \log(x + a)$, $\log(x + e^x)$, $e^x/\log x$, $\log(xe^x)$, $\sin(e^x)\log x$, $e^{\cos x}\log(\cos x)$, $\log_a \tan x$, $3^{\log x}$, $5^{\sin ax}$, $\log_{x^2}(\cos ax)$.

25. $D\sin[\cos(ax + b)^n] = -na(a + bx)^{n-1}\sin(ax + b)^n \cos[\cos(ax + b)^n]$.

26. If $y = \frac{1}{2}(e^x - e^{-x})$, show that

$$x = \log\left(y + \sqrt{1 + y^2}\right),$$

and that $D_x y \, D_y x = 1$.

27. In Ex. 1, § 41, differentiate x as a function of y and check the result there given.

CHAPTER III.

ON THE DIFFERENTIAL OF A FUNCTION.

45. Definition.—The *differential* of a function is defined to be the product of the derivative of the function and an arbitrary difference of the variable.

If $f(x)$ represents any function of x, and $x_1 - x$ any difference of the variable, then

$$(x_1 - x)f'(x)$$

is the differential of $f(x)$ at x.

The value of the differential at a particular value x depends on the value assigned to the arbitrary number x_1.

We use, after Leibnitz, the characteristic letter d to represent the differential, and write $df(x)$ to represent the differential of the function $f(x)$ at x. Thus

$$\begin{aligned} df(x) &\equiv (x_1 - x)f'(x), \\ &\equiv f'(x)\Delta x. \end{aligned}$$

46. Theorem.—The differential of a function is equal to the product of the derivative of the function into the differential of the variable.

For, let $f(x) \equiv x$, then $f'(x) = 1$, and

$$\begin{aligned} dx &= Dx \cdot \Delta x \\ &= \Delta x. \end{aligned}$$

Therefore we can write dx for Δx, and have

$$df(x) = f'(x)\,dx.$$

The differential of the variable is then any arbitrary difference or increment of the variable we choose to assign. In writing the differential of a variable we choose to assign to it *always* a finite number as its value. In fact we cannot *assign* to it any other value.

47. The Differential-Quotient of a Function.*—Since the differential of the variable is a finite number we can divide by it, and have

$$\frac{df(x)}{dx} = f'(x),$$

---

* By some writers the derivative $f'(x)$ is called the *differential-coefficient* of the function $f(x)$, because of its relation to the differentials in the equation

$$df(x) = f'(x)\,dx.$$

or, the *differential-quotient* of a function, which is the quotient of the differential of the function by the differential of the variable, is equal to the derivative of the function.

This furnishes another notation, due to Leibnitz, for the value of the derivative, and expresses that number as the quotient of two numbers. The advantages of this notation will appear continually in the sequel, in the symmetry of the equations, and in the analogy and relation of differentials to differences.

We frequently abbreviate the differential-quotient into

$$\frac{df}{dx} = f', \quad \text{or} \quad \frac{dy}{dx},$$

where $y = f(x)$. Also, when $f(x)$ is a complicated function we frequently write

$$\frac{df(x)}{dx} \equiv \frac{d}{dx}f(x).$$

48. Geometrical Illustration.—We have seen, § 31, that if $y = f(x)$ is represented by a curve PP_1, then the derivative $f'(x)$ or Dy is represented by $\tan\theta$, where θ is the angle made by the tangent PT to the curve at P with the axis Ox.

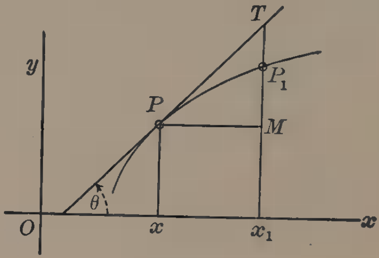

FIG. 8.

Assign any arbitrary number x_1, and let P_1 represent $f(x_1)$, and T the corresponding point on the tangent to the curve at P. Then we have

$$\begin{aligned} PM &= x_1 - x = \Delta x = dx. \\ df(x) &= (x_1 - x)f'(x), \\ &= PM \tan MPT, \\ &= MT. \end{aligned}$$

MT therefore represents the differential of the function $f(x)$ at x corresponding to x_1. While

$$MP_1 = f(x_1) - f(x) = \Delta f(x).$$

df and Δf are more nearly equal when Δx or dx is a small number.

Observe that for a particular x the differential-quotient

$$\frac{df(x)}{dx} = f'(x) = \tan\,\theta$$

is constant for all values of x_1.

49. Relation of Differentials to Differences.—Since the difference-quotient has the derivative for its limit, we can put

$$\frac{\Delta f(x)}{\Delta x} = f'(x) + \alpha,$$

where $\alpha (=) 0$, when $\Delta x (=) 0$. Therefore

$$\begin{aligned} \Delta f(x) &= f'(x)\Delta x + \alpha\,\Delta x, \\ &= f'(x)\,dx + \alpha\,\Delta x. \\ \therefore\ \frac{\Delta f(x)}{df(x)} &= 1 + \frac{\alpha}{f'(x)}. \end{aligned}$$

Hence, when $f'(x) \neq 0$, we have

$$\mathop{\pounds}_{\Delta x (=) 0} \frac{\Delta f(x)}{df(x)} = 1.$$

This substantiates the remark made in § 48 that the difference and the differential of a function are more nearly equal the smaller we take dx.

50. Differentiation with Differentials.—Observe that all the formulæ in the derivative table, § 44, are immediately true in differentials when we change D into d. For we need only multiply such derivative equation through by dx in order to make it read differentials instead of derivatives.

We have

$$df(u) = D_x f(u)\,dx.$$

For, by definition,

$$\begin{aligned} df(u) &= D_u f(u)\,du, \\ &= D_u f(u) \cdot D_x u\,dx, \\ &= D_x f(u)\,dx, \end{aligned}$$

since $\quad D_x f(u) = f'(u)\,D_x u, \quad$ and $\quad du = D_x u\,dx.$

$$\therefore\ D_u f(u)\,du = D_x f(u)\,dx,$$

or the first differential of a function is the same whatever be the variable.

More generally, let u, v, and w be functions of x. Distinguishing differentials like derivatives by subscripts, we have

$$\begin{aligned} d_v f(u) &= D_v f(u)\,dv = D_u f(u)\,D_v u\,dv, \\ &= D_u f(u)\,du = d_u f(u). \end{aligned}$$

In like manner, $d_w f(u) = d_u f(u)$. Therefore

$$d_v f(u) = d_w f(u),$$

or the differential of a function is independent of the variable employed. It is not necessary, therefore, to indicate the variable by subscripts or in any other way; in fact the variable need not be specified. It is due to this that frequently the use of differentials has marked advantages over that of derivatives.

51. We add a further list of exercises in differentiation, using indifferently the notations of derivatives, differential-quotients, and differentials in order to insure familiarity in their use. The sequel will show the advantage of each in its appropriate place.

EXERCISES.

1. If x, y, are the coordinates of a point on a curve, show that

$$Y - y = (X - x)\frac{dy}{dx}$$

or

$$(Y - y)\,dx = (X - x)\,dy$$

is the equation of the tangent at x, y, where X, Y are the current coordinates on the tangent. This equation can also be written

$$\frac{Y - y}{dy} = \frac{X - x}{dx}.$$

2. Show, with the above notation, that

$$(Y - y)\,dy + (X - x)\,dx = 0$$

is the equation of the normal at x, y.

3. Show that $d(x^3 \log x) = x^2(\log x^3 + 1)\,dx$.

4. $d(\cos mx \cos nx) = -m \cos nx \sin mx\,dx - n \cos mx \sin nx\,dx$.

5. $\dfrac{d}{dx}\sin^n x = n \sin^{n-1} x \cos x$.

6. $d \sin(1 + x^2) = 2x \cos(1 + x^2)\,dx$.

7. If $y = \sin^m x \sin mx$, show that

$$\sin^2 x \frac{dy}{dx} = m \sin^{m+1} x \sin(m + 1)x.$$

8. $D(a \sin^2 x + b \cos^2 x)^n = n(a - b) \sin 2x\,(a \sin^2 x + b \cos^2 x)^{n-1}$.

9. $d \sin(\sin x) = \cos x \cos(\sin x)\,dx$.

10. $f(x) = \sin^{-1}(x^n)$, show $f'(x) = nx^{n-1}(1 - x^{2n})^{-\frac{1}{2}}$.

11. $d \sin^{-1}(1 - x^2)^{\frac{1}{2}} = -(1 - x^2)^{-\frac{1}{2}}dx$.

12. $\dfrac{d}{dx}\cos^{-1}\dfrac{b + a \cos x}{a + b \cos x} = \dfrac{\sqrt{a^2 - b^2}}{a + b \cos x}$.

13. $d \sec^n x = n \sec^n x \tan x\,dx$.

14. $d \sec^{-1}(x^2) = \dfrac{2\,dx}{x\sqrt{x^4 - 1}}$.

15. $d(a^2 + x^2)^{\frac{1}{2}} = x(a^2 + x^2)^{-\frac{1}{2}}dx$.

16. $d(a^2 - x^2)^{-\frac{1}{2}} = x(a^2 - x^2)^{-\frac{3}{2}}dx$.

17. $\dfrac{d}{dx}x(x^2 + a^2)^{-\frac{1}{2}} = \dfrac{a^2}{\sqrt{(x^2 + a^2)^3}}$.

18. $D_x(2ax - x^2)^{\frac{1}{2}} = (a - x)(2ax - x^2)^{-\frac{1}{2}}$.

19. If $f(x) = \frac{1}{2}x - \frac{1}{4}\sin 2x$, then $f'(x) = \sin^2 x$.

20. Show that $d(\frac{1}{2}x + \frac{1}{4}\sin 2x) = \cos^2 x\,dx$.

21. $\dfrac{d}{dx}\left(\dfrac{\cos^3 x}{3} - \cos x\right) = \sin^3 x$.

22. If $y = \sin x - \frac{1}{3}\sin^3 x$, then $\dfrac{dy}{dx} = \cos^3 x$.

23. $d \log \cos x = -\tan x\,dx$.

24. $D \log \sin x = \cot x$.

25. $y = \tan x - x$. $dy = \tan^2 x\,dx$.

26. $\phi(t) = \cot t + t$, then $\phi'(t) = -\cot^2 t$.

27. If $z = \log \tan \frac{1}{2} y$, show that

$$\frac{dz}{dy} = \csc y = D \log \sqrt{\frac{1 - \cos y}{1 + \cos y}}.$$

28. $D_m \log \tan (\frac{1}{4}\pi + \frac{1}{2}m) = \sec m.$

29. $\dfrac{d}{d\alpha} \log \sqrt{\dfrac{1 + \sin \alpha}{1 - \sin \alpha}} = \sec \alpha.$

30. $d \sin^{-1}(3\theta - 4\theta^3) = 3(1 - \theta^2)^{-\frac{1}{2}}\, d\theta.$

31. $\dfrac{d}{d\phi} \tan^{-1} \dfrac{a\phi + b}{\sqrt{ac - b^2}} = \dfrac{\sqrt{ac - b^2}}{a\phi^2 + 2b\phi + c}.$

32. $\dfrac{d}{du} \log \left(\dfrac{u - 1}{u + 1}\right)^{\frac{1}{2}} = \dfrac{1}{u^2 - 1}.$

33. $\dfrac{d}{ds} \left\{ \dfrac{1}{6} \log \dfrac{(s + 1)^3}{s^3 + 1} + \dfrac{1}{\sqrt{3}} \tan^{-1} \dfrac{2s - 1}{\sqrt{3}} \right\} = \dfrac{1}{1 + s^3}.$

34. $d \log \dfrac{\sqrt{a^2 + \zeta^2} - a}{\zeta} = \dfrac{a\, d\zeta}{\zeta \sqrt{a^2 + \zeta^2}}.$

35. $\dfrac{d}{d\eta} \left\{ \eta \sqrt{a^2 - \eta^2} + a^2 \sin^{-1} \dfrac{\eta}{a} \right\} = 2 \sqrt{a^2 - \eta^2}.$

36. $\dfrac{d}{dp} \sin^{-1} \sqrt{\dfrac{p - \beta}{\alpha - \beta}} = \dfrac{1}{2 \sqrt{(\alpha - p)(p - \beta)}}.$

37. $\dfrac{d}{dz} \cos^{-1} \dfrac{a \cos z + b}{a + b \cos z} = \dfrac{\sqrt{a^2 - b^2}}{a + b \cos z}.$

38. $de^x(1 - x^3) = e^x(1 - 3x^2 - x^3)\, dx.$

39. $\dfrac{d}{dv} \dfrac{(\sin mv)^n}{(\cos nv)^m} = \dfrac{mn\, (\sin mv)^{n-1} \cos (nv - mv)}{(\cos nv)^{m+1}}.$

40. $\dfrac{d}{d\theta} \dfrac{\sin^m \theta}{\cos^n \theta} = \dfrac{\sin^{m-1} \theta}{\cos^{n+1} \theta} (m \cos^2 \theta + n \sin^2 \theta).$

41. $dx^x = x^x(1 + \log x)\, dx.$

42. $De^{x^x} = e^{x^x} x^x(1 + \log x).$

43. $\dfrac{d}{dy} \left(\dfrac{y}{n}\right)^{ny} = n \left(\dfrac{y}{n}\right)^{ny} \left(1 + \log \dfrac{y}{n}\right).$

44. $\dfrac{d}{dx} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{4}{(e^x + e^{-x})^2}.$

45. $d \log (e^x + e^{-x}) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}\, dx.$

46. $De^{(a+x)^2} \sin x = e^{(a+x)^2} [2(a + x) \sin x + \cos x].$

47. $\dfrac{d}{dz} \dfrac{z}{e^z - 1} = \dfrac{e^z(1 - z) - 1}{(e^z - 1)^2}.$

48. $\dfrac{d}{dt} \left(\dfrac{t}{1 + \sqrt{1 + t^2}}\right)^n = \dfrac{n}{t \sqrt{1 + t^2}} \left(\dfrac{t}{1 + \sqrt{1 + t^2}}\right)^n.$

49. If $\psi(t) = a^{(a^2-t^2)^{-\frac{1}{2}}}$, show that

$$\psi'(t) = \frac{t}{(a^2 - t^2)^{\frac{3}{2}}} a^{(a^2-t^2)^{-\frac{1}{2}}} \log a.$$

50. $d \tan a^{\frac{1}{u}} = - a^{\frac{1}{u}} u^{-2} \sec^2 a^{\frac{1}{u}} \log a\, du.$

51. $d\, [\theta + \log \cos (\frac{1}{4}\pi - \theta)] = 2(1 + \tan \theta)^{-1}\, d\theta.$

52. $D(\psi \sin^{-1}\psi) = \sin^{-1} \psi + \psi(1 - \psi^2)^{-\frac{1}{2}}.$

53. $D(\tan \theta \tan^{-1} \theta) = \sec^2 \theta \tan^{-1} \theta + (1 + \theta^2)^{-1} \tan \theta.$

54. $De^{-a^2x^2} \cos bx = - e^{a^2x^2}(2a^2x \cos bx + b \sin bx).$

55. $dx^{\frac{1}{x}} = x^{\frac{1}{x}-2}(1 - \log x)\, dx.$

56. $de^{e^x} = e^{e^x} e^x\, dx.$

57. $Dx^{x^x} = x^{x^x} x^x [x^{-1} + \log x + (\log x)^2].$

58. $dx^{e^x} = x^{e^x} e^x x^{-1} (1 + x \log x)\, dx.$

59. $D(1 - \tan x) \cos x = - \cos x - \sin x.$

60. $D \log (\log t) = 1/\log t^t.$

61. If $\phi(t) = e^{at} \sin bt$, show that

$$\phi'(t) = e^{at} (a^2 + b^2)^{\frac{1}{2}} \sin (bt + \theta),$$

where $\tan \theta = b/a.$

62. If $\sin y = x \sin (a + y)$, prove that

$$\frac{dy}{dx} = \frac{\sin^2 (a + y)}{\sin a}.$$

63. If $x(1 + y)^{\frac{1}{2}} + y(1 + x)^{\frac{1}{2}} = 0$, show that

$$D_x y = - (1 + x)^{-2} \quad \text{or} \quad 1.$$

64. If $y = f(t)$ and $x = F(t)$, show that

$$D_x y = \frac{f_t'(t)}{F_t'(t)}.$$

65. If $x^y = e^{x-y}$, show that

$$\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}.$$

66. $d\, (\sin x)^x = (\sin x)^x (\log \sin x + x \cot x)\, dx.$

67. $\frac{d}{dt} (\log \tan t)^2 = 4 \frac{\log \tan t}{\sin 2t}.$

CHAPTER IV.

ON SUCCESSIVE DIFFERENTIATION

52. The Second Derivative.—The derivative $f'(x)$ of a function $f(x)$ is itself a function of x, which is, in general, also differentiable.

The derivative of the derivative $f'(x)$ of a function $f(x)$ we call the *second* derivative of $f(x)$, and write it $f''(x)$.

Thus

$$f''(x) = \mathop{\mathcal{L}}_{x_1 (=) x} \frac{f'(x_1) - f'(x)}{x_1 - x}.$$

For example, if $f(x) \equiv x^n$, the first derivative $f'(x)$ is nx^{n-1}, and in the same way we find the second derivative

$$f''(x) = n(n - 1)x^{n-2}.$$

Again, if $f(x) \equiv \sin x$, then

$$f'(x) = \cos x \quad \text{and} \quad f''(x) = - \sin x.$$

If we use the symbol $Df(x)$ to represent the operation of differentiation performed on $f(x)$, then two successive differentiations of $f(x)$, which result in the second derivative, are represented by $D^2f(x)$.

$$\therefore \; D[Df(x)] \equiv D^2f(x) = f''(x).$$

EXAMPLES.

1. $D(a + bx + cx^2) = b + 2cx,$
$D^2(a + bx + cx^2) = D(b + 2cx),$
$= 2c.$

2. $D \cos ax = - a \sin ax,$
$D^2 \cos ax = - aD \sin ax = - a^2 \cos ax.$

3. $D \log ax = a/x; \quad D^2 \log ax = - a/x^2.$

4. $D \sqrt{a^2 - x^2} = - x(a^2 - x^2)^{-\frac{1}{2}},$
$D^2 \sqrt{a^2 - x^2} = - (a^2 - x^2)^{-\frac{1}{2}} + x^2(a^2 - x^2)^{-\frac{3}{2}}.$

53. Successive Differentiation.—The second derivative like the first is, in general, a differentiable function. Its derivative is called the third derivative of the function, and written

$$\begin{aligned} f'''(x) &\equiv \mathop{\mathcal{L}}_{x_1 (=) x} \frac{f''(x_1) - f''(x)}{x_1 - x}, \\ &\equiv D^3f(x). \end{aligned}$$

In general, if the operation of differentiation be repeated n times on a function $f(x)$, we call the result the nth derivative of the function. We write the nth derivative in either of the equivalent symbols

$$D^n f(x) \equiv f^{(n)}(x).$$

It is customary to omit the parenthesis in $f^{(n)}(x)$, including the index of the order of the derivative attached to the functional symbol f when there is no danger of mistaking it for a power, and write

$$D^n f(x) \equiv f^n(x).$$

The index of either D or f in D^n, f^n denotes merely the order of the derivative and number of times the operation is performed.

54. Successive Differentials.—In defining the first differential of a function, the differential of the independent variable was taken to be an arbitrary number. In repeating this operation it is convenient to take the same value of the differential of the independent variable in the second operation as that in the first. In other words, we make the differential of the independent variable constant during the successive differentiations.

Thus the second differential of $f(x)$ is

$$\begin{aligned} d^2 f(x) &= d\,[d f(x)], \\ &= d[f'(x)\,dx], \\ &= d[f'(x)]\cdot dx, \end{aligned} \tag{i}$$

since dx is constant. But, by the definition of the differential,

$$\begin{aligned} d[f'(x)] &= D[f'(x)]\,dx, \\ &= f''(x)\,dx. \end{aligned} \tag{ii}$$

Substituting in (i), we have for the second differential

$$d^2 f(x) = f''(x)(dx)^2,$$

or the second differential of a function is equal to the product of the second derivative into the *square* of the differential of the variable.

It is customary to write the square of the differential of the variable in the conventional form dx^2 instead of $(dx)^2$, whenever there is no danger of confounding

$$dx^2 \equiv (dx)^2$$

with $d(x)^2$, the differential of the square of x. We shall write then

$$d^2 f(x) = f''(x)\,dx^2.$$

In like manner for the third differential of $f(x)$

$$\begin{aligned} d[d^2 f(x)] &= d[f''(x)\,dx^2], \\ &= d[f''(x)]\cdot dx^2, \end{aligned}$$

since dx is constant; and since by definition

$$\begin{aligned} d[f''(x)] &= D[f''(x)]\,dx, \\ &= f'''(x)\,dx, \end{aligned}$$

we have for the third differential

$$d^3 f(x) = f'''(x)\, dx^3,$$

and so on.

In general, the nth differential of a function is equal to the product of the nth derivative of the function into the nth *power* of the differential of the independent variable. In symbols

$$d^n f(x) = f^n(x)\, dx^n,$$

where it is always to be remembered that dx^n means $(dx)^n$, and d^n, f^n indicate the number of operations and order of the derivative respectively.

EXAMPLES.

1. We have $d \sin x = \cos x\, dx$, and

$$d^2 \sin x = d(\cos x\, dx) = d(\cos x) \cdot dx = - \sin x\, dx^2.$$

2. $d^2(a + bx^2) = d(2bx) \cdot dx = 2b\, dx^2.$

3. $d^2 \log x = d\left(\frac{1}{x}\right) \cdot dx = - \frac{dx^2}{x^2}.$

55. The Differential-Quotients.—The nth differential-quotient of a function is the quotient of the nth differential of the function by the nth power of the differential of the independent variable.

In symbols we have, from § 54,

$$\frac{d^n f(x)}{dx^n} = f^n(x).$$

This symbol is also written, for convenience, in the forms

$$\frac{d^n f(x)}{dx^n} \equiv \frac{d^n}{dx^n} f(x) \equiv \left(\frac{d}{dx}\right)^n f(x),$$

all of which notations are equivalent to either of

$$D^n f(x) \equiv f^n(x),$$

and are used indifferently according to convenience.

56. Observations on Successive Differentiation.—In practice or in the applications of the Calculus we require, in general, only the first few derivatives of a function for solving the ordinary problems that are proposed. But, in the theory of the subject, i.e., the theory of functions, we are required to deal with the general or nth derivative of a function in order to know all the properties of the function.

The formation of the nth derivative of a given function presents no theoretical difficulty, but owing to the fact that differentiation, in general, produces a function of more complicated form (owing to the introduction of more terms) than the primitive function from which it was derived, the successive derivatives soon become so complicated that the practical limitations (of our ability to handle them) are soon reached.

The Differential Calculus as an instrument for investigating functions finds its limitations fixed by the complexity of the general or *n*th derivative of the function whose properties we wish to investigate.

There are a few functions whose *n*th derivatives can be obtained in simple form, as will be shown below.

We are aided in forming the *n*th derivatives of functions by the following:

(1). The *n*th derivative of the sum of a finite number of functions is equal to the sum of their *n*th derivatives.

(2). The *n*th derivative of the product of a finite number of functions can be determined by a formula due to Leibnitz, which we shall deduce presently.

(3). The *n*th derivative of the quotient of two functions can be expressed in the form of a determinant and in a recurrence formula, directly from Leibnitz's formula. This is done in the Appendix, Note 3.

(4). The *n*th derivative of a function of a function can be expressed in terms of the successive derivatives of the functions involved. This is also given in the Appendix, Note 4.

In the application of the Calculus to the solution of ordinary geometrical questions, we need the first, frequently the second, and but rarely the third derivative of a function. When the function is given explicitly in terms of the variable, these derivatives are found by the direct processes as heretofore applied. If the derivatives are to be found from an implicit relation, such as $\phi(x, y) = 0$, we can of course solve for y, when possible, and differentiate as before. It is generally, however, better to differentiate $\phi(x, y)$ with respect to x and then solve for Dy. If we wish D^2y, we can either differentiate Dy with respect to x, or differentiate $\phi(x, y) = 0$ twice with respect to x and solve the equations for D^2y.

In illustration,

$$2x^3 - 3y^3 - axy = 0.$$

$$\therefore \quad 6x^2 - ay - (9y^2 + ax)Dy = 0,$$

$$12x - aDy - (18y\,Dy + a)Dy - (9y^2 + ax)D^2y = 0.$$

Therefore

$$Dy = \frac{6x^2 - ay}{9y^2 + ax},$$

$$D^2y = \frac{12x(9y^2 + ax)^2 - 2a(6x^2 - ay)\,9y^2 + ax) - 18y(6x^2 - ay)^2}{(9y^2 + ax)^3}.$$

Again, we frequently require the derivatives $D_x y$ and $D_x^2 y$, when we have given the polar equation $\phi(\rho, \theta) = 0$, where $x = \rho \cos\theta$, $y = \rho \sin\theta$.

We have

$$D_x y = D_\theta y\, D_x\theta = \frac{D_\theta y}{D_\theta x},$$

$$= \frac{\sin\theta\, D_\theta\rho + \rho\cos\theta}{\cos\theta\, D_\theta\rho - \rho\sin\theta}. \qquad (1)$$

Also,

$$\begin{aligned} D_x^2 y &= D_x(D_x y) = D_\theta(D_x y)\cdot D_x\theta, \\ &= \frac{D_\theta^2 y\cdot D_\theta x - D_\theta y\cdot D_\theta^2 x}{(D_\theta x)^3}, \\ &= \frac{\rho^2 + 2(D_\theta\rho)^2 - \rho D_\theta^2\rho}{(\cos\theta\, D_\theta\rho - \rho\sin\theta)^3}. \end{aligned}$$

$$D_\theta \frac{D_\theta y}{D_\theta x} \cdot D_x\theta$$

$$\frac{D_\theta x\, D_\theta^2 y - D_\theta y \cdot D_\theta^2 x}{(D_\theta x)^2} \cdot \frac{1}{D_\theta x}$$

In which $D_\theta\rho$ and $D_\theta^2\rho$ must be determined from the polar equation $\phi(\rho, \theta) = 0$.[*]

EXAMPLES.

1. The nth derivative of x^a, a being constant.

(1). Let $a \equiv m$ be a positive integer. Then

$$\begin{aligned} Dx^m &= mx^{m-1}, \\ D^2x^m &= m(m-1)x^{m-2}, \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ D^nx^m &= m(m-1)\,.\,.\,.\,(m-n+1)x^{m-n}, \end{aligned}$$

for all values of $n < m$. If $n = m$, then

$$D^mx^m = m(m-1)\,.\,.\,.\,3\cdot2\cdot1 = m!$$

This being a constant, all higher derivatives are 0.

$$\therefore\quad D^{m+p}x^m = 0$$

for all positive integers p.

Also, when $x = 0$,

$$D^nx^m = 0, \qquad\qquad n < m.$$

(2). Let the constant a be not a positive integer. Then, as before,

$$D^nx^a = a(a-1)\,.\,.\,.\,(a-n+1)x^{a-n}.$$

Whatever be the assigned constant a, we can continue the process until $n > a$, when the exponent of x will be negative and continue negative for all higher derivatives.

Consequently, when $x = 0$,

$$D^nx^a = 0, \qquad\qquad n < a.$$

$$D^nx^a = \infty, \qquad\qquad n > a.$$

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[*] The differentiation of an implicit function $\phi(x, y) = 0$ is, properly speaking, the differentiation of a function of two variables, and a simpler treatment will be given in Book II.

It will be shown in Book II that the derivative of y with respect to x, when $\phi(x, y) = 0$, is

$$\frac{dy}{dx} = -\frac{\dfrac{\partial\phi}{\partial x}}{\dfrac{\partial\phi}{\partial y}},$$

where $\dfrac{\partial\phi}{\partial x}$ means the derivative of $\phi(x, y)$ with respect to x, x being the *only* variable; $\dfrac{\partial\phi}{\partial y}$ means the derivative of ϕ with respect to y, y being the only variable.

For example, if $\phi(x, y) \equiv 2x^3 - 3y^3 - axy = 0$,

then $$\frac{\partial\phi}{\partial x} = 6x^2 - ay; \quad \frac{\partial\phi}{\partial y} = -9y^2 - ax.$$

Therefore, as in the text,

$$\frac{dy}{dx} = \frac{6x^2 - ay}{9y^2 + ax}.$$

2. Deduce the binomial formula for $(1 + x)^n$, when the exponent n is a positive integer.

We have

$$(1 + x)(1 + x) = (1 + x)^2 = 1 + 2x + x^2,$$
$$(1 + x)(1 + x)^2 = (1 + x)^3 = 1 + 3x + 3x^2 + x^3.$$

By an easy induction we see that $(1 + x)^n$ must be a polynomial in x of degree n. It is our object to find the numerical coefficients of the various powers of x in this function. Let

$$(1 + x)^n = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n.$$

Differentiating this r times with respect to x, we have

$$n(n-1)\ldots(n-r+1)(1+x)^{n-r} = r!\, a_r + \ldots + n(n-1)\ldots(n-r+1)a_n\, x^{n-r}.$$

This equation is true for all assigned values of x and r, and when $x = 0$,

$$a_r = \frac{n(n-1)\ldots(n-r+1)}{r!},$$
$$= \frac{n!}{r!(n-r)!},$$

a number which it is customary to represent conventionally by either of the symbols

$$C_{n,r} \qquad \text{or} \qquad \binom{n}{r}.$$

This number is of frequent occurrence in analysis. In Algebra, when n is an integer, it represents the number of combinations of n things taken r at a time.

Hence we have the binomial formula of Newton,

$$(1 + x)^n = \sum_{r=0}^{n} C_{n,r}\, x^r. \tag{1}$$

Corollary. If we wish the corresponding expression for $(a + y)^n$, then

$$(a + y)^n \equiv a^n\left(1 + \frac{y}{a}\right)^n.$$

Put y/a for x in (1), and multiply both sides by a^n.

$$\therefore\ (a + y)^n = \sum_{r=0}^{n} C_{n,r}\, a^{n-r}x^r.$$

This can be written more symmetrically thus:

$$\frac{(a + y)^n}{n!} = \sum_{0}^{n} \frac{a^{n-r}}{(n-r)!}\,\frac{x^r}{r!}.$$

3. The nth derivative of $\log x$. We have

$$D \log x = \frac{1}{x} = x^{-1}.$$

Therefore, by Ex. 1,

$$D^n \log x = (-1)^{n-1}(n-1)!\,\frac{1}{x^n}.$$

4. The nth derivative of a^x. We have

$$Da^x = a^x \log a.$$
$$\therefore\ D^na^x = a^x (\log a)^n.$$

In particular, $De^x = e^x$; $D^ne^x = e^x$. This remarkable function is not changed by differentiation.

5. The nth derivative of $\sin x$ and $\cos x$.

We observe that

$$\begin{aligned} D\sin x &= +\cos x; & D\cos x &= -\sin x; \\ D^2\sin x &= -\sin x; & D^2\cos x &= -\cos x; \\ D^3\sin x &= -\cos x; & D^3\cos x &= +\sin x; \\ D^4\sin x &= +\sin x; & D^4\cos x &= +\cos x. \end{aligned}$$

Thus four differentiations reproduce the original functions and therefore the higher derivatives repeat in the same order, so that

$$\begin{aligned} D^{2n-1}\sin x &= (-1)^{n-1}\cos x; & D^{2n-1}\cos x &= (-1)^n\sin x; \\ D^{2n}\sin x &= (-1)^n\sin x; & D^{2n}\cos x &= (-1)^n\cos x. \end{aligned}$$

In virtue of the relations

$$\cos x = \sin\left(\tfrac{1}{2}\pi + x\right), \qquad \sin x = -\cos\left(\tfrac{1}{2}\pi + x\right),$$

these formulæ can be expressed in the compact forms

$$D^n\sin x = \sin\left(x + \frac{n}{2}\pi\right),$$

$$D^n\cos x = \cos\left(x + \frac{n}{2}\pi\right).$$

6. Given $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, find $D_x^2 y$, $D_x^3 y$.

Differentiating with respect to x,

$$\frac{x}{a^2} + \frac{y}{b^2}\frac{dy}{dx} = 0.$$

Differentiating again,

$$\frac{1}{a^2} + \frac{1}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{y}{b^2}\frac{d^2y}{dx^2} = 0.$$

$$\begin{aligned} \therefore\quad \frac{d^2y}{dx^2} &= -\frac{b^2}{y}\left\{\frac{1}{a^2} + \frac{1}{b^2}\left(\frac{dy}{dx}\right)^2\right\}, \\ &= -\frac{b^4}{a^2y^3}, \quad \text{since} \quad \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}. \end{aligned}$$

Differentiating again, we can find

$$\frac{d^3y}{dx^3} = -\frac{3b^6x}{a^4y^5}.$$

7. If $y^2 = 4ax$, show that

$$\frac{dy}{dx} = \frac{2a}{y}; \quad \frac{d^2y}{dx^2} = -\frac{4a^2}{y^3}.$$

8. If $y^2 - 2xy = a^2$, show that

$$\frac{dy}{dx} = \frac{y}{y-x}, \quad \frac{d^2y}{dx^2} = \frac{a^2}{(y-x)^3}, \quad \frac{d^3y}{dx^3} = -\frac{3a^2x}{(y-x)^5}.$$

9. From the relation $x^3 + y^3 - 3axy = 0$, show that

$$\frac{dy}{dx} = -\frac{x^2 - ay}{y^2 - ax}, \quad \frac{d^2y}{dx^2} = -\frac{2a^3xy}{(y^2 - ax)^3}.$$

10. If $\sec x \cos y = a$, show that

$$\frac{dy}{dx} = \frac{\tan x}{\tan y}, \quad \frac{d^2y}{dx^2} = \frac{\tan^2 y - \tan^2 x}{\tan^3 y}$$

57. Leibnitz's Formula for the nth Derivative of the Product of Two Functions.—Let u, v be any two functions of x. For sake of brevity, let us represent the successive derivatives of u and v by these letters with indices, thus :

$$u',\quad u'',\quad u''',\ .\ .\ .\ ,\quad u^n,\ .\ .\ .$$
$$v',\quad v'',\quad v''',\ .\ .\ .\ ,\quad v^n,\ .\ .\ .$$

Then

$$\begin{aligned} D(uv) &= u'v + v'u, \\ D^2(uv) &= u''v + u'v' + u'v' + v''u, \\ &= u''v + 2u'v' + uv''. \end{aligned}$$

In like manner, differentiating again this sum of products, we find on simplification

$$D^3(uv) = u'''v + 3u''v' + 3u'v'' + uv'''.$$

Observing, when we use indices to indicate the derivatives, the symbols D^0u, $f^0(x)$, v^0, mean that no differentiation has been performed and the function itself is unchanged,

$$\therefore\quad D^0u \equiv u^0 \equiv u, \quad \text{and} \quad f^0(x) \equiv f(x).$$

In the above successive derivatives of uv we observe that the indices representing differentiation follow the law of the powers of $u + v$ when expanded by the binomial formula, and the numerical coefficients are the same as those in the corresponding formula of that expansion

In order to find if this law is generally true, let us assume it true for the nth derivative and then differentiate again to see if it be true, in consequence of that assumption, for $n + 1$.

Assume that (see Ex. 2, § 56)

$$\begin{aligned} D^n(uv) &= \sum_{r=0}^{n} C_{n,r}\ u^{n-r}v^r, \\ &= u^nv + C_{n,1}\ u^{n-1}v' + \ldots + C_{n,r}\ u^{n-r}v^r + \ldots + uv^n. \end{aligned}$$

Differentiating this, we have

$$\begin{aligned} D^{n+1}(uv) &= u^{n+1}v + C_{n,1}\ u^nv' + \ldots + C_{n,r}\ u^{n-r+1}v^r + \ldots + u'v^n \\ &\quad + \qquad\quad u^nv' + \ldots + C_{n,r-1}\ u^{n-r+1}v^r + \ldots + nu'v^n + uv^{n+1}, \\ &= u^{n+1}v + C_{n+1,1}\ u^nv' + \ldots + C_{n+1,r}\ u^{n+1-r}v^r + \ldots + uv^{n+1}, \\ &= \sum_{r=0}^{n+1} C_{n+1,r}\, u^{n+1-r}\, v^r, \end{aligned}$$

in virtue of the relation* $C_{n,r} + C_{n,r-1} = C_{n+1,r}$.

Therefore, when the law is true for any integer n, it is also true for $n + 1$. But, being true for $n = 2$, 3, it is true for any assigned integer whatever.

---

* $$\frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} = \frac{n!}{(r-1)!(n-r)!}\left(\frac{1}{r} + \frac{1}{n-r+1}\right) = \frac{(n+1)!}{r!(n+1-r)!}$$

We can express the nth derivative of the product uv symbolically, thus:

$$D^n(uv) = (u + v)^n,$$

in which $(u + v)^n$ is to be expanded by the binomial formula, and the powers of u and v in the expansion are taken to indicate the orders of differentiation of these functions. Remembering that when the index is a power we have $u^0 \equiv 1$, but when it means differentiation, $u^0 \equiv u$.

EXAMPLES.

1. To differentiate the product of a linear function by any function $f(x)$.

Let $$u = (ax + b)f(x).$$

Then $$D(ax + b) = a, \qquad D^2(ax + b) = 0,$$

$$\therefore\quad D^n u = (ax + b)f^n(x) + na\, f^{n-1}(x).$$

2. In like manner show that the nth derivative of the product of a quadratic function of x, say y, by any other function f, is

$$yf^n + ny'f^{n-1} + \tfrac{1}{2}n(n - 1)y''f^{n-2}.$$

3. Show, if $\phi(x)$ and $\psi(x)$ are differentiable functions of x,

$$\frac{D^n[\phi(x)\psi(x)]}{n!} = \sum_{r=0}^{n} \frac{\phi^{n-r}(x)}{(n - r)!}\,\frac{\psi^r(x)}{r!}.$$

58. Function of a Function.—A formula for the nth derivative of a function of a function will be deduced in the supplementary notes.* However, the simple case of a function of a linear function of the independent variable is so useful and of such frequent occurrence that we give it here.

Let $u = ax + b$, and $f(u)$ be any differentiable function of u. Then

$$\begin{aligned} D_x f(u) &= f_u'(u)\, D_x u, \\ &= af_u'(u). \\ D_x^2 f(u) &= aD_x[f_u'(u)], \\ &= af_u''(u)\, Du, \\ &= a^2 f_u''(u), \end{aligned}$$

and generally

$$D_x^n f(u) = a^n f_u^n(u).$$

EXAMPLES.

1. Show that

$$D^n \sin ax = a^n \sin\left(ax + \frac{n}{2}\pi\right),$$

$$D^n \cos ax = a^n \cos\left(ax + \frac{n}{2}\pi\right).$$

2. $D^n e^{ax} = a^n e^{ax}.$

3. Show that $D_a^n\left(\dfrac{1}{x - a}\right) = \dfrac{n!}{(x - a)^{n+1}}.$

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* Appendix, Note 4.

EXERCISES.

Show that

1. $D^r\left(\frac{1}{x}\right) = (-1)^r \frac{r!}{x^{r+1}}$.

2. $D^r\left(\frac{a}{x^n}\right) = (-1)^r a \frac{n(n+1)\dots(n+r-1)}{x^{n+r}}$.

3. $D^r\left(\frac{a}{c-x}\right) = r!\frac{a}{(c-x)^{r+1}}$.

4. $D^n \log(1+x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$.

5. $D^4(x^3 \log x) = 6x^{-1}$.

6. $D^5(x^4 + a\sin 2x) = 32a\cos 2x$.

7. $D_x^r(xu) = x\,D^r u + r\,D^{r-1}u$, where u is any function of x.

8. $D_x^r(a-x)u = (a-x)\,D^r u - r\,D^{r-1}u$.

9. $D^6(x^4 \log x) = -4!\,x^{-2}$.

10. $D^n(x\log x) = (-1)^n(n-2)!\,x^{-n+1}$.

11. $D^2x^x = x^x(1+\log x)^2 + x^{x-1}$.

12. $D^3 \log(\sin x) = 2\cos x\csc^3 x$.

13. $D^5(x^4 \log x^{\frac{2}{3}}) = 2^4 x^{-1}$. 7 a

14. $D^n a^{cx} = a^x(\log a^c)^n$.

15. $D^n \frac{ax+b}{x^2-c^2} = (-1)^n \frac{n!}{2c}\left\{\frac{ac+b}{(x-c)^{n+1}} + \frac{ac-b}{(x+c)^{n+1}}\right\}$.

Observe that by the method of partial fractions we can write

$$\frac{ax+b}{(x-c)(x+c)} = \frac{1}{2c}\left\{\frac{ac+b}{x-c} + \frac{ac-b}{x+c}\right\}.$$

16. $D^n \frac{ax+b}{(x-p)(x-q)} = D^n \frac{1}{p-q}\left(\frac{ap+b}{x-p} - \frac{aq+b}{x-q}\right) = ?$

17. Make use of the method of partial fractions, to find the nth derivative of

$$\frac{ax^2+bx+c}{(x-p)(x-q)} \equiv \frac{1}{p-q}\left(\frac{ap^2+pb+c}{x-p} - \frac{aq^2+bq+c}{x-q}\right) + a.$$

18. Show that $\left(\frac{d}{dx}\right)^n \frac{2x^2-4x-6}{x^2-5x+6} = -6\frac{n!}{(2-x)^{n+1}}$.

19. If $y = a(1+x^2)^{-1}$, show that

$$(1+x^2)y^{(n)} + 2nxy^{(n-1)} + n(n-1)y^{(n-2)} = 0.$$

20. If $f(x) \equiv a\cos(\log x) + b\sin(\log x)$, show that

$$x^2f''(x) + xf'(x) + f(x) = 0.$$

21. Show in 20 that the following equation is true:

$$x^2f^{n+2} + x(2n+1)f^{n+1} + (n^2+1)f^n = 0.$$

22. If $y = e^{a\sin^{-1}x}$, show that

$$(1-x^2)d^2y - x\,dy\,dx = a^2y\,dx^2,$$

thence find, as in 21, an equation in y^{n+2}, y^{n+1}, y_n.

23. If $y = (x + \sqrt{x^2 - 1})^m$, show that

$$(x^2 - 1)d^2y + x\,dy\,dx - m^2y\,dx^2 = 0.$$

24. If $y = \sin(\sin x)$, show that

$$d^2y + \tan x\,dy\,dx + y\cos^2 x\,dx^2 = 0.$$

25. If $y = A\cos nx + B\sin nx$, then

$$D^2y + n^2y = 0.$$

26. $D^n e^{ax}\cos bx = (a^2 + b^2)^{\frac{n}{2}} e^{ax}\cos(bx + n\phi)$, where $\tan\phi = b/a$. Differentiate once, twice, and observe that the law follows directly by induction.

27. $D^n \tan^{-1}x^{-1} = (-1)^n(n-1)!\,\sin^n(\tan^{-1}x^{-1})\sin n(\tan^{-1}x^{-1})$.

Put $y = \tan^{-1}x^{-1}$. Then $x = \cot y$, and

$$D_x y = -(1 + x^2)^{-1} = -\sin^2 y.$$

$$D_x^2 y = -D_x \sin^2 y = -D_y \sin^2 y\,Dy = \sin^2 y \sin 2y$$

The rest follows by an easy induction.

28. If $x = \phi(t)$ and $y = \psi(t)$, then y is a function of x

Required $D_x y$, $D_x^2 y$.

We have $dy = \psi'(t)\,dt$, $dx = \phi'(t)\,dt$.

$$\therefore \frac{dy}{dx} = \frac{\psi'(t)}{\phi'(t)}.$$

Also,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{\psi_t'}{\phi_t'} = \frac{d}{dt}\frac{\psi_t'}{\phi_t'}\cdot\frac{dt}{ax},$$

$$= \frac{\psi_t''\phi_t' - \psi_t'\phi_t''}{(\phi_t')^3}, \quad \because \frac{dt}{dx} = \frac{1}{\phi_t'}.$$

$$= \frac{\dfrac{d^2y}{dt^2}\dfrac{dx}{dt} - \dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{\left(\dfrac{dx}{dt}\right)^3}.$$

29. If $x = \sin 3t$, $y = \cos 3t$, show that

$$D_x^2 y = -y^{-3}.$$

30. $D^n \tan^{-1}x = (-1)^{n-1}(n-1)!\,\dfrac{\sin(n\tan^{-1}x^{-1})}{(1 + x^2)^{\frac{1}{2}n}}$.

This follows immediately from Ex. 27, since

$$\tan^{-1}x = \tfrac{1}{2}\pi - \tan^{-1}x^{-1}.$$

31. If $y = \tan^{-1}x$, show that

$$(1 + x^2)y^{(n+1)} + 2nxy^{(n)} + n(n-1)y^{(n-1)} = 0.$$

32. $D^n(a^2 + x^2)^{-1} = (-1)^n n!\,a^{-n-2}\sin^{n+1}\phi\,\sin(n+1)\phi$, where $\tan\phi = a/x$. Hint. Use Ex. 30, and

$$D\tan^{-1}(a/x) = -a(a^2 + x^2)^{-1}.$$

33. $D^n x(a^2 + x^2)^{-1} = (-1)^n a^{-n-1} n!\,\sin^{n+1}\phi\,\cos(n+1)\phi$ where $\tan\phi = a/x$. Use Ex. 32 and Leibnitz's Formula.

34. $\left(\dfrac{d}{dx}\right)^n \dfrac{1 - x}{1 + x} = 2(-1)^n \dfrac{n!}{(1 + x)^{n+1}}$.

35. $D^2 e^{2\sqrt{x}} = \dfrac{2\sqrt{x} - 1}{2x\sqrt{x}} e^{2\sqrt{x}}.$

36. If $y^2(1 + x^2) = (1 - x + x^2)^2$, then

$$\frac{d^2y}{dx^2} = \frac{1 + 3x + x^2}{(1 + x^2)^{\frac{5}{2}}}.$$

37. If $y = \sin(m \sin^{-1} x)$, prove

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2 y = 0.$$

38. If $y = \sin^{-1} x$, deduce

$$(1 - x^2)y'' - xy' = 0,$$

and
$$(1 - x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n + 1)x\frac{d^{n+1}y}{dx^{n+1}} - n^2\frac{d^n y}{dx^n} = 0,$$

by applying Leibnitz's Formula to the above. The deduction of such differential equations is of fundamental importance for the expansion of functions in series.

39. Show that

$$\frac{(a - x)^{n+1}}{n!}\left(\frac{d}{dx}\right)^n \frac{f(x)}{a - x} = \sum_{r=0}^{n} \frac{(a - x)^r}{r!} f^r(x).$$

Apply Leibnitz's Formula to the product $f(x) \cdot (a - x)^{-1}$.

40. Show that

$$\frac{(x - y)^{n+1}}{n!}\left(\frac{\partial}{\partial y}\right)^n\left(\frac{f(x) - f(y)}{x - y}\right) = f(x) - \sum_{r=0}^{n} \frac{(x - y)^r}{r!} f^r(y),$$

where, in the differentiation indicated by $\dfrac{\partial}{\partial y}$, x is constant and y the variable. The result follows at once when Leibnitz's Formula is applied to the product of the two functions $f(x) - f(y)$ and $(x - y)^{-1}$.

This is one of the most important formulæ in the Calculus. Observe that it is obtained by successive differentiation of the difference-quotient.

41. Show that the derivative of the right member of the equation in Ex. 40, with respect to y (x being considered constant during the operation), is

$$-\frac{(x - y)^n}{n!} f^{n+1}(y).$$

Hint. Differentiating each product in the sum, we find that the terms all cancel out except the last.

CHAPTER V.

ON THE THEOREM OF MEAN VALUE.

59. Increasing and Decreasing Functions.

Definition.—A function $f(x)$ is said to be an increasing function when it *increases* as its variable *increases*. A function is said to be a *decreasing* function when it *decreases* as its variable *increases*.

In symbols, $f(x)$ is an increasing function at $x = a$ when

$$f(x) - f(a) \qquad (1)$$

changes from *negative* to *positive* (less to greater) as x increases through the neighborhood, $(a - \epsilon, a + \epsilon)$, of a. In like manner $f(x)$ is a decreasing function at a when the difference (1) changes from *positive* to *negative* (greater to less) as x increases through the neighborhood of a.

60. Theorem.—A function $f(x)$ is an *increasing* or *decreasing* function at a according as its derivative $f'(a)$ is *positive* or *negative* respectively.

Proof: If $f(x)$ is an increasing function at a, the difference-quotient

$$\frac{f(x) - f(a)}{x - a}$$

is always positive for x in the neighborhood of a, consequently its limit $f'(a)$ cannot be negative. If $f'(a)$ is a positive number, then for all values of x in the neighborhood of a the difference-quotient must be in the neighborhood of its limit $f'(a)$, and therefore positive. The function is therefore increasing at a.

In like manner, if $f(x)$ is decreasing at a, the difference-quotient is negative for all values of x in the neighborhood of a and therefore its limit cannot be positive. Hence, if $f'(a)$ is a negative number, the difference-quotient must be negative for x in the neighborhood of a, and therefore $f(x)$ is decreasing at a.

GEOMETRICAL ILLUSTRATION.

Let $y = f(x)$ be represented by the curve A_1A_2. The function is increasing at A_1 and decreasing at A_2.

We have

$$f'(a_1) = \tan \theta_1 = +,$$

for θ_1 is acute, while

$$f'(a_2) = \tan \theta_2 = -,$$

since θ_2 is obtuse. Remembering that, under the convention of Cartesian coordinates, the angle which a tangent to a curve makes with the x-axis is the angle between that part of the tangent *above* Ox and the positive direction of Ox.

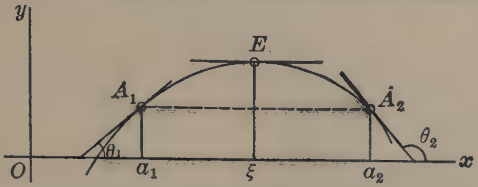

FIG. 9.

61. Rolle's Theorem.—If a function $f(x)$ is one-valued and differentiable in (α, β), and we have $f(\alpha) = f(\beta)$, then there is a value ξ of x in (α, β) at which we have

$$f'(\xi) = 0,$$

provided $f'(x)$ is continuous in (α, β).

If $f(x)$ is constant in any subinterval of (α, β), its derivative there is 0 and the theorem is proved.

If $f(x)$ is not constant in (α, β), then at some value x' in (α, β) we shall have $f(x') \neq f(\alpha)$. If $f(x') > f(\alpha) = f(\beta)$, the function must increase between α and x' and decrease between x' and β, in order to pass from $f(\alpha)$ to the greater value $f(x')$, and from $f(x')$ to the lesser value $f(\beta)$. Also, if $f(x') < f(\alpha) = f(\beta)$, then the function must decrease in (α, x') and increase in (x', β), for like reasons. In either case the derivative $f'(x_1)$ at some point x_1 in (α, x') must have contrary sign, § 60, to the derivative $f'(x_2)$ at some value x_2 in (x', β).

Since $f'(x_1)$ and $f'(x_2)$ have opposite signs, and $f'(x)$ is, by hypothesis, continuous in (x_1, x_2), then there is, § 23, I, a number ξ in (x_1, x_2), and therefore in (α, β), at which we have

$$f'(\xi) = 0.$$

In particular, if $f(\alpha) = 0$ and $f(\beta) = 0$, then there is a number ξ between α and β at which

$$f'(\xi) = 0.$$

Rolle's Theorem is usually enunciated: If a function vanishes for two values of the variable, its derivative vanishes for some value of the variable between the two. Or, the derivative has a root between each pair of roots of the function.

The figure in § 60 illustrates the theorem.

62. Particular Theorem of Mean Value.—If $f(x)$ is a one-valued differentiable function having a continuous derivative in (α, β), and if a and b are any two values of x in (α, β), then

$$f(b) - f(a) = (b - a)f'(\xi),$$

where ξ is some number in (a, b).

The truth of this theorem follows immediately from Rolle's Theorem.

Let k represent the difference-quotient

$$k = \frac{f(b) - f(a)}{b - a}.$$

Then

$$f(b) - f(a) = (b - a)k,$$

or

$$f(b) - kb = f(a) - ka. \qquad (1)$$

The function $f(x) - kx$ is equal to the number on the left of (1) when $x = b$, and to the number on the right when $x = a$. Therefore, by Rolle's Theorem, having equal values when $x = a$ and when $x = b$, its derivative must vanish for $x = \xi$ between a and b. Differentiating, $f(x) - kx$, k being independent of x, we have at ξ

$$f'(\xi) - k = 0,$$

which proves the theorem.

Another way of establishing the result is to observe that the function

$$(a - b)f(x) - (x - b)f(a) + (x - a)f(b)$$

vanishes when $x = a$, also when $x = b$. Therefore its derivative must vanish for some value of x, say ξ, between a and b.

$$\therefore\ (a - b)f'(\xi) - f(a) + f(b) = 0.$$

GEOMETRICAL ILLUSTRATION.

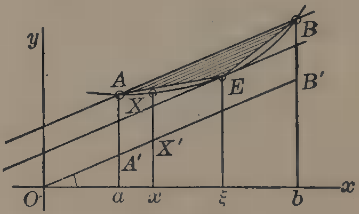

FIG. 10.

Each of these processes admits of geometrical illustration.

(1). k is the trigonometrical tangent of the angle which the secant AB makes with Ox. Draw $OA'B'$ parallel to AB. Then

$$BB' = f(b) - kb = AA' = f(a) - ka.$$

$XX' = f(x) - kx$ is equal to AA' when $x = a$, and to BB' when $x = b$. The theorem asserts that there is a point E on the curve $y = f(x)$ having abscissa ξ at which $f'(\xi) = k$, or the tangent at E is parallel to the chord AB.

(2). The function

$$(a - b)f(x) - (x - b)f(a) + (x - a)f(b)$$

is nothing more than the determinant

$$\begin{vmatrix} f(x), & x, & 1 \\ f(a), & a, & 1 \\ f(b), & b, & 1 \end{vmatrix}$$

which is the well-known formula in Analytical Geometry for twice the area of the triangle AXB, in terms of the coordinates of its corners. This vanishes when X coincides with A or B. It attains a maximum when the distance of X from the base AB is greatest, or when X is at E, where the tangent is parallel to the chord.

This theorem amounts to nothing more than Rolle's Theorem when the axes of coordinates are changed.

63. Lemma.—Ex. 39, § 58, forms the basis of the most important theorem in the Differential Calculus, i.e., the Theorem of Mean Value for a function of one variable. On account of its usefulness, we interpolate its solution here.

The starting point of the Differential Calculus is the difference-quotient. On that is based the derivative of the function. We shall now use it in presenting the Theorem of Mean Value.

Let $f(x)$ be a one-valued successively differentiable function of x in a given interval (α, β). Let x represent any *arbitrary* value of the variable, and y some *particular* value of the variable at which the derivatives of f are known.

(1). Consider the difference-quotient

$$\frac{f(x)-f(y)}{x-y}.$$

If we hold x constant while we differentiate this n times with respect to the variable y by Leibnitz's Formula, § 57, and then multiply both sides by

$$\frac{(x-y)^{n+1}}{n!},$$

we shall obtain

$$f(x)-f(y)-(x-y)f'(y)-\ldots-\frac{(x-y)^n}{n!}f^n(y)$$
$$=\frac{(x-y)^{n+1}}{n!}\left(\frac{\partial}{\partial y}\right)^n\left(\frac{f(x)-f(y)}{x-y}\right).$$

For, we have

$$D_y^r[f(x)-f(y)]=-f^r(y),$$
$$D_y^{n-r}(x-y)^{-1}=(n-r)!(x-y)^{r-(n+1)},$$

which values substituted in the form of Leibnitz's Formula in Ex. 3, § 57, give the result.

(2). On account of the importance of this formula we give another deduction which does not use Leibnitz's Formula directly.

Let

$$\frac{f(x)-f(y)}{x-y}=Q.$$

Then

$$f(x)-f(y)=(x-y)Q.$$

To introduce the known derivatives at y, let x be constant and differentiate this last equation successively with respect to y. Thus

$$f(x) - f(y) = (x - y)Q, \tag{1}$$

$$-f'(y) = (x - y)Q'_y - Q, \tag{2}$$

$$-f''(y) = (x - y)Q''_y - 2Q'_y, \tag{3}$$

$$\vdots \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$-f^n(y) = (x - y)Q_y^{(n)} - nQ_y^{(n-1)}. \tag{$n+1$}$$

Multiply (2) by $(x - y)$, (3) by $\frac{1}{2!}(x - y)^2$, . . . , and $(n + 1)$ by $\frac{1}{n!}(x - y)^n$, and add the $n + 1$ equations. There results

$$f(x) - f(y) - (x - y)f'(y) - \ldots - \frac{(x-y)^n}{n!}f^n(y) = \frac{(x-y)^{n+1}}{n!}Q_y^{(n)}, \tag{q}$$

the same formula as in (1).

64. The Theorem of Mean Value. Lagrange's Form.—The Theorem of Mean Value, which we now present, is the most important theorem in the Differential Calculus. The applications of the Differential Calculus depend on it as do also its generalizations. It is but a direct modification of the differential identity (q) established in § 63, and consists in the evaluation of the nth derivative, $Q_y^{(n)}$, of the difference-quotient Q in a different form.

Consider the arbitrarily laid down function of z,

$$F(z) \equiv f(x) - f(z) - (x - z)f'(z) - \ldots - \frac{(x-z)^n}{n!}f^n(z) - \frac{(x-z)^{n+1}}{n!}Q_y^{(n)},$$

in which, as in § 63,

$$Q_y^{(n)} \equiv D_y^n\left(\frac{f(x) - f(y)}{x - y}\right)$$

does not contain z and is constant with respect to z.

Observe that this function $F(z)$ is 0 when $z = x$, because the first two terms cancel and all the others vanish when $z = x$. Also, $F(z)$ is 0 when $z = y$, by reason of the identity (q).

Consequently, by Rolle's Theorem, § 61, the derivative $F'(z)$ must be 0 for some value ξ of z between x and y. Differentiating with respect to z, and observing that the terms on the right, after differentiation, cancel except the last two, we have

$$F'_z(z) = -\frac{(x - z)^n}{n!}f^{n+1}(z) + (n + 1)\frac{(x - z)^n}{n!}Q_y^{(n)}.$$

Hence, when $z = \xi$, at which $F'(\xi) = 0$,

$$Q_y^{(n)} = \frac{f^{n+1}(\xi)}{n + 1}.$$

Substituting this value in (q), we have Lagrange's form of the Theorem of Mean value,*

$$f(x)=f(y)+(x-y)f'(y)+\ldots+\frac{(x-y)^n}{n!}f^n(y)+\frac{(x-y)^{n+1}}{(n+1)!}f^{n+1}(\xi),$$

$$=\sum_{r=0}^{n}\frac{(x-y)^r}{r!}f^r(y)+\frac{(x-y)^{n+1}}{(n+1)!}f^{n+1}(\xi). \tag{L}$$

65. Theorem of Mean Value. Cauchy's Form.—Cauchy has given another form to the evaluation of the difference

$$f(x)-\sum_{r=0}^{n}\frac{(x-y)^r}{r!}f^r(y),$$

which for some purposes is more useful than that of Lagrange. Its deduction is somewhat simpler.

Let x be constant and z a variable. Consider the function

$$F(z)\equiv f(z)+(x-z)f'(z)+\ldots+\frac{(x-z)^n}{n!}f^n(z). \tag{i}$$

By the Theorem of Mean Value, § 62,

$$F(x)-F(a)=(x-a)F'(\xi), \tag{ii}$$

where ξ is some number between x and a.

When $z=x$, we have from (i)

$$F(x)=f(x).$$

When $z=a$, then from (i)

$$F(a)=f(a)+(x-a)f'(a)+\ldots+\frac{(x-a)^n}{n!}f^n(a).$$

Differentiating (i),

$$F_z'(z)=\frac{(x-z)^n}{n!}f^{n+1}(z),$$

and

$$F'(\xi)=\frac{(x-\xi)^n}{n!}f^{n+1}(\xi).$$

Substituting in (ii), we have Cauchy's form

$$f(x)=\sum_{r=0}^{n}\frac{(x-a)^r}{r!}f^r(a)+(x-a)\frac{(x-\xi)^n}{n!}f^{n+1}(\xi). \tag{C}$$

---

* In order that this result shall be true, it is necessary that the function $f(x)$ and its first $n+1$ derivatives shall be finite and determinate at x and at y, and also for *all* values of the variable between x and y. This important formula will be presented in another form in the Integral Calculus, Chapter XIX, § 152.

For a proof of the Theorem: If a function becomes ∞ at a given value of the variable, then all its derivatives are ∞ there, and also the quotient of the derivative by the function is ∞, see Appendix, Note 5.

The numbers represented by ξ in (C) and in (L) are not equal numbers. All we know about ξ in either case is that it is some number between certain limits.

66. Observations on the Theorem of Mean Value.—The formula (L) or (C) is a generalization of the theorem of mean value stated in § 62; that theorem corresponds to the particular value $n = 0$.

The Theorem of the Mean is the basis of the expansion of a function in positive integral powers of the variable. When this expansion in an infinite series is possible, it solves the problem: Given the value of a function and of its derivatives at any one particular value of the variable, to compute the value of the function and of its derivatives at another given value of the variable.

The Theorem of Mean Value is the basis of the application of the Differential Calculus to Geometry in the study of curves and of surfaces, as will be amply illustrated in the sequel.

It solves the problem: To find a polynomial in the variable which shall have the same value and the same first n derivatives at a given value of the variable as a given function. This polynomial, therefore, has the same properties as the given function at the given value of the variable, so far as those properties are dependent on the first n derivatives. This is a most important and valuable property of the formula, for it enables us to study a proposed function by aid of the polynomial, and we know more about the polynomial than about any other function.

67. In Chapters I, . . . , IV, we may be said to have designed the tools of the Differential Calculus, for functions of one variable, in the derivatives on which the properties of functions depend.

In the present chapter this design may be said to have culminated in the presentation of the Theorem of Mean Value.

The subject has been developed continuously and harmoniously from the difference-quotient. The difference-quotient is the foundation-stone from which the derivatives have been evaluated, and by successive differentiation of the difference-quotient we have been led to the Theorem of Mean Value.

It is not necessary to add here any exercises or examples of the application of the Theorem of the Mean, since it will be employed so frequently in what follows. We merely notice other forms under which the formula may be expressed.

68. Forms of the Theorem of Mean Value.

(1). It is customary to write R_n as a symbol of the difference between the functions

$$f(x) \quad \text{and} \quad \sum_{0}^{n} \frac{(x-a)^r}{r!} f^r(a),$$

so that

$$f(x) = \sum_{0}^{n} \frac{(x-a)^r}{r!} f^r(a) + R_n.$$

Or, more briefly,

$$f(x) = S_n + R_n,$$

where S_n represents the Σ function.

(2). In particular, if $a = 0$, and $f(x)$ is differentiable, [illegible] $+ 1$ times at 0 and in $(0, x)$, we have

$$f(x) = f(0) + xf'(0) + \ldots + \frac{x^n}{n!} f^n(0) + R_n,$$

where, using Lagrange's form,

$$R_n = \frac{x^{n+1}}{(n+1)!} f^{n+1}(\xi), \qquad \xi \text{ in } (0, x),$$

or, using Cauchy's form,

$$R_n = x\frac{(x-\xi)^n}{n!} f^{n+1}(\xi), \qquad \xi \text{ in } (0, x).$$

(3). If we write the difference $x - y = h$, so that $x = y + h$,

$$f(y+h) = f(y) + hf'(y) + \ldots + \frac{h^n}{n!} f^n(y) + R_n.$$

(4). Again, since h is arbitrary we can put $h = dy$. Then

$$f(y + dy) = f(y) + df(y) + \ldots + \frac{d^n f(y)}{n!} + R_n,$$

or

$$\Delta f = df + \frac{d^2 f}{2!} + \ldots + \frac{d^n f}{n!} + R_n.$$

EXERCISES.

1. If $f(x) = 0$ when $x = a_1, \ldots, x = a_n$, where

$$a_1 < a_2 < \ldots < a_n,$$

and $f(x)$ and its first n derivatives are continuous in (a_1, a_n), show that

$$f(x) = (x - a_1) \ldots (x - a_n) \frac{f^n(\xi)}{n!},$$

where ξ is some number between the greatest and the least of the numbers $x, a_1, \ldots, a_n$.

2. In particular, if $a_1 = a_2 = \ldots = a_n = a$, then

$$f(x) = \frac{(x-a)^n}{n!} f^n(\xi),$$

where ξ lies between x and a.

CHAPTER VI.

ON THE EXPANSION OF FUNCTIONS.

69. The Power-Series.—To expand a proposed function, in general, means to express its value in terms of a series of given functions. This series has, in general, an infinite number of terms, and when so must be convergent.

We confine our attention here to the expansion of a proposed function in a series of positive integral powers of the variable, based on the Theorem of Mean Value.

The problem of the expansion of a proposed function in an infinite series of positive integral powers of the variable does not admit of complete solution in general, when we are restricted to real values of the variable, for the reason that the values of the variable at which the function becomes infinite enter into the problem, whether these values of the variable be real or imaginary. In the present chapter we shall confine the attention to those simple functions whose expansions can be readily demonstrated in real variables, relegating to the Appendix * a more complete discussion of the general problem.

70. Taylor's Series.—If in the formula of the Theorem of Mean Value,

$$f(x) = \sum_{r=0}^{n} \frac{(x-a)^n}{n!} f^n(a) + R_n, \tag{1}$$

the derivatives $f^r(a)$, $r = 1, 2, \ldots,$ at a, are such that the series

$$S_n \equiv \sum_{r=0}^{n} \frac{(x-a)^n}{n!} f^n(a),$$

has a finite limit when $n = \infty$, and we also have

$$\mathop{\pounds}_{n=\infty} R_n = 0,$$

then for the values of x and a involved we have

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots \tag{T}$$

This is called *Taylor's formula* or series.

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* See Appendix, Notes 6, 7, 8.

We may use any of the different forms of R_n we choose in showing $\pounds R_n = 0$.

71. Maclaurin's Series.—Under the same conditions as in § 70, if $a = 0$,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots \tag{M}$$

This is called Maclaurin's formula * or series.

The series (M) generally admits calculation more readily than does Taylor's (T), because usually the derivatives at 0 are of simpler form than those at an arbitrarily selected value of the variable a.

EXAMPLES.

1. Any rational integral function or polynomial $f(x)$ can always be expressed as

$$f(a) + (x-a)f'(a) + \dots + \frac{(x-a)^n}{n!}f^n(a),$$

where n is the degree of the polynomial $f(x)$.

For, since f is of the nth degree, all derivatives of order higher than f^n are 0. Consequently the theorem of mean value gives

$$f(x) = \sum_{r=0}^{n} \frac{(x-a)^r}{r!} f^r(a),$$

whatever values be assigned to x and a.

In particular, we may put $a = 0$, and have

$$f(x) = f(0) + xf'(0) + \dots + \frac{x^n}{n!}f^n(0),$$

and this must be the polynomial considered when arranged according to the powers of x.

2. We may define as a *transcendental integral function* one such that *all* of its derivatives remain determinate and *non-infinite* for any assigned value of the variable.

Any such function can be calculated by either Taylor's or Maclaurin's series for any finite value of the variable, whatever.

For if f be such a function, then, whatever be the assigned number a, we have

$$\underset{n=\infty}{\pounds} \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(\xi) = 0,$$

since $f^{n+1}(\xi)$ is finite for any ξ between x and a, for all values of n. Also, $(x-a)^{n+1}/(n+1)!$ has the limit 0 when $n = \infty$ (see § 15, Ex. 9).

Moreover, the series is absolutely convergent (Introd., § 15, Ex. 10), since

$$\sum_{r=0}^{\infty} \frac{(x-a)^r}{r!} f^r(a)| \leqq |\sum_{r=0}^{\infty} \frac{|x-a|^r}{r!} M^r,$$

where M is a finite absolute number not less than the absolute value of any derivative of f at a. The series on the right is absolutely convergent, since

$$\underset{n=\infty}{\pounds} \frac{|x-a|}{n+1} M| < |1,$$

see § 15, Ex. 10.

---

* This formula is really due to Stirling.

Therefore, if $f(x)$ be any *transcendental integral* function, we have for any assigned value of x or a

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots$$

Also,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots$$

Such functions are $\sin x$, $\cos x$, e^x.

3. Show that if $f(x)$ is any transcendental integral function as defined in Ex. 2, then $f(px+q)$ can be expanded in Taylor's series for any assigned values of p, q, x and a.

This follows immediately from 2, since

$$\left(\frac{d}{dx}\right)^n f(px+q) = p^n f^n(px+q).$$

4. To expand e^x by Maclaurin's formula.

We have $D^r e^x = e^x$ for all values of r. At 0 we have $D^r e^x = e^0 = 1$. Also,

$$\underset{n=\infty}{\mathcal{L}} \frac{x^{n+1}}{(n+1)!}e^{\xi} = 0.$$

Hence, substituting in Maclaurin's series, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= \sum_{r=0}^{\infty} \frac{x^r}{r!}.$$

In particular, when $x = 1$,

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots,$$

which gives a simple and easy method of computing e to any degree of approximation we choose.

5. To compute $\sin x$, given x, by Maclaurin's formula.

$$\sin 0 = 0, \qquad D^{2n-1}\sin 0 = (-1)^{n-1}, \qquad \text{and} \qquad D^{2n}\sin 0 = 0,$$

by Ex. 5, § 56. Therefore

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

6. To compute in the same way $\cos x$, given x.

By Ex. 5, § 56, $\cos 0 = 1$, $D^{2n-1}\cos 0 = 0$, $D^{2n}\cos 0 = (-1)^n$.

$$\therefore \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

The derivatives of $\sin x$ and $\cos x$ being always finite, these functions are transcendental integral functions and it is unnecessary to examine the terminal term R_n. The limit of R_n, however, is very readily seen to be 0, since we have respectively

$$R_n = \frac{x^{n+1}}{(n+1)!}\sin\left(\xi + \frac{n}{2}\pi\right), \quad \text{for} \quad \sin x,$$

$$= \frac{x^{n+1}}{(n+1)!}\cos\left(\xi + \frac{n}{2}\pi\right), \quad \text{for} \quad \cos x.$$

7. The binomial formula for any real exponent.

Consider the expansion of $(1 + x)^a$ by Maclaurin's series, when a is any assigned real number.

We have

$$D^n(1 + x)^a = a(a - 1) \dots (a - n + 1)(1 + x)^{a-n}.$$

$$\therefore \quad [D^n(1 + x)^a]_{x=0} = a(a - 1) \dots (a - n + 1).$$

Substituting in Maclaurin's series, we have

$$1 + ax + \frac{a(a - 1)}{2!}x^2 + \frac{a(a - 1)(a - 2)}{3!}x^3 + \dots$$

The quotient of convergency, § 15, Ex. 9, of this series is

$$\underset{n=\infty}{\mathcal{L}} \left|\frac{a - n}{n + 1}x\right| = |x|. \tag{1}$$

Therefore the series is absolutely convergent when $|x| < 1$, or for all values of x in $)-1, +1($. For $|x| > 1$, the series is ∞.

Also, by (C), § 65, or § 68, (2),

$$R_n = x\frac{(x - \xi)^n}{n!}\,\frac{a(a - 1) \dots (a - n)}{(1 + \xi)^{n+1-a}}. \tag{2}$$

Whatever be the value of ξ between x and 0, so long as $|x| < 1$ we have

$$\underset{n=\infty}{\mathcal{L}} \frac{a - n}{n + 1}\,\frac{x - \xi}{1 + \xi}\ | < |\ 1. \tag{3}$$

For this limit is the same as

$$\mathcal{L}\left|\frac{x - \xi}{1 + \xi}\right|,$$

which is less than 1 when $0 < x < 1$. If $x < 0$, put $x = -x'$ and $\xi = -\xi'$. Then the limit is equal to

$$\mathcal{L}\left|\frac{x' - \xi'}{1 - \xi'}\right|.$$

But $x' - \xi' < 1 - \xi'$, since $0 < x' < 1$ and $0 < \xi' < x'$.

Inequality (3) being true, $\mathcal{L}R_n = 0$, in (2). Therefore the series is equal to the function for the same values of x for which the series is absolutely convergent.

$$\therefore \quad (1 + x)^a = 1 + ax + \frac{a(a - 1)}{2!}x^2 + \frac{a(a - 1)(a - 2)}{3!}x^3 + \dots$$

for all values of x in $)-1, +1($, and the equality does not exist for any value of x for which $|x| > 1$.

8. Expand $\log(1 + x)$ by Maclaurin's series.

Let $$f(x) = \log(1 + x).$$

$$\therefore \quad f^n(x) = (-1)^{n+1}\frac{(n - 1)!}{(1 + x)^n},$$

and $$f^n(0) = (-1)^{n+1}(n - 1)!.$$

Substituting in Maclaurin's series, we get

$$x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \dots$$

The convergency quotient of this is

$$\underset{n=\infty}{\mathcal{L}} \left|-\frac{n}{n + 1}x\right| = |x|.$$

The series is therefore absolutely convergent for $|x| < 1$, and is ∞ for $|x| > 1$. Also, we have, by (C), § 65,

$$R_n = (-1)^n x \frac{(x-\xi)^n}{(1+\xi)^{n+1}}.$$

Whatever may be ξ between x and 0 when $|x| < 1$, we have, as in Ex. 7,

$$\mathcal{L}_{n=\infty} \left|\frac{x-\xi}{1+\xi}\right| < 1.$$

Therefore $\mathcal{L} R_n = 0$, and

$$\log (1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots \qquad (1)$$

This series converges too slowly for convenience, that is, too many terms have to be calculated to get a close approximation to the value of $\log (1+x)$.

By changing the sign of x,

$$\log (1-x) = -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \dots \qquad (2)$$

By subtracting (2) from (1),

$$\log \frac{1+x}{1-x} = 2(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \dots) \qquad (3)$$

If n and m are any positive numbers, put

$$\frac{1+x}{1-x} = \frac{n+m}{n}, \qquad \text{then} \qquad x = \frac{m}{2n+m}.$$

Substituting in (3),

$$\log \left(\frac{n+m}{n}\right) = 2\left(\frac{m}{2n+m} + \frac{1}{3}\frac{m^3}{(2n+m)^3} + \frac{1}{5}\frac{m^5}{(2n+m)^5} + \dots\right),$$

a series which converges rapidly when $n > m$, and gives the logarithm of $m+n$ when $\log n$ is known.

The logarithms thus computed are of course calculated to the base e. To find the logarithm to any other base, we have

$$\log_a y = \frac{\log_e y}{\log_e a}.$$

72. Observations on the Expansion of Functions by Taylor's Series.—The expansion of a given function by the law of the mean is rendered difficult, in general, because of the complicated character of the nth derivative which it is necessary to know in order to get the law of the series and test of its convergency.

Still more difficult is the investigation of the limit of R_n. This latter investigation is usually more troublesome than the question of convergency of the series because of the uncertainty regarding the value of the number ξ. The only information we have with regard to ξ is that it is some number which lies between two given numbers. Moreover, we know that ξ is a function of n and in general changes its value with n. It is therefore necessary that we should show that $\mathcal{L} R_n = 0$ for *all* values of ξ between x and a, in order to be sure that $\mathcal{L} R_n$ is 0 for the particular value ξ involved in the law of the mean whatever may be that number ξ between x and a. In the deduction of the form R_n in the Integral Calculus, Chapter XIX,

§ 152, it is there shown that not only is it sufficient that we should consider all values of ξ in the interval (a, x), but it is also necessary. The equality of the function and the series depends on R_n vanishing for *all* values of ξ in (a, x).*

It is desirable therefore, that we should have such general laws with regard to the expansion of functions as will enable us, as far as it is possible, to avoid the formation of the nth derivative and the investigation of the remainder term R_n, and which will permit us to state for certain classes of functions determined by general properties that the equivalence of Taylor's or Maclaurin's series with the function is true for a certain definite interval of the variable. The general discussion of this subject is too extensive for this course. We give in the next article some observations which will be of assistance in simplifying the problem. In the Appendix a more general treatment of the question is discussed.

73. Consider a function $f(x)$ and its derivative $f'(x)$. We can state certain relations between a primitive and its derivative, with regard to the corresponding power series as follows:

Cauchy's form of the law of the mean value applied to each of the functions $f(x)$ and $f'(x)$ gives

$$f(x) = f(a) + (x-a)f'(a) + \ldots + \frac{(x-a)^n}{n!}f^n(a) + R_n, \qquad (1)$$

$$f'(x) = f'(a) + (x-a)f''(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!}f^n(a) + R'_n, \quad (2)$$

where

$$R_n = (x-a)\frac{(x-\xi)^n}{n!}f^{n+1}(\xi), \qquad (3)$$

$$R'_n = (x-a)\frac{(x-\xi')^{n-1}}{(n-1)!}f^{n+1}(\xi'). \qquad (4)$$

I. We observe that the quotients of convergency of (1) and (2), as obtained by taking the limit of the quotient of the $(n+1)$th term to the nth term, have the same value, for

$$\underset{n=\infty}{\mathcal{L}}\frac{x-a}{n+1}\frac{f^{n+1}(a)}{f^n(a)} = \underset{n=\infty}{\mathcal{L}}\frac{x-a}{n}\frac{f^{n+1}(a)}{f^n(a)},$$

$$= \underset{n=\infty}{\mathcal{L}}\frac{x-a}{n+1}\frac{f^{n+1}(a)}{f^n(a)}\left(1 - \frac{1}{n+1}\right)^{-1}.$$

---

* In the theorem of the mean, (I), § 70, the series

$$\sum_0^\infty \frac{(x-a)^n}{n!}f^n(a)$$

may be *absolutely convergent* and yet not equal to the function $f(x)$. For Pringsheim's example, see Appendix, Note 8.

Therefore, if

$$\mathcal{L}\left|n\frac{f^{n}(a)}{f^{n+1}(a)}\right| = R \tag{5}$$

is a finite determinate number, then the two series

$$f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots$$

and

$$f'(a) + (x-a)f''(a) + \frac{(x-a)^2}{2!}f'''(a) + \dots$$

are absolutely convergent in the common interval $)a - R,\ a + R($, and are both ∞ for any value of x outside of this interval.

The number a is called the base of the expansion, or the centre of the interval of convergence. The number R is called the radius of convergence.

II. We observe that if, for *all* values of ξ between x and a, we have

$$\mathop{\mathcal{L}}_{n=\infty} \frac{x-\xi}{n}\frac{f^{n+1}(\xi)}{f^{n}(\xi)} \,|<|\, 1, \tag{6}$$

in which ξ has the same value wherever it occurs, then, § 15, Ex. 9, must (3) and (4) be 0 when $n = \infty$ whatever be the value of ξ between x and a in (3) or (4).

Consequently, if we have determined (5) for any function and shown that (6) is true for values of x in the interval of convergence, then this function, its derivative or its primitive is equal to the corresponding Taylor's series in the common interval

$$)a - R, \quad a + R(.$$

EXAMPLES.

1. Having proved that the requirements in § 73 are satisfied for $(1 + x)^a$, and this function is equal to its Maclaurin's series for all values of x between -1 and $+1$, and for no values of x outside these limits, it follows immediately, in virtue of § 73, that $\log(1 + x)$ is equal to its Maclaurin's series in the same interval, since

$$D\log(1 + x) = (1 + x)^{-1}.$$

2. The function $\tan^{-1}x$ is equal to its Maclaurin's series for $x^2 < 1$. For

$$D\tan^{-1}x = \frac{1}{1 + x^2},$$

and $x^2 < 1$ is the interval of equivalence of $(1 + x^2)^{-1}$ with its Maclaurin's series. Moreover, since

$$(1 + x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \dots,$$

and the primitive of $(1 + x^2)^{-1}$ is $\tan^{-1}x$, and $\tan^{-1}0 = 0$, we have, by § 73,

$$\tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \dots,$$

for

$$-1 < x < +1.$$

We can verify this result directly, for

$$D^n \tan^{-1}x = (-1)^{n-1}\frac{(n-1)!}{(1+x^2)^{\frac{1}{2}n}}\sin(n\tan^{-1}x^{-1}).$$

$$\therefore\quad [D^n \tan^{-1}x]_0 = (-1)^{n-1}(n-1)!\,\sin(\tfrac{1}{2}n\pi).$$

Also, $\sin\left(2m\,\dfrac{\pi}{2}\right) = 0,\quad \sin\,(2m+1)\dfrac{\pi}{2} = (-1)^n.$

Therefore the Maclaurin's series for $\tan^{-1}x$ is

$$x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \ldots,$$

which has the interval of absolute convergence $)-1, +1($.

For R_n, in Lagrange's form, we have

$$R^n = \frac{x^n}{n}\,\frac{\sin\,(n\tan^{-1}x^{-1})}{(1+\xi^2)^{\frac{1}{2}n}},$$

the limit of which, for $n = \infty$, is 0 when $|x| < 1$.

In particular, if $x = \tan\frac{1}{6}\pi = 1/\sqrt{3}$, then

$$\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{3}\,\frac{1}{3} + \frac{1}{5}\,\frac{1}{3^2} - \frac{1}{7}\,\frac{1}{3^3} + \ldots,$$

which can be used to compute the number π. A better method, however, is given below

3. For all values of $|x| < 1$ we have shown that

$$(1-x^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \ldots$$

But a primitive of $(1-x^2)^{-\frac{1}{2}}$ is $\sin^{-1}x$, and since $\sin^{-1}0 = 0$, we have, by § 73,

$$\sin^{-1}x = x + \frac{1}{2}\,\frac{1}{3}\,x^3 + \frac{1\cdot 3}{2\cdot 4}\,\frac{1}{5}\,x^5 + \ldots$$

for x in $)-1, +1($.

In particular, since $\frac{1}{6}\pi = \sin^{-1}\frac{1}{2}$, we have

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2\cdot 3}\,\frac{1}{2^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5}\,\frac{1}{2^5} + \ldots,$$

from which π can be computed rapidly.

4. Determine the Maclaurin's series for $\cos^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\csc^{-1}x$. In each case determine the interval for which the function is equal to the series.

74. We can find the nth derivative of $\sin^{-1}x$ without difficulty, but it would be difficult to evaluate the corresponding limit of R_n by the direct processes of Maclaurin's formula.

Observe that the coefficients in the power series for $\sin^{-1}x$ can be determined from Ex. 38, § 58, where we have

$$(1-x^2)D^{n+2}\sin^{-1}x - (2n+1)x\,D^{n+1}\sin^{-1}x - n^2D^n\sin^{-1}x = 0.$$

$$\therefore\quad D^{n+2}\sin^{-1}0 = n^2D^n\sin^{-1}0.$$

When we have found $D\sin^{-1}0$, $D^2\sin^{-1}0$, the other derivatives at 0 can be found directly, and the interval of the convergence of the series established. The interval of equivalence of the function and the series by evaluating $\pounds R_n$ is a matter of considerable difficulty.

In the text we go no further into this matter of the expansion of functions by Taylor's formula. We have made use of it to show how the tables of the ordinary functions and of logarithms can be computed, and the numbers e and π evaluated.

We add a few exercises in the application of the formula. The cases in which the remainder term R_n is inserted are those for which we have not established either the convergence of the infinite series or its equivalence with the function; they may be regarded as exercises in differentiation or as applications of the Law of Mean Value. Some of these results will be useful later in the evaluation of indeterminate forms and approximate calculations.

We observe that for the purpose of approximate calculations, if M be the greatest and m the least absolute value of the $(n+1)$th derivative in the interval (a, x), the error committed in taking

$$f(x) = \sum_{1}^{n} \frac{(x-a)^r}{r!} f^r(a)$$

lies in absolute value between

$$\frac{|x-a|^{n+1}}{(n+1)!} m \quad \text{and} \quad \frac{|x-a|^{n+1}}{(n+1)!} M,$$

by Lagrange's form of R_n. When we know the nth derivative of the function to be calculated, we can thus determine beforehand how many terms of the series will have to be taken in order that the error shall not exceed a given number.

EXERCISES.

1. If c is the chord of a circular arc a, and b the chord of half the arc, show that the error in taking

$$a = \frac{1}{3}(8b - c)$$

is less than $\frac{a}{7680}$, where $a <$ radius of the arc.

2. If d is the distance between the middle points of the chord c and arc a, in Ex. 1, show that the error in taking

$$c = a - \frac{8}{3}\frac{d^2}{a}$$

is less than $\frac{32}{3}\frac{d^4}{a^3}$.

3. The series $1 + x + x^2 + \dots$ is convergent for $|x| < 1$. It is infinite when $x \geqq 1$, and also ∞ when $x < -1$. Show that we can make x converge to -1 in such a way as to make the sum of the series equal to any assigned number we choose.

Let $x = \frac{a}{n+1} - 1$, where a is any assigned number. Then we have for the sum of $n+1$ terms of the series

$$\frac{1 - x^{n+1}}{1 - x} = \frac{1 + (-1)^n\left(1 - \dfrac{a}{n+1}\right)^{n+1}}{2 - \dfrac{a}{n+1}}.$$

If $n = 2m$ or $2m + 1$, and $m = \infty$, this sum is respectively equal to

$$\tfrac{1}{2}(1 + e^{-a}) \quad \text{or} \quad \tfrac{1}{2}(1 - e^{-a}),$$

one or the other of which can be made equal to any given number by properly assigning a.

Show that

4. $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + R_7.$

5. $\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + R_8.$

6. $\log(1 + \sin x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + R_6.$

7. $e^x \sec x = 1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{2}x^4 + \frac{3}{10}x^5 + R_6.$

8. Show that for $|x| < 1$ we have

$$\log\left(x + \sqrt{1 + x^2}\right) = x - \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} - \cdots$$

Hint. $D \log\left(x + \sqrt{1 + x^2}\right) = (1 + x^2)^{-\frac{1}{2}}.$

9. Expand $\sin^{-1}\dfrac{2x}{1 + x^2}$ and $\tan^{-1}\dfrac{x}{\sqrt{1 - x^2}}$, in powers of x, determining the intervals of equivalence, §§ 72, 75.

10. Expand $x\sqrt{x^2 + a^2} + a^2 \log\left(x + \sqrt{x^2 + a^2}\right)$, in powers of x and determine the interval of equivalence.

Hint. The derivative is $2\sqrt{a^2 + x^2}$.

11. Expand in like manner

$$\frac{1}{4\sqrt{2}} \log \frac{1 + x\sqrt{2} + x^2}{1 - x\sqrt{2} + x^2} + \frac{1}{2\sqrt{2}} \tan^{-1} \frac{x\sqrt{2}}{1 - x^2}$$

by using its derivative $(1 + x^4)^{-1}$.

12. Show that the nth derivative of $(x^2 + 6x + 8)^{-1}$ at 0 is

$$(-1)^n n! \frac{1}{2^{n+2}}\left(1 - \frac{1}{2^{n+1}}\right).$$

Expand the function in integral powers of x and determine the interval of equivalence.

13. Show by Maclaurin's formula that

$$(1 + x)^{\frac{1}{x}} = e\left\{1 - \tfrac{1}{2}x + \tfrac{11}{24}x^2 - \tfrac{7}{16}x^3\right\} + R_4.$$

Hint. If $y = (1 + x)^{\frac{1}{x}}$, then $\log y = \dfrac{\log(1 + x)}{x}$.

$$\therefore \; y = e^{\phi(x)}, \quad \phi(x) = 1 - \tfrac{1}{2}x + \tfrac{1}{3}x^2 - \tfrac{1}{4}x^3 + \ldots,$$

and the first few derivatives can be found.

14. Compute the following numbers to six decimal places: e, π, $\log 2$, $\log_e 10$, $\sin 10°$.

CHAPTER VII.

ON UNDETERMINED FORMS.

75. When u and v are functions of x, they are also functions of each other. If, when $x(=)a$, we have $u(=)0$ and $v(=)0$, the *quotient*

$$\frac{u}{v}$$

will in general have a determinate limit when $x(=)a$. This limit will depend on the law of connectivity between u and v. The evaluation of the derivative is but a particular and simple case of the evaluation of the limit of the quotient of two functions which have a common root as the variable converges to that root. For, in the derivative, we are evaluating the limit of the quotient

$$\frac{f(x)-f(a)}{x-a}$$

when $$f(x)-f(a)(=)0 \quad \text{and} \quad x-a(=)0.$$

The evaluation of the quotient u/v when x converges to the common root a of u and v, is but a generalization of the idea involved in the evaluation of the derivative. For, let $\phi(x)$ and $\psi(x)$ be two functions which vanish when $x=a$, or, as we say, have a common root a. Then

$$\phi(a)=0 \quad \text{and} \quad \psi(a)=0.$$

We wish to evaluate the limit of the quotient

$$\frac{\phi(x)}{\psi(x)}$$

when $x(=)a$.

Since $\phi(a)=0$, $\psi(a)=0$, we have

$$\frac{\phi(x)}{\psi(x)}=\frac{\phi(x)-\phi(a)}{\psi(x)-\psi(a)},$$

$$=\frac{\dfrac{\phi(x)-\phi(a)}{x-a}}{\dfrac{\psi(x)-\psi(a)}{x-a}}.$$

Consequently if $\phi(x)$ and $\psi(x)$ are differentiable functions at a, and the member on the left has a determinate limit when $x(=)a$, we have

$$\mathop{\pounds}_{x(=)a} \frac{\phi(x)}{\psi(x)} = \frac{\phi'(a)}{\psi'(a)}.$$

For example,

$$\mathop{\pounds}_{x(=)1} \frac{\log x}{x - 1} = 1.$$

It may happen that a is a common root of $\phi'(x)$ and $\psi'(x)$, then $\phi'(a) = 0$ and $\psi'(a) = 0$. In this case we shall require a further investigation in order to evaluate the quotient ϕ/ψ. For this purpose we require the following theorems:

76. A Theorem due to Cauchy.—Let $\phi(x)$ and $\psi(x)$ be two functions which vanish at a, as also do their first n derivatives, but the $(n+1)$th derivatives of both $\phi(x)$ and $\psi(x)$ do not vanish at a. Then we shall have

$$\phi(x)\psi^{n+1}(\xi) = \psi(x)\phi^{n+1}(\xi),$$

where ξ is some number between x and a.

Let z be a variable in the interval determined by the two fixed numbers x and a. Then the function

$$J(z) \equiv \phi(z)\,\psi(x) - \psi(z)\,\phi(x),$$
$$= 0 \text{ when } z = a, \text{ also when } z = x.$$

By the law of the mean, § 62, $J'(z) = 0$ for some number $z = \xi_1$ between x and a. But, in virtue of the fact that $\phi'(a) = \psi'(a) = 0$, we have $J'(a) = 0$. Consequently $J''(z) = 0$ for some number ξ_2 between ξ_1 and a.

In like manner $J'''(z) = 0$ for $z = \xi_3$ between ξ_2 and a, and so on until finally we have

$$J^{n+1}(\xi) = \phi^{n+1}(\xi)\,\psi(x) - \psi^{n+1}(\xi)\,\phi(x) = 0,$$

where ξ is some number between x and a.

If $\psi^{n+1}(z)$ is not 0 between x and a, we can divide by it. Hence

$$\frac{\phi(x)}{\psi(x)} = \frac{\phi^{n+1}(\xi)}{\psi^{n+1}(\xi)}.$$

This theorem is of great generality and usefulness.

For example, the functions $(x-a)^{n+1}/(n+1)!$ and

$$F(x) \equiv f(x) - \sum_{r=0}^{n} \frac{(x-a)^r}{r!} f^r(a)$$

are such that they and their first n derivatives vanish at $x = a$, while the $(n+1)$th derivative of the first function is 1. Therefore, by the theorem just proved, we have

$$F(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(\xi),$$

which is Lagrange's formula for the law of the mean.

This theorem can be utilized for finding many of the different forms of the remainder in the law of the mean. It has, however, its chief application in:

77. The Theorem of l'Hôpital.—If $\phi(x)$ and $\psi(x)$ are two functions which vanish at a, as also do their first n derivatives, then we shall have

$$\underset{x(=)a}{\mathcal{L}} \frac{\phi(x)}{\psi(x)} = \underset{x(=)a}{\mathcal{L}} \frac{\phi^{n+1}(\xi)}{\psi^{n+1}(\xi)} = \underset{x(=)a}{\mathcal{L}} \frac{\phi^{n+1}(x)}{\psi^{n+1}(x)},$$

$$= \frac{\phi^{n+1}(a)}{\psi^{n+1}(a)}.$$

For, by Cauchy's theorem, § 76,

$$\frac{\phi(x)}{\psi(x)} = \frac{\phi^{n+1}(\xi)}{\psi^{n+1}(\xi)},$$

where ξ lies between x and a. Hence, since ξ and x converge to a together, we have for $x(=)a$

$$\mathcal{L} \frac{\phi(x)}{\psi(x)} = \frac{\phi^{n+1}(a)}{\psi^{n+1}(a)}.$$

Moreover, Cauchy's theorem shows that the quotients

$$\frac{\phi^{r}(x)}{\psi^{r}(x)}, \qquad (r = 1, 2, \ldots, n)$$

all have this same limit.

Therefore, to find the value of the undetermined form, we evaluate successively the quotients of the successive derivatives until we arrive at a quotient no longer indeterminate.

EXAMPLES.

1. Evaluate, when $x(=)1$, the quotient

$$\frac{x^2 - 3x + 2}{x^2 - 1}.$$

$$x^2 - 3x + 2 = 0, \quad \text{when } x = 1.$$

$$D(x^2 - 3x + 2) = 2x - 3, \quad = -1, \quad \text{when } x = 1.$$

$$x^2 - 1 = 0, \quad \text{when } x = 1.$$

$$D(x^2 - 1) = 2x, \quad = 2, \quad \text{when } x = 1.$$

$$\therefore \quad \underset{x(=)1}{\mathcal{L}} \frac{x^2 - 3x + 2}{x^2 - 1} = \underset{x(=)1}{\mathcal{L}} \frac{2x - 3}{2x} = -\frac{1}{2}.$$

2. Show that

$$\underset{x(=)1}{\mathcal{L}} \frac{x - 1}{x^n - 1} = \frac{1}{n}.$$

3. Evaluate, when $x(=)0$, the following:

$$\underset{x(=)0}{\pounds}\frac{e^x - e^{-x}}{\sin x} = 2\ ; \quad \underset{x(=)0}{\pounds}\frac{x \sin x}{x - 2\sin x} = 0.$$

4. Show that for $x(=)0$, we have

$$\pounds\frac{e^x - e^{-x} - 2x}{x - \sin x} = 2\ ; \quad \pounds\frac{e^x + e^{-x} - 2}{\operatorname{vers} x} = 2.$$

5. Evaluate, when $x(=)0$,

$$\pounds\frac{x - \sin^{-1}x}{\sin^3 x} = -\frac{1}{6}\ ; \quad \pounds\frac{a^x - b^x}{x} = \log\frac{a}{b}\ ; \quad \pounds\frac{\tan x - x}{x - \sin x} = 2.$$

6. Find the limits, when $x(=)0$,

$$\pounds\frac{x - \sin x}{x^3} = \frac{1}{6}\ ; \quad \pounds\frac{\sin 3x}{x - \frac{3}{2}\sin 2x} = -\frac{3}{2}\ ; \quad \pounds\frac{x^2}{1 - \cos mx} = \frac{2}{m^2}.$$

78. The Illusory Forms.—When u and v are two functions of x, which are such that the functions

$$u/v, \qquad uv, \qquad u - v, \qquad u^v,$$

tend to take any of the forms

$$0/0, \quad \infty/\infty, \quad 0 \times \infty, \quad \infty - \infty, \quad 0^0, \quad \infty^0, \quad 1^\infty,$$

as x converges to a; then when these functions have determinate limits for $x(=)a$, the theorem of l'Hôpital will evaluate these limits.

All these forms can be reduced to the evaluation of the first, $0/0$, as follows:

(1). ∞/∞ and $0 \times \infty$ reduce directly to $0/0$.

For, if $u_a = \infty$, $v_a = \infty$, then

$$\frac{u_a}{v_a} = \frac{\infty}{\infty} = \frac{1/v_a}{1/u_a} = \frac{0}{0},$$

and we evaluate

$$\frac{1/v_x}{1/u_x}.$$

If $u_a = 0$, $v_a = \infty$, then

$$u_a v_a = \frac{u_a}{1/v_a} = \frac{0}{0},$$

and we evaluate

$$\frac{u_x}{1/v_x}.$$

(2). In like manner, if $u_a = \infty$, $v_a = \infty$, then

$$u_a - v_a = u_a\left(1 - \frac{v_a}{u_a}\right) = \frac{1 - v_a/u_a}{1/u_a} = \frac{0}{0},$$

provided $\pounds(v_x/u_x) = 1$, otherwise this form has no determinate finite limit and is ∞.

This illusory form can also be reduced to the evaluation of the form $0/0$ when $x(=)a$, thus:

$$e^{u-v} = \frac{e^{-v}}{e^{-u}},$$

which takes the form $0/0$ when $x = a$. Therefore, if $\mathcal{L}\, e^{u-v} = e^{c}$, $\mathcal{L}(u - v) = c$, for $x(=)a$.

(3). The last three forms, 0^0, ∞^0, 1^∞, arise from the function u^v, which can be reduced to $0/0$, thus:

Since $$u = e^{\log u},$$

$$\therefore \quad u^v = e^{v \log u}.$$

In each of the cases 0^0, ∞^0, 1^∞, the function $v \log u$ takes the form $0 \times \infty$, which can be turned directly into $0/0$ and evaluated as in (1).

EXAMPLES OF ∞/∞ and $0/\infty$.

The evaluation of u/v, when $u = \infty$, $v = \infty$, for $x = a$, is carried out in the same way as for $0/0$. For we have

$$\mathcal{L}\frac{\phi(x)}{\psi(x)} = \mathcal{L}\frac{1/\psi(x)}{1/\phi(x)} = \mathcal{L}\frac{-\psi'(x)/[\psi(x)]^2}{-\phi'(x)/[\phi(x)]^2},$$

$$= \mathcal{L}\left\{\frac{\phi(x)}{\psi(x)}\right\}^2 \frac{\psi'(x)}{\phi'(x)},$$

when $x(=)a$. If now $\phi(x)/\psi(x)$ has a determinate limit $A \neq 0$, when $x(=)a$, then

$$A = A^2 \,\mathcal{L}\frac{\psi'(x)}{\phi'(x)}.$$

Therefore, for $x(=)a$, when $\phi(x) = \infty$, $\psi(x) = \infty$,

$$\mathcal{L}\frac{\phi(x)}{\psi(x)} = A = \mathcal{L}\frac{\phi'(x)}{\psi'(x)} = \frac{\phi'(a)}{\psi'(a)},$$

if $\phi'(a)/\psi'(a)$ is determinate.

1. $$\mathcal{L}\frac{\tan x}{\sec x} = \mathcal{L}\frac{\sec^2 x}{\sec x \tan x} = \mathcal{L}\frac{\sec x}{\tan x}.$$

$$\therefore \quad \mathcal{L}\left(\frac{\tan x}{\sec x}\right)^2 = 1, \quad \text{when} \quad x(=)\tfrac{1}{2}\pi.$$

Or immediately, by Trigonometry, $\dfrac{\tan x}{\sec x} = \sin x$.

2. Show that

$$\mathcal{L}_{x=\infty}\frac{x^n}{e^x} = \mathcal{L}\frac{n!}{e^x} = 0,$$

when n is a positive integer. Also when n is not an integer.

3. Show that $\mathcal{L}_{x(=)0}\, x^m(\log x)^n = 0$.

4. Show that $\underset{\theta(=)\frac{1}{2}\pi}{\mathcal{L}} \dfrac{\log(\theta - \frac{1}{2}\pi)}{\tan\theta} = 0.$

5. Evaluate, when $x(=)\frac{1}{2}\pi$,

$$\frac{\tan x}{\tan 3x}; \quad \frac{\log\tan 2x}{\log\tan x}; \quad \frac{1 - \sin x + \cos x}{\sin x + \cos x - 1};$$

$$\frac{\log\sin x}{(\pi - 2x)^2}; \quad \frac{\sec x}{\sec 3x}; \quad \frac{\tan x}{\tan 5x}.$$

6. Show that $\underset{x(=)1}{\mathcal{L}}(1 - x)\tan\frac{1}{2}(\pi x) = \dfrac{2}{\pi}.$

EXAMPLES OF $\infty - \infty$.

7. $\mathcal{L}(\sec x - \tan x) = 0$, for $x(=)\frac{1}{2}\pi$.

8. $\mathcal{L}(x^{-1} - \cot x) = 0$, for $x(=)0$.

9. $x\tan x - \frac{1}{2}\pi\sec x\,(=) - 1$, when $x(=)\frac{1}{2}\pi$.

10. $\dfrac{x - \sin x}{x^3}\,(=)\dfrac{1}{6}$, when $x(=)0$.

11. $(a^x - 1)/x\,(=)\log a$, when $x(=)0$.

EXAMPLES OF u^v.

12. $(1 + x)^{\frac{1}{x}}(=)e, \quad x(=)0.$

13. $(1 + x^2)^{\frac{1}{x}}(=)1, \quad x(=)0.$

14. $(e^x + 1)^{\frac{1}{x}}(=)e, \quad x = \infty.$

15. $(\cos 2x)^{\frac{1}{x^2}}(=)e^{-2}, \quad x(=)0.$

16. $x^{\frac{1}{1-x}}(=)e^{-1}, \quad x(=)1.$

79. General Observations on Illusory Forms.—In evaluating illusory forms, we may at any stage of the process suppress any common factors in the numerator and denominator, and evaluate independently any factor which has a determinate limit. We can frequently make use of algebraic and trigonometric transformations which will simplify and sometimes permit the evaluation without use of the Calculus.

In illustration consider the limit of

$$(x - 1)^{\frac{a}{\log\sin\pi x}}, \quad \text{when} \quad x(=)1.$$

This takes the form 0^0. To evaluate, equate the function to y and take the logarithm,

$$\therefore \quad \log y = a\,\frac{\log(x - 1)}{\log\sin\pi x}.$$

$$\mathcal{L}\frac{\log(x - 1)}{\log\sin\pi x} = \mathcal{L}\frac{\dfrac{1}{x - 1}}{\dfrac{\pi\cos\pi x}{\sin\pi x}} = \frac{1}{\pi}\mathcal{L}\frac{\sin\pi x}{x - 1}\sec\pi x.$$

But $\pounds \sec \pi x = -1$, and

$$\pounds \frac{\sin \pi x}{x-1} = \pounds \frac{\pi \cos \pi x}{1} = -\pi.$$

$$\therefore \quad \pounds \log y = \log \pounds y = a.$$

Hence $\pounds y = e^a$, when $x(=)1$.

Frequently the evaluation can be simplified by substituting for the functions involved their values in terms of the law of the mean.

For example, evaluate for $x(=)0$,

$$\frac{(1+x)^{\frac{1}{x}} - e}{x}.$$

Differentiating numerator and denominator, we have

$$(1+x)^{\frac{1}{x}} \frac{x - (1+x)\log(1+x)}{x^2(1+x)}.$$

$\pounds(1+x)^{\frac{1}{x}} = e$, and the limit of the other factor is, by the ordinary process, readily found to be $-\frac{1}{2}$. Hence the limit is $-e/2$.

Otherwise, put for $(1+x)^{\frac{1}{x}}$ its value, Ex 13, Chap. VI,

$$e\left\{1 - \frac{x}{2} + \frac{11}{24}x^2 - \frac{7}{16}x^3 + R_4\right\},$$

and the result appears immediately without differentiation.

GEOMETRICAL ILLUSTRATIONS.

(1). If $f(a) = 0$, $\phi(a) = 0$, $f'(a) \neq 0$, $\phi'(a) \neq 0$, consider the curves representing $y = f(x)$, $y = \phi(x)$.

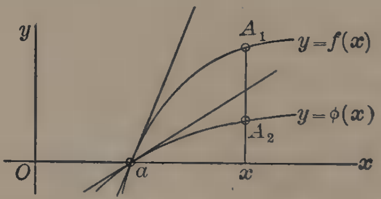

FIG. 11.

These curves cross Ox at $x = a$ at angles whose tangents are equal to $f'(a)$, $\phi'(a)$, or

$$f'(a) = \tan\theta_1, \quad \phi'(a) = \tan\theta_2.$$

$$\pounds \frac{f(x)}{\phi(x)} = \pounds\left(\frac{xA_1}{x-a} \Big/ \frac{xA_2}{x-a}\right) = \frac{\tan\theta_1}{\tan\theta_2}.$$

The limit of the quotient $f(x)/\phi(x)$ is represented by the quotient of the slopes of these curves at their common point of intersection with Ox.

(2). Consider the functions x and y in

$$(x^2 + y^2)^2 = a^2(x^2 - y^2). \tag{i}$$

Differentiate with respect to x and solve for Dy.

$$\therefore \quad - Dy = \frac{2x^3 + 2xy^2 - a^2x}{2x^2y + 2y^3 + a^2y}.$$

Dy takes the form 0/0 when $x = 0$, for then also $y = 0$ by (i). To evaluate this, differentiate the numerator and denominator with respect to x.

$$\therefore \quad - \pounds\, Dy = \pounds \frac{6x^2 + 2y^2 + 4xy\, Dy - a^2}{4xy + (2x^2 + 6y^2 + a^2)Dy},$$

$$= \frac{-a^2}{a^2 \pounds Dy}.$$

$$\therefore \quad (\pounds Dy)^2 = 1, \quad \text{or} \quad \pounds Dy = \pm 1.$$

This means that the curve whose equation is (i) in Cartesian coordinates has two branches passing through the origin $x = 0$, $y = 0$, which is a singular point. There the slopes of the two branches to Ox are $+ 1$ and $- 1$. The curve is the *lemniscate.*

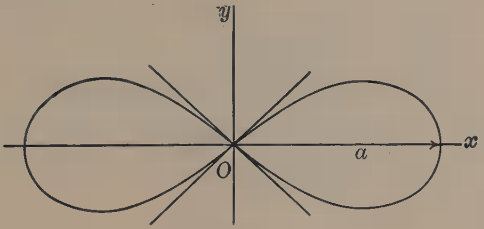

FIG. 12.

We can find Dy at $x = 0$, $y = 0$ for the curve (i), without indetermination by differentiating the equation (i) twice with respect to x. Thus

$$(2a^2 - 12x^2 - 4y^2) = 16xy\, Dy + (4x^2 + 12y^2 + 2a^2)(Dy)^2 + (4x^2y + 4y^3 + 2a^2y)D^2y,$$

which gives, as before, $Dy = \pm 1$, when $x = 0$, $y = 0$.

(3). We know from trigonometry, that the radius ρ of the circle circumscribing a triangle ABC with sides a, b, c having area S, is

$$\rho = \frac{abc}{4S}.$$

Also, from Analytical Geometry, we have

$$2S = \begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix},$$

where x, y; x_1, y_1; x_2, y_2, are the coordinates of the corners of ABC. Show that if A, B, C are three points on a curve $y = f(x)$, then the radius of the circle through these three points, when $x_1(=)x$, $x_2(=)x$, is

$$\rho = \frac{[1 + (Dy)^2]^{\frac{3}{2}}}{D^2y}.$$

We have

$$c^2 = (x_1 - x)^2 + (y_1 - y)^2,$$
$$b^2 = (x_2 - x)^2 + (y_2 - y)^2,$$
$$a^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Also,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = xy_1 - yx_1 + y_2(x_1 - x) - x_2(y_1 - y).$$

Substitute these values in the expression for ρ. Observe, when $x_1(=)x$, ρ is of the form 0/0. Divide the numerator and denominator by $x_1 - x$ and let $x_1(=)x$.

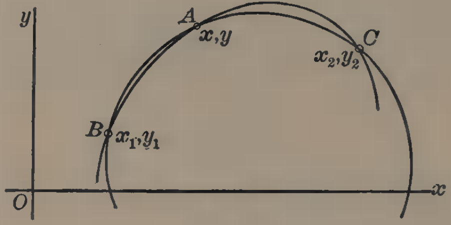

FIG. 13.

To evaluate

$$\frac{xy_1 - yx_1}{x_1 - x},$$

for $x_1(=)x$, differentiate the numerator and denominator with respect to x_1 and then let $x_1(=)x$.

$$\therefore \quad \underset{x_1(=)x}{\pounds} \frac{xy_1 - yx_1}{x_1 - x} = \pounds(x\,Dy_1 - y) = x\,Dy - y.$$

Therefore, when $B(=)A$,

$$\rho = \frac{1}{2} \frac{(x_2 - x)^2}{y_2 - y - (x_2 - x)Dy} \left\{ 1 + \left(\frac{y_2 - y}{x_2 - x}\right)^2 \right\} [1 + (Dy)^2]^{\frac{1}{2}}$$

The first factor takes the form 0/0 when $x_2 = x$. To evaluate it, differentiate the numerator and denominator with respect to x_2, and we have

$$\frac{1}{2} \frac{2(x_2 - x)}{Dy_2 - Dy};$$

this is again 0/0 when $x_2 = x$. To find its limit when $x_2(=)x$, differentiate the numerator and denominator with respect to x_2, and there results

$$\frac{1}{D^2 y_2},$$

which has the limit $1/D^2 y$ when $x_2(=)x$.

Therefore when the points B and C converge to A along the curve, the circle ABC converges to a fixed circle passing through A which has the radius

$$R = \frac{\{1 + [f'(x)]^2\}^{\frac{3}{2}}}{f''(x)} = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

This circle is called the circle of curvature of the curve $y = f(x)$ at the point x, y, and R is called the radius of curvature. Observe that when $x_1(=)x$ and $x_2 \neq x$, the circle and curve have a common tangent at A, or, as we say, are tangent at A. When this is the case the curve and circle both lie on the same side of the tangent at A. Also the circle lies on the same side of the curve in the neighborhood of A. But when also $x_2(=)x$ the circle crosses over the curve at A. The circle of curvature is said to cut a curve in three coincident points at the point of contact, in the same sense that a tangent straight line to a curve is said to cut the curve in two coincident points at the point of contact. Remembering that all points in the same neighborhood are consecutive, the above statement has definite meaning.

Much shorter ways of finding the expression for the radius of curvature will be given hereafter, but none more instructive.

EXERCISES.

1. Evaluate, when $x(=)0$,

$$\mathcal{L}\frac{e^x - 2\cos x + e^{-x}}{x \sin x} = 2; \qquad \mathcal{L}\frac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x} = 4.$$

2. Also, for the same limit of x,

$$\mathcal{L}\frac{\sin 4x \cot x}{\operatorname{vers} 2x \cot^2 2x} = 8; \qquad \mathcal{L}\frac{\sin \frac{1}{2}x \cos 2x}{\operatorname{vers} x \cot x} = 1.$$

3. Show, when $x(=)0$,

$$\mathcal{L}\frac{m \sin x - \sin mx}{x(\cos x - \cos mx)} = \frac{m}{3}; \qquad \underset{x(=)1}{\mathcal{L}}\frac{\tan nx - n \tan x}{n \sin x - \sin nx} = 2.$$

4. If $x(=)0$, then

$$\mathcal{L}\frac{(x-2)e^x + x + 2}{(e^x - 1)^3} = \frac{1}{6}; \qquad \underset{x(=)1}{\mathcal{L}}(1-x)\tan\frac{\pi x}{2} = \frac{2}{\pi}.$$

5. Evaluate for $x = \infty$,

$$\left(\cos\frac{a}{x}\right)^x, \quad \left(\cos\frac{a}{x}\right)^{x^2}, \quad \left(\cos\frac{a}{x}\right)^{x^3}, \quad \left(\cos\frac{a}{x}\right)^{\log x}.$$

6. Find the limits, when $x(=)0$, of

$$\left(\frac{1}{x}\right)^{\tan x}, \quad \left(\frac{1}{x}\right)^{\sin x}, \quad (\sin x)^{\sin x}, \quad (\sin x)^{\tan x}.$$

7. Find the radius of curvature of the parabola $y^2 = 4px$ at any point x, y, and show that at the origin it is equal to $2p$.

8. Evaluate

$$\underset{\theta(=)a}{\mathcal{L}}\frac{(a^2-\theta^2)^{\frac{1}{2}} + (a-\theta)^{\frac{3}{2}}}{(a^3-\theta^3)^{\frac{1}{2}} + (a-\theta)^{\frac{1}{2}}} = \frac{\sqrt{2a}}{1 + a\sqrt{3}}.$$

9. $\dfrac{a^{\sin x} - a}{\log \sin x}(=)a \log a$, when $x(=)\frac{1}{2}\pi$.

10. $\underset{x(=)0}{\mathcal{L}}\dfrac{e^x - 4 + e^{-x} + 2\cos x}{x^4} = \dfrac{1}{6}$.

11. $\underset{x(=)0}{\mathcal{L}}\, xe^{\frac{1}{x}} = \infty$.

12. $\underset{x(=)0}{\mathcal{L}}\dfrac{(1+x)^{\frac{1}{x}} - e + \frac{1}{2}ex}{x^2} = \dfrac{11}{24}e.$

13. $$\underset{x(=)\infty}{\mathcal{L}} \left\{ x^2\left(1+\frac{1}{x}\right)^x - ex^3 \log\left(1+\frac{1}{x}\right)\right\} = \frac{1}{8}e.$$

14. $$\underset{x=\infty}{\mathcal{L}} \frac{d}{dx}\left(\frac{ax^2+bx+c}{px+q}\right) = \frac{a}{p}.$$

15. Evaluate, when $x = \infty$,

$$\sqrt{x+a}-\sqrt{x+b}, \quad \sqrt{x^2+ax}-x, \quad a^x \sin(c/a^x).$$

16. Find where the quadratrix

$$y = x \cot \frac{\pi x}{2a}$$

crosses the y axis.

17. Show that $$\underset{x(=)0}{\mathcal{L}} \left(\frac{1}{2x^2} - \frac{1}{2x \tan x}\right) = \frac{1}{6}.$$

CHAPTER VIII.

ON MAXIMUM AND MINIMUM.

80. Definition.—A function $f(x)$ is said to have a *maximum* value at $x = a$ when the value of the function, $f(a)$, at a is *greater* than the values of the function corresponding to all other values of x in the neighborhood of a.

The function is said to have a *minimum* value at a when $f(a)$ is *less* than $f(x)$ for all values of x in the neighborhood of a.

In symbols, $f(x)$ is a *maximum* or a *minimum* at a according as

$$f(x) - f(a)$$

is *negative* or *positive*, respectively, for all values of $x \neq a$ in $(a - \epsilon, a + \epsilon)$ the neighborhood of a.

81. Theorem—At a value a of the variable for which the function $f(x)$ is differentiable and has a maximum or a minimum, the derivative $f'(a)$ is 0.

At a value a at which $f(x)$ is a maximum or a minimum, by definition the differences

$$f(x') - f(a) \quad \text{and} \quad f(x'') - f(a),$$

where $x' < a < x''$, have the same sign.

Consequently the difference-quotients

$$q' = \frac{f(x') - f(a)}{x' - a} \quad \text{and} \quad q'' = \frac{f(x'') - f(a)}{x'' - a}$$

have opposite signs for all values of x' and x'' in the neighborhood of a, since $x' - a$ is negative and $x'' - a$ is positive. Therefore, since q' and q'' have the common limit $f'(a)$ when $x'(=)a$ and $x''(=)a$, we have

$$\begin{aligned} |\pounds\,(q' \sim q'')| &= \pounds\,(|q'| + |q''|), \\ &= 2|f'(a)| = 0. \end{aligned}$$

Hence $$f'(a) = 0.$$

Notice that at a maximum value of the function the derivative is 0, and since, by definition, the function must increase up to its maximum value and then decrease as x increases through the neighborhood of a, the derivative on the inferior side of a is positive and on the superior side is negative, § 60.

Hence, at a maximum, a, the derivative, $f'(a)$, is 0 and $f'(x)$ changes from *positive* to *negative* as x increases through a.

In like manner, at a minimum, $x = a$, the derivative, $f'(a)$, is 0, and $f'(x)$ changes from *negative* to *positive* as x increases through a.

Conversely, whenever these conditions hold, then the function has a maximum or a minimum value at a, accordingly.

For example:

1. Let $f(x) \equiv x^2 - 2x + 3$.
$\therefore \quad f'(x) = 2(x - 1)$.
We have $f'(1) = 0$. Also for $x < 1$, we have $f'(x)$ negative, and for $x > 1$, $f'(x)$ positive.

Hence $f(1) = 2$ is a minimum value of $f(x)$.

2. Let $f(x) = -2x^2 + 8x - 9$.
$\therefore \quad f'(x) = 4(2 - x)$.
We have $f'(2) = 0$, $f'(2 - \epsilon) = +$, $f'(2 + \epsilon) = -$.
$\therefore \quad f(2) = -1$ is a maximum.

82. The condition $f'(a) = 0$ is necessary, but it is not sufficient, in order that the function $f(x)$ shall have a maximum or a minimum value at a. For the derivative $f'(x)$ may not change sign as x increases through a. It may continue positive, in which case $f(x)$ continues to increase as x increases through a; or $f'(x)$ may be negative throughout the neighborhood of a, in which case the function continually diminishes as x increases through a. These conditions can be illustrated geometrically thus:

GEOMETRICAL ILLUSTRATION.

Represent $y = f(x)$ by the curve $ABCDE$. Then $f'(x)$ is represented by the slope of the tangent to the curve to the x-axis. At a maximum or a minimum, $f'(x) = 0$ or the tangent to the curve is parallel to Ox. In the neighborhood of a maximum point, such as A or C, the curve lies below the tangent, and the ordinate there is greater than any other ordinate in its neighborhood. In like manner at a minimum point, such as B or D, the points B, D are the lowest points in their respective neighborhoods. At a point E the tangent is parallel to Ox, and $f'(x) = 0$, but the curve crosses over the tangent and is an increasing function at E, also the derivative $f'(x)$ is positive for all values of x in the neighborhood.

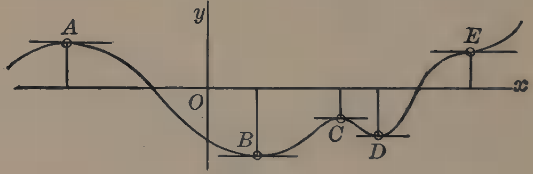

FIG. 14.

It will frequently be impracticable to examine the signs of the derivative in the neighborhood of a value of x at which $f'(x) = 0$. A more general and satisfactory investigation is required to discriminate as to maximum and minimum at such a point.

83. Study of a Function at a Value of the Variable at which the First n Derivatives are Zero.

(1). Let $f(x)$ be a function such that $f'(a) \neq 0$. Then by the law of the mean, §§ 62, 64,

$$f(x) - f(a) = (x - a)f'(\xi).$$

By hypothesis, $f'(a) \neq 0$ is the limit of $f'(x)$ and of $f'(\xi)$ as $x(=)a$, since ξ lies between x and a. Consequently we can always take x so near a that throughout the neighborhood of a we have $f'(\xi)$ of the same sign as $f'(a)$ for all values of x in that neighborhood. Hence, as x increases through the neighborhood of a, the difference $f(x) - f(a)$ changes sign with $x - a$; and by definition $f(x)$ is an increasing or decreasing function at a according as $f'(a)$ is positive or negative respectively.

(2). Let $f'(a) = 0$ and $f''(a) \neq 0$. Then

$$f(x) - f(a) = \frac{(x - a)^2}{2!} f''(\xi).$$

Throughout the neighborhood of a, $f''(\xi)$ has the same sign as its limit $f''(a) \neq 0$, and therefore does not change its sign as x increases through a. But, as $(x - a)^2$ also does not change sign as x passes through a, we have the difference

$$f(x) - f(a),$$

retaining the same sign for all values of x in the neighborhood of a, and having the same sign as $f''(a)$. Consequently, by definition, the function $f(x)$ has a *maximum* or a *minimum* value $f(a)$ at a according as $f''(a)$ is *negative* or *positive* respectively.

(3). Let $f'(a) = 0$, $f''(a) = 0$, $f'''(a) \neq 0$. Then

$$f(x) - f(a) = \frac{(x - a)^3}{3!} f'''(\xi).$$

As before, in the neighborhood of a, $f'''(\xi)$ has the same sign as its limit $f'''(a) \neq 0$. But $(x - a)^3$ changes its sign from $-$ to $+$ as x increases through a. Therefore the difference

$$f(x) - f(a)$$

must change sign as x increases through a, and $f(x)$ is an *increasing* or *decreasing* function at a according as $f'''(a)$ is *positive* or *negative.*

(4). Let $f'(a) = f''(a) = \dots = f^n(a) = 0$, but $f^{n+1}(a) \neq 0$. Then, by the law of the mean,

$$f(x) - f(a) = \frac{(x - a)^{n+1}}{(n + 1)!} f^{n+1}(\xi).$$

In the neighborhood of a, $f^{n+1}(\xi)$ has the same sign as $f^{n+1}(a)$. If $n + 1$ is *odd*, then $(x - a)^{n+1}$ and therefore $f(x) - f(a)$ change sign as x increases through a; and $f(x)$ is an *increasing* or *decreasing* function at a according as $f^{n+1}(a)$ is *positive* or *negative*. If, however, $n + 1$ is *even*, then $(x - a)^{n+1}$ does not change sign, nor does the difference $f(x) - f(a)$, as x increases through a; consequently $f(x)$ is a *maximum* or a *minimum* at a according as $f^{n+1}(a)$ is *negative* or *positive*. Hence the following

84. Rule for Maximum and Minimum.—To find the maxima and minima values of a given function $f(x)$, solve the equation $f'(x) = 0$. If a be a root of the equation $f'(x) = 0$, and the first derivative of $f(x)$ which does not vanish at a is of *even* order, say $f^{2n}(a) \neq 0$, then $f(a)$ is a *maximum* if $f^{2n}(a)$ is *negative*, or a *minimum* if $f^{2n}(a)$ is *positive*.

EXAMPLES.

1. Find the max. and min. values, if any exist, of

$$\phi(x) = x^3 - 9x^2 + 24x - 7.$$

We have $\phi'(x) = 3(x^2 - 6x + 8) = 3(x - 2)(x - 4)$.

$\therefore$ $\phi'(2) = 0$, $\phi'(4) = 0$.

Also, $\phi''(x) = 6(x - 3)$.

$\therefore$ $\phi''(2) = -$, $\phi''(4) = +$.

$\therefore$ $\phi(2) = +13$ is a maximum, $\phi(4) = 9$ a minimum.

2. Investigate for maxima and minima values the function

$$\phi(x) \equiv e^x + e^{-x} + 2 \cos x.$$

We have $\phi'(0) = \phi''(0) = \phi'''(0) = 0$, $\phi^{\text{IV}}(0) = 4$.

$\therefore$ $\phi(0) = 4$ is a minimum. Show that 0 is the only root of $\phi'(x)$.

3. Investigate $x^5 - 5x^4 + 5x^3 - 1$, at $x = 1$, $x = 3$.

4. Investigate $x^3 - 3x^2 + 3x + 7$, at $x = 1$.

5. Investigate for max. and min. the functions

$$x^3 - 3x^2 + 6x + 7, \qquad x^3 - 9x + 15x - 3.$$
$$3x^5 - 125x^3 + 2160x, \qquad x^3 + 3x^2 + 6x - 15.$$

6. Show that $(1 - x + x^2)/(1 + x - x^2)$ is min. at $x = \frac{1}{2}$.

7. If $xy(y - x) = 2a^3$, show that y has a minimum value when $x = a$.

8. If $3a^2y^2 + xy^3 + 4ax^3 = 0$, show that when $x = 3a/2$, then $y = -3a$ is a maximum. D^2y being then $-$

9. If $2x^5 + 3ay^4 - x^2y^3 = 0$, then $x = 5^{\frac{4}{3}}a$ makes $y = 5^{\frac{5}{3}}a$ a minimum.

85. Observations on Maximum and Minimum.

(1). We can frequently detect the max. or min. value of a function by inspection, making use of the definition that there the neighboring values are greater or less than the min. or max. value respectively.

For example, consider the function

$$ax^2 + bx + c.$$

Substitute $y - b/2a$ for x. The function becomes

$$\frac{4ac - b^2}{4a} + ay^2,$$

which is evidently a maximum when $y = 0$ and a is negative, and a minimum when $y = 0$ and a is positive.

(2). Labor is frequently saved by considering the behavior of the first derivative in the neighborhood of its roots, instead of finding the values of the higher derivatives there.

For example, see Ex. 6, § 85, and also

$$\phi(x) = (x - 4)^5(x + 2)^4.$$

Here
$$\phi'(x) = 3(3x - 2)(x - 4)^4(x + 2)^3.$$

ϕ' passes through 0, changing from + to − as x increases through -2; therefore $\phi(-2)$ is a maximum.

ϕ' passes through 0, but is always positive as x increases through 4; therefore $\phi(4)$ is an increasing value of $\phi(x)$. Also ϕ' passes through 0, changing from − to + as x increases through 2/3, and the function is a minimum there.

(3). The work of finding maximum and minimum values is frequently simplified by observing that

Any value of x which makes $f(x)$ a maximum or a minimum also makes $Cf(x)$ a maximum or a minimum when C is a positive constant, and a minimum or a maximum when C is a negative constant.

$f(x)$ and $C + f(x)$ have max. and min. values for the same values of x.

(4). If n is an integer, positive or negative, $f(x)$ and $\{f(x)\}^n$ have max. and min. values at the same values of the variable. In particular, a function is a maximum or a minimum when its reciprocal is a minimum or a maximum respectively.

(5). The maximum and minimum values of a continuous function must occur alternately.

(6). A function $f(x)$ may be continuous throughout an interval (α, β), and have a maximum or a minimum value at $x = a$ in the interval, while its derivative $f'(x)$ is ∞ at a, but continuous for all values of (x) on either side of a.

In this case, to determine the character of $f(x)$ at a, we can use (1) or (2) as a test. Otherwise we can consider the reciprocal $1/f'(x)$, which passes through 0 and must change sign as x passes through a, for a maximum or a minimum of $f(x)$ at a.

EXAMPLES.

1. Consider $\phi(x) \equiv (x - 2)^{\frac{2}{3}} + 1.$

ϕ is a one-valued and continuous function and is always positive. It clearly has a minimum at $x = 2$, where $\phi(x) = 1$. We have

$$\phi'(x) = \frac{2}{3} \frac{1}{(x - 2)^{\frac{1}{3}}},$$

and $\phi'(2) = \infty$. Also, $\phi'(2 - h)$ is negative and $\phi'(2 + h)$ is positive.

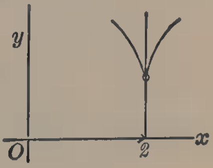

FIG. 15.

2. In like manner

$$\psi(x) = 1 - (x - 2)^{\frac{2}{3}}$$

has a maximum at $x = 2$.

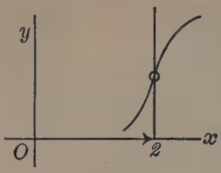

FIG. 16.

3. Consider $\phi(x) \equiv 1 + (x-2)^{\frac{3}{5}}$, which is also uniform and continuous.

We have

$$\phi'(x) = \frac{3}{5}\,\frac{1}{(x-2)^{\frac{2}{5}}},$$

which is $+\infty$ when $x = 2$, but is always $+$ in the neighborhood of 2. Therefore, at $x = 2$, $\phi(x)$ is an increasing function.

In like manner $1 - (x-2)^{\frac{3}{5}}$ is a decreasing function at $x = 2$.

(7). In problems involving more than one variable we reduce the conditions to a function of one variable by algebraic considerations. Otherwise, we can frequently make a problem involving more than one variable depend on one which can be solved by elementary considerations.

For example, the sum of several numbers is constant; show that their product is greatest when the numbers are equal.

First, take two numbers, and let

$$x + y = c.$$

Then $$4xy = (x+y)^2 - (x-y)^2 = c^2 - (x-y)^2,$$

which is evidently greatest when $x = y$.

Let $$x + y + z = c.$$

Then, as long as any two of x, y, z are unequal, we can increase the product xyz without changing the third, by the above result. Therefore xyz is greatest when $x = y = z$. The method and result is general, whatever be the number of variables.

EXERCISES.

1. Find the maximum and minimum values of y, where

$$y = (x-1)(x-2)^2.$$

2. Find the max. and min. values of

(1). $2x^3 - 15x^2 + 36x + 6$.

(2). $(x-2)(x-3)^2$.

(3). $x^3 - 3x^2 + 6x + 3$.

(4). $3x^5 - 25x^3 + 60x$.

3. Show that $(x^2+x+1)/(x^2-x+1)$ has 3^{+1} for max. and 3^{-1} for min.

4. Find the greatest and least values of

$$a \sin x + b \cos x \quad \text{and} \quad a \sin^2 x + b \cos^2 x.$$

5. Investigate $(x^2+2x-15)/(x-5)$, and also

$$\frac{x^2-7x+6}{x-10},$$

for maximum and minimum values.

6. The derivative of a certain function is

$$(x-1)(x-2)^2(x-3)^3(x-4)^4;$$

discuss the function at $x = 1, 2, 3, 4$.

7. Find the max. and min. values of

(a). $(x-1)(x-2)(x-3)$,

(b). $x^4 - 8x^3 + 22x^2 - 24x$,

(c). $(x-a)^2(x-b)$,

(d). $(x-a)^4(x-b)^3$,

(e). $x(1-x)(1-x^2)$,

(f). $(x^2-1)/(x^2+3)^3$,

(g). $\sin x \cos^3 x$,

(h). $(\log x)/x$.

8. Show that the shortest distance from a given point to a given straight line is the perpendicular distance from the point to the straight line.

9. Given two sides a and b of a triangle, construct the triangle of greatest area.

10. Construct a triangle of greatest area, given one side and the opposite angle.

11. If an *oval* is a plane closed curve such that a straight line can cut it in only two points, show that if the triangle of greatest area be inscribed in an oval, the tangents at the corners must be parallel to the opposite sides.

12. The sum of two numbers is given; when will their product be greatest? The product of two numbers is given; when will their sum be least?

13. Extend 12 by elementary reasoning to show that if

$$\sum_{1}^{n}(x_r) \equiv x_1 + \dots + x_n = c,$$

then
$$\prod_{1}^{n}(x_r) \equiv x_1 \dots x_n$$

is greatest when $x_1 = x_2 = \dots = x_n$.

14. Apply 13 to show that if. $+ y + z = c$, the maximum value of xy^2z^3 is $c^6/432$.

15. Show that if $x + y + z = c$, the maximum value of $x^l y^m z^n$ is

$$\frac{l^l m^m n^n c^{l+m+n}}{(l + m + n)^{l+m+n}}.$$

16. Find the area of the greatest rectangle that can be inscribed in the ellipse. (Use the method of Ex. 15.)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad [\text{Ans. } 2ab.]$$

17. Find the greatest value of $8xyz$, if

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \qquad \left[\text{Ans. } 8\frac{abc}{3\sqrt{3}}.\right]$$

This is the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid.

18. Show that the greatest length intercepted by two circles on a straight line passing through a point of their intersection is when the line is parallel to their line of centres.

19. From a point C distant c from the centre O of a given circle, a secant is drawn cutting the circle in A and B. Draw the secant when the area of the triangle AOB is the greatest. [With C as a centre and radius equal to the diagonal of the square on c, draw an arc cutting the parallel tangent to OC in D. Then DC is the required secant. Prove it.]

20. A piece of wire is bent into a circular arc. Find the radius when the segmental area under the arc is greatest and least. [$r = a/\pi$, $r = \infty$.]

21. Find when a straight line through a fixed point P makes with two fixed straight lines AC, AB, a triangle of minimum area. [P bisects that side.]

22. The product xy is constant; when is $x + y$ least?

23. An open tank is to be constructed with a square base and vertical sides, and is to contain a given volume; show that the expense of lining it with sheet lead will be least when the depth is one half the width.

24. Solve 23 when the base is a regular hexagon.

25. From a fixed point A on the circumference of a circle of radius a, a perpendicular AY is drawn to the tangent at a point P; show that the maximum area of the triangle APY is $3\sqrt{3}\,a^2/8$.

26. Cut four equal squares from the corners of a given rectangle so as to construct a box of greatest content.

27. Construct a cylindrical cup with least surface that will hold a given volume.

28. Construct a cylindrical cup with given surface that will hold the greatest volume.

29. Find the circular sector of given perimeter which has the greatest area.

30. Find the sphere which placed in a conical cup full of water will displace the greatest amount of liquid.

31. A rectangle is surmounted by a semicircle. Given the outside perimeter of the whole figure, construct it when the area is greatest.

32. A person in a boat 4 miles from the nearest point of the beach wishes to reach in the shortest time a place 12 miles from that point along the shore; he can ride 10 miles an hour and can sail 6 miles an hour: show that he should land at a point on the beach 9 miles from the place to be reached.

33. The length of a straight line, passing through the point a, b, included between the axes of rectangular coordinates is l. The axial intercepts of the line are α, β, and it makes the angle θ with Ox. Show that

(*a*). l is least when $\tan\theta = (b/a)^{\frac{1}{3}}$.

(*b*). $\alpha + \beta$ is least when $\tan\theta = (b/a)^{\frac{1}{2}}$.

(*c*). $\alpha\beta$ is least when $\tan\theta = b/a$.

34. Find what sector must be taken out of a given circle in order that the *remainder* may form the curved surface of a cone of maximum volume.

[Angle of sector $= 2\pi(1 - \sqrt{2/3})$.]

35. Of all right cones having the same slant height, that one has the greatest volume whose semi-vertical angle is $\tan^{-1}\sqrt{2}$.

36. The intensity of light varies inversely as the square of the distance from the source. Find the point in the line between two lights which receives the least illumination.

37. Find the point on the line of centres between two spheres from which the greatest amount of spherical surface can be seen.

38. Two points are both inside or outside a given sphere. Find the shortest route from one point to the other via the surface of the sphere.

39. Find the nearest point on the parabola $y^2 = 4px$ to a given point on the axis.

40. The sum of the perimeters of a circle and a square is l. Show that when the sum of the areas is least, the side of the square is double the radius of the circle.

41. The sum of the surfaces of a sphere and a cube is given. Show that when the sum of the volumes is least, the diameter of the sphere is equal to the edge of the cube.

42. Show that the right cone of greatest volume that can be inscribed in a given sphere is such that three times its altitude is twice the diameter of the sphere.

Also show that this is the cone of greatest convex surface that can be inscribed in the sphere.

43. Find the right cylinder of greatest volume that can be inscribed in a given right cone.

44. Show that the right cylinder of given surface and maximum volume has its height equal to the diameter of its base.

45. Show that the right cone of maximum entire surface inscribed in a sphere of radius a has for its altitude $(23 - \sqrt{17})a/16$; while that of the corresponding right cylinder is $(2 - 2/\sqrt{5})^{\frac{1}{2}}a$.

46. Show that the altitude of the cone of least volume circumscribed about a sphere of radius a is $4a$, and its volume is twice that of the sphere.

47. The altitude of the right cylinder of greatest volume inscribed in a given sphere of radius a is $2a/\sqrt{3}$.

48. The corner of a rectangle whose width is a is folded over to touch the other side. Show that the area of the triangle folded over is least when $\frac{2}{3}a$ is folded over, and the length of the crease is least when $\frac{3}{4}a$ is folded over.

49. Show that the altitude of the least isosceles triangle circumscribed about an ellipse whose axes are $2a$ and $2b$, is $3b$. The base of the triangle being parallel to the major axis.

50. Find the least length of the tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$, intercepted between the axes. [Ans. $a + b$.]

51. A right prism on a regular hexagonal base is truncated by three planes through the alternate vertices of the upper base and intersecting at a common point on the axis of the prism prolonged. The volume remains unchanged. Show that the inclination of the planes to the axis is $\sec^{-1}\sqrt{3}$ when the surface is least.

[This is the celebrated bee-cell problem.]

52. Show that the piece of square timber of greatest volume that can be cut from a sawmill log L feet long of diameters D and d at the ends has the volume

$$\frac{2}{27}\frac{LD^3}{D - d}.$$

53. A man in a boat off shore wishes to reach an inland station in the shortest time. He can row u miles per hour and walk v miles per hour. Show that he should land at a point on the straight shore at which

$$\cos\alpha : \cos\beta = u : v,$$

approaching the shore at an angle α and leaving it at an angle β.

[This is the law of refraction.]

54. From a point O outside a circle of radius r and centre C, and at a distance a from C, a secant is drawn cutting the circumference at R and R'. The line OC cuts the circle in A and B.

Show that the inscribed quadrilateral $ARR'B$ is of maximum area when the projection of RR' on AB is equal to the radius of the circle.

55. Design a sheet-steel cylindrical stand-pipe for a city water-supply which shall hold a given volume, using the least amount of metal. The uniform thickness of the metal to be a.

If H is the height and R the radius of the base, then $H = R$.

56. If a chord cuts off a maximum or minimum area from a simple closed curve when the chord passes through a fixed point, show that the point must bisect the chord.

PART II.

APPLICATIONS TO GEOMETRY.

CHAPTER IX.

TANGENT AND NORMAL.

86. The application of the Differential Calculus to geometry is limited mainly to the discussion of properties at a point on the curve. Of chief interest are the contact problems, or the relations of a proposed curve to straight lines and other curves touching the proposed curve at a point. The application of the Calculus to curves is best treated after the development of the theory for functions of two variables.

87. The Tangent (Rectangular Coordinates).—Let $y = f(x)$, or $\phi(x, y) = 0$, be the equation to any curve. The equation to the secant through the points x, y and x_1, y_1 on the curve is

$$\frac{Y - y}{X - x} = \frac{y_1 - y}{x_1 - x}, \qquad (1)$$

X, Y being the coordinates of an arbitrary point on the secant. By definition, the tangent to a curve at P is the straight line which is the limiting position of the secant PP_1 when $P_1 (=) P$. But when $P_1 (=) P$ we have $x_1 (=) x$ and $y_1 (=) y$. The member on the right of equation (1), being the difference-quotient of y with respect to x, has for its limit the derivative of y with respect to x. At the same time the arbitrary point X, Y on the secant becomes an arbitrary point on the tangent. Therefore we have for the equation to the tangent at P

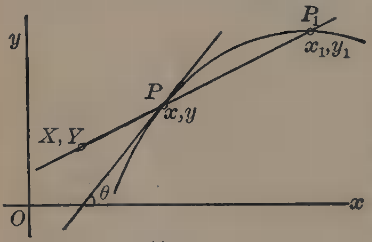

FIG. 17.

$$\frac{Y - y}{X - x} = \frac{dy}{dx} = Dy, \qquad (2)$$

in terms of the coordinates x, y of the point of contact.

The equation to the tangent (2) can be written

$$Y - y = (X - x)\frac{dy}{dx}, \tag{3}$$

or in differentials

$$(Y - y)\,dx - (X - x)dy = 0, \tag{4}$$

or in the symmetrical form

$$\frac{X - x}{dx} = \frac{Y - y}{dy}. \tag{5}$$

EXAMPLES.

1. Find the equation to the tangent to the circle $x^2 + y^2 = a^2$.
Differentiating, we have

$$2x + 2y\,Dy = 0.$$

$\therefore$ $Dy = -x/y$, and the tangent at x, y is

$$Y - y + (X - x)\frac{x}{y} = Yy + Xx - (x^2 + y^2) = 0,$$

or

$$Yy + Xx = a^2.$$

2. Find the tangent at x, y to $x^2/a^2 + y^2/b^2 = 1$.

3. Find the tangent at x, y to $x^2/a^2 - y^2/b^2 = 1$.

4. Find the tangent at x, y to $y^2 = 4px$.

5. Find the tangent at x, y to $x^2 + y^2 + 2fy + 2gx + d = 0$.

6. Show that the equation to the tangent at x, y to the conic

$$\phi(x, y) \equiv ax^2 + by^2 + 2hxy + 2fx + 2gy + d = 0$$

is

$$(ax + hy + f)X + (hx + by + g)Y + (fx + gy + d) = 0.$$

7. Show that the equation to the tangent at x, y to the curve

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$$

is

$$\frac{x^{m-1}X}{a^m} + \frac{y^{m-1}Y}{b^m} = 1.$$

8. Find the tangent at x, y to $x^5 = a^3y^2$. [$5X/x - 2Y/y = 3$.]

9. The tangent at x, y to $x^3 - 3axy + y^3 = 0$ is

$$(y^2 - ax)Y + (x^2 - ay)X = axy.$$

10. Find the equation to the tangent to the hypocycloid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

and prove that the portion of the tangent included between the axes is of constant length.

88. If the equation to a curve is given by

$$x = \phi(t), \quad y = \psi(t),$$

then, since $dx = \phi'(t)dt$, $dy = \psi'(t)dt$, we have for the equation to the tangent

$$(Y - y)\phi'(t) = (X - x)\psi'(t). \tag{1}$$

EXAMPLES.

1. If the coordinates of any point on a curve satisfy the cycloid

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta),$$

show that the tangent at x, y makes an angle $\frac{1}{2}\theta$ with Oy, and has for its equation

$$Y - y = (X - x)\cot\tfrac{1}{2}\theta.$$

2. In like manner, if

$$x = c\sin 2\theta(1 + \cos 2\theta), \quad y = c\cos 2\theta(1 - \cos 2\theta),$$

the tangent makes the angle θ with Ox, and its equation is

$$Y - y = (X - x)\tan\theta.$$

89. The angle at which two curves intersect is defined as the angle between their tangents at the point of intersection.

If $y = \phi(x)$ and $y = \psi(x)$ are two curves, and these equations be solved for x and y, we find the coordinates of the points of intersection. If the curves intersect at an angle ω, then since $\phi'(x)$ and $\psi'(x)$ are the tangents of the angles which the tangents to the curves make with Ox, we have

$$\tan\omega = \frac{\phi'_x - \psi'_x}{1 + \phi'_x\psi'_x}. \tag{1}$$

The two lines cut at right angles when $\phi'_x\psi'_x = -1$.

Ex. Show that $x^2 + y^2 = 8ax$ and $y^2(2a - x) = x^3$ cut at right angles and at 45°.

90. The Normal (Rectangular Coordinates).—The normal at a point of a curve is the straight line perpendicular to the tangent at that point.

If θ_t and θ_n are the angles which the tangent and normal at a point make with Ox respectively, then since one is always equal to the sum of $\frac{1}{2}\pi$ and the other, we have $\tan\theta_t \tan\theta_n = -1$. Therefore

$$\tan\theta_n = -\frac{dx}{dy} = -D_y x.$$

Hence the equation to the normal at x, y to a curve is

$$Y - y + (X - x)D_y x = 0, \tag{1}$$

or

$$(Y - y)D_x y + X - x = 0, \tag{2}$$

or in differentials

$$(Y - y)dy + (X - x)dx = 0, \tag{3}$$

where $D_x y$ or $D_y x$ must be found from the equation to the curve.

EXAMPLES.

1. The equation to the normal at x, y to $x^2/a^2 + y^2/b^2 = 1$ is

$$\frac{a^2X}{x} - \frac{b^2Y}{y} = a^2 - b^2.$$

2. The normal at x, y to $y^m = ax^n$ is

$$nyY + mxX = ny^2 + mx^2.$$

3. Show that the tangent and normal to the cissoid

$$y^2(2a - x) = x^3, \quad \text{at} \quad x = a, \quad \text{are,}$$

$$\text{at } (a, a), \qquad y = 2x - a, \quad 2y + x = 3a;$$

$$\text{at } (a, -a), \quad y + 2x = a, \quad 2y = x - 3a.$$

4. In the Witch of Agnesi, $y(4a^2 + x^2) = 8a^3$, the tangent and normal at $x = 2a$, are

$$x + 2y = 4a, \quad y = 2x - 3a.$$

5. Show that the maximum or minimum distance from a point to a curve is measured along the normal to the curve through the point.

Let α, β be a point in the plane of a curve $\phi(x, y) = 0$.

If δ is the distance from α, β to a point x, y on the curve, then

$$\delta^2 = (\alpha - x)^2 + (\beta - y)^2.$$

When this is a maximum or minimum,

$$d\delta^2 = -2(\alpha - x)dx - 2(\beta - y)dy = 0,$$

which is the equation (3), § 90, to the normal through α, β.

91. Subtangent and Subnormal (Rectangular Coordinates).— The portion of the tangent, PT, included between the point of contact, P, and the x-axis, is called the *tangent-length.* The portion of the normal between the point of contact and the x-axis is called the *normal-length.* The projections TM and MN of the tangent-length and normal-length on the x-axis respectively are called the subtangent and subnormal corresponding to the point P.

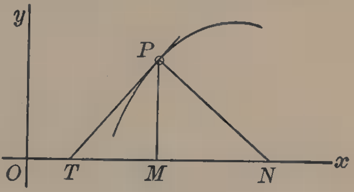

FIG. 18.

If t, n, S_t, S_n represent the tangent-length, normal-length, subtangent, and subnormal respectively, then we have directly from the figure

$$S_t = y \bigg/ \frac{dy}{dx}, \quad t = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \bigg/ \frac{dy}{dx},$$

$$S_n = y\frac{dy}{dx}, \qquad n = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

S_t is measured from T to the right or left according as S_t is $+$ or $-$, and S_n is measured from M to the right or left according as S_n is $+$ or $-$.

EXAMPLES.

1. Show that the subnormal in the ellipse $x^2/a^2 + y^2/b^2 = 1$ is

$$S_n = -b^2x/a^2.$$

2. Show that S_t in $y = a^x$ is constant.

3. In $y^2 = 2mx$, show that $S_n = m$ is constant.

4. In the catenary $y = \frac{1}{2}a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$, $n = y^2/a$.

5. Show that $\phi(x, y) = 0$ must be a straight line if S_t/S_n is constant.

6. Show, in the cissoid $x^3 = (2a - x)y^2$, that

$$S_t = (2ax - x^2)/(3a - x).$$

7. Show that the circle $x^2 + y^2 = a^2$ has n constant.

92. Tangent, Normal, Subtangent, Subnormal (Polar Coordinates).—Let $f(\rho, \theta) = 0$ be the equation to any curve in polar coordinates, ψ the angle which the tangent at any point makes with the radius vector, and ϕ the angle which the tangent makes with the initial line. From the figure we have

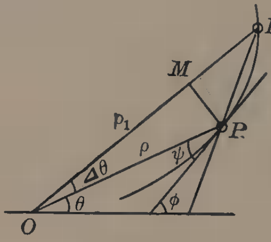

FIG. 19.

$$PM \perp OP_1 = \rho_1 = \rho + \Delta\rho,$$

$$\tan MP_1P = \frac{\rho \sin \Delta\theta}{\rho + \Delta\rho - \rho\cos\Delta\theta},$$

$$= \rho\frac{\Delta\theta}{\Delta\rho}\,\frac{\dfrac{\sin\Delta\theta}{\Delta\theta}}{1 + \rho\dfrac{\Delta\theta}{\Delta\rho}\,\dfrac{1-\cos\Delta\theta}{\Delta\theta}}.$$

When $\Delta\theta (=) 0$, we have, passing to limits,

$$\tan\psi = \rho\frac{d\theta}{d\rho}, \tag{1}$$

since $\mathcal{L}\dfrac{1-\cos x}{x} = \mathcal{L}\sin x = 0$, when $x (=) 0$.

Also, since $\phi = \theta + \psi$, we have

$$\tan\phi = \frac{\rho D_\rho\theta + \tan\theta}{1 - \rho\tan\theta D_\rho\theta},$$

$$= \frac{\rho + \tan\theta D_\theta\rho}{D_\theta\rho - \rho\tan\theta}. \tag{2}$$

Observe that (2) is the same value as that obtained for $D_x y$ in § 56.

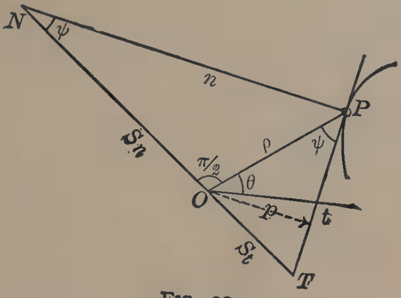

FIG. 20.

Draw a straight line through the origin perpendicular to the radius vector, cutting the tangent in T and the normal in N. We call PN and PT, the portions of the normal and tangent intercepted between the point of contact, P, and the perpendicular through the origin, O, to the radius vector, OP, the polar *normal-length* and polar *tangent-length* respectively; and their projections, ON and OT, on this perpendicular are called respectively the polar *subnormal* and *subtangent*.

We have directly from the figure

$$t = \rho \sec \psi = \rho \sqrt{1 + \rho^2 (D_\rho \theta)^2}, \tag{3}$$

$$n = \rho \csc \psi = \sqrt{\rho^2 + (D_\theta \rho)^2}. \tag{4}$$

$$S_t = \rho \tan \psi = \rho^2 D_\rho \theta, \quad S_n = \rho \cot \psi = D_\theta \rho. \tag{5}$$

When $D_\rho \theta$ is positive (negative), S_t is to be measured from O to the right (left) of an observer looking from O to P.

Putting $\rho' \equiv D_\theta \rho$, we have for the perpendicular from the origin on the tangent

$$p = \frac{\rho^2}{\sqrt{\rho^2 + \rho'^2}}, \tag{6}$$

since $pt = \rho S_t$. This can be written

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \tag{7}$$

if we put $\rho = 1/u$, for then

$$\frac{d\rho}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}.$$

EXAMPLES.

1. In the spiral of Archimedes $\rho = a\theta$, show that $\tan \psi = \theta$, and S_n is constant.

2. Show that S_t is constant in the reciprocal or hyperbolic spiral $\rho\theta = a$.

3. In the equiangular spiral $\rho = ae^{\theta \cot \alpha}$, show that $\psi = \alpha$, $S_t = \rho \tan \alpha$, $S_n = \rho \cot \alpha$.

4. If $\rho = a^\theta$, show that $\tan \psi = (\log a)^{-1}$.

5. Show that the perpendicular from the focus to the tangent in the ellipse

$$(1 - e \cos \theta)\rho = a(1 - e^2)$$

is

$$p^2 = \rho a^2 \frac{1 - e^2}{2a - \rho}.$$

6. Determine the points in the curve $\rho = a(1 + \cos \theta)$, the cardioid, at which the tangent is parallel to the initial line.

7. If $\rho = a(1 - \cos \theta)$, show that

$$\psi = \tfrac{1}{2}\theta, \quad p = 2a \sin^3 \tfrac{1}{2}\theta, \quad S_t = 2a \sin^2 \tfrac{1}{2}\theta \tan \tfrac{1}{2}\theta.$$

EXERCISES.

1. Show that in $\phi(x, y) = 0$, the intercepts of the tangent at any point x, y on the axes are

$$X_i = x - y\, D_y x, \quad Y_i = y - x\, D_x y.$$

2. The length of the perpendicular from the origin on the tangent is

$$p = \frac{x\, Dy - y}{\sqrt{1 + (Dy)^2}}.$$

3. Show that when the area of the triangle formed by the tangent to a given curve and the axes of coordinates is a maximum or a minimum, the point of contact is the middle point of the hypothenuse.

Indicate $D_x y$ by y', and the area by Ω. Then

$$2\Omega = X_i Y_i = -\frac{(y - xy')^2}{y'}.$$

Also,

$$2\frac{d\Omega}{dx} = \frac{(y - xy')(y + xy')y''}{y'^2},$$

where $y'' \equiv D_x^2 y$. For a maximum or a minimum $D\Omega = 0$. The conditions

$$y' \neq 0, \quad y'' \neq 0, \quad y - xy' \neq 0, \quad y + xy' = 0$$

show, by Ex. 1, that $X_i = 2x$, $Y_i = 2y$.

4. Find when the area of the triangle formed by the coordinate axes and the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is a minimum.

5. Show that the tangent at the point $(2, -1)$ of the curve

$$x^3 + 2x^2y - 3y^2 + 4x + y - 4 = 0$$

is

$$8x + 15y = 1.$$

6. The line $ex + y = e(1 + \pi)$ is tangent to the curve

$$\sin x - \cos x = \log y, \quad \text{at } (\pi, e).$$

7. The line $y + 1 = 0$ is tangent at $(+1, -1)$ to

$$x^4 - 2x^2y^2 - 3y^3 + 4xy + 4x + 5y + 3 = 0.$$

8. Determine the points at which the tangents to

$$x^3 + y^3 = 3x$$

are parallel to the coordinate axes. $(x = 0, y = 0)$, $(x = \pm 1, y = \pm \sqrt[3]{2})$.

9. At what point of $x^4 + 4y - 9 = 0$ is the tangent parallel to $x - y = 0$?
$(x = -1, y = 2.)$

10. The tangents from the origin to

$$x^4 - y^3 + 3x^2y + 2xy^2 = 0$$

are $\qquad y = 0, \qquad 3x - y = 0, \qquad x + y = 0.$

11. The perpendicular from the origin to the tangent at x, y of the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \quad \text{is} \quad p = \sqrt[3]{axy}.$$

12. Show that the slope of the curve $x^2y^2 = a^3(x + y)$ to the x-axis is $\frac{3}{4}\pi$ at 0, 0.

13. If x, y are rectangular coordinates and ρ, θ the polar coordinates of a point on a curve, show geometrically that when $D_xy = 0$ we have $D_\theta\rho = \rho \tan \theta$, and verify from the formulæ in the text.

14. Show that the curves

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$$

cut at right angle if $a^2 - b^2 = a'^2 - b'^2$.

15. In the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, show that at x, y the tangent is

$$Xy^{\frac{1}{2}} + Yx^{\frac{1}{2}} = (axy)^{\frac{1}{2}},$$

and that the sum of its intercepts is constant and equal to a.

16. The tangent at x, y to $(x/a)^2 + (y/b)^{\frac{2}{3}} = 1$ is

$$Xx/a^2 + (Y + 2y)/3b^{\frac{2}{3}}y^{\frac{1}{3}} = 1.$$

Also find the normal.

17. The tangent and normal to the ellipse

$$x^2 + 2y^2 - 2xy - x = 0$$

at $x = 1$ are,

at $(1, 0)$, $\qquad 2y = x - 1, \qquad y + 2x = 2;$
at $(1, 1)$, $\qquad 2y = x + 1, \qquad y + 2x = 3.$

18. In the curve $y(x - 1)(x - 2) = x - 3$, show that the tangent is parallel to the x-axis at $x = 3 \pm \sqrt{2}$.

19. In the curve $(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1$, show that (see Ex. 1.)

$$\frac{X_i^2}{a^2} + \frac{Y_i^2}{b^2} = 1.$$

20. Show that the tractrix

$$x + \sqrt{c^2 - y^2} = \frac{c}{2} \log \frac{c - \sqrt{c^2 - y^2}}{c + \sqrt{c^2 - y^2}}$$

has a constant tangent-length.

21. In the curve $y^n = a^{n-1}x$, find the equation to the tangent; and determine the value of n when the area included between the tangent and axes is constant.

22. In $\rho(ae^\theta + be^{-\theta}) = ab$, show that

$$S_t = -ab/(ae^\theta - be^{-\theta}).$$

23. If $\rho^2 \cos 2\theta = a^2$, show that $\sin \psi = a^2/\rho^2$.

24. If two points be taken, one on the curve and one on the tangent, the points being equidistant from the point of contact, show that the normal to the curve is the limit of the straight line passing through the two points as they converge to the point of contact.

25. If Q, P, R are three points on a curve, P the mid-point of the arc QR, and V the middle point of the chord QR, show that the normal at P is the limit of the line PV as $Q(=)P$, $R(=)P$.

26. Prove that the limit of any secant line through any two points R, Q on a curve is the tangent at a point P as $R(=)P$, $Q(=)P$.

27. Show that as a variable normal converges to a fixed normal, their intersection converges, in general, to a definite point, and find its coordinates.

Let $$(Y-y)y'+X-x=0$$
and $$(Y-y_1)y_1'+X-x_1=0,$$
where y', y_1' represent D_xy at x, y and x_1, y_1, be the equations of a fixed normal at x, y and a variable normal at x_1, y_1. Eliminating X, we have

$$\begin{aligned} Y(y_1'-y') &= y_1y_1'-yy'+x_1-x, \\ &= y_1(y_1'-y')+y'(y_1-y)+(x_1-x). \end{aligned}$$

$$\therefore\quad Y = y_1+\frac{1+\dfrac{y_1-y}{x_1-x}y'}{\dfrac{y_1'-y'}{x_1-x}},$$

$$= y+\frac{1+y'^2}{y''},\quad \text{when}\quad x_1(=)x.$$

Also,

$$X = x-y'\,\frac{1+y'^2}{y''},\quad \text{where}\quad y''\equiv\frac{d^2y}{dx^2}.$$

This point is the center of curvature of the curve for x, y.

CHAPTER X.

RECTILINEAR ASYMPTOTES.

93. Definition.—An *asymptote* to a curve is the limiting position of the tangent as the point of contact moves off to an infinite distance from the origin.

Or, an asymptote is the limiting position of a secant which cuts the curve in two infinitely distant points on an infinitely extended branch of the curve.

94. We have the following methods of determining the asymptotes to a curve $f(x, y) = 0$:

I. The equations to the tangent at x, y and its axial intercepts are

$$Y - y = (X - x)\frac{dy}{dx},$$

$$Y_i = y - x\frac{dy}{dx},$$

$$X_i = x - y\frac{dx}{dy}.$$

If we determine, for $x = y = \infty$,

$$\pounds X_i = a, \qquad \pounds Y_i = b,$$

then the equation to the asymptote is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Or, if we determine, for $x = y = \infty$,

$$\pounds \frac{dy}{dx} = m,$$

and either a or b as above, we have for the asymptote

$$y = mx + b \quad \text{or} \quad x = \frac{y}{m} + a.$$

This method involves the evaluation of indeterminate forms, which must be evaluated either by purely algebraic principles or by aid of the method of the Calculus prescribed for such forms. The algebraic evaluations are of more or less difficulty, and another method will be given in III for algebraic curves.

EXAMPLES.

1. Find the asymptotes to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

We have $X_i = a^2/x$, $Y_i = -b^2/y$. These are 0 when $x = y = \infty$. Therefore the asymptotes pass through the origin. Also,

$$\frac{dy}{dx} = \frac{b^2x}{a^2y} = \pm\frac{b}{a}\frac{1}{\sqrt{1 - a^2/x^2}},$$

the limits of which are $\pm b/a$ when $x = \infty$. The equations to the two asymptotes are $ay = \pm bx$.

2. Find the asymptotes to the curves

(a). $y = \log x$. $x = 0$.
(b). $y = e^x$. (Fig. 33.) $y = 0$.
(c). $y = e^{-x^2}$. (Fig. 34.) $y = 0$.
(d). $y^2e^{2x} = x^2 - 1$. $y = 0$.
(e). $1 + y = e^{\frac{1}{x}}$. $x = 0$, $y = 0$.
(f). $y = \tan ax$, $y = \cot ax$, $y = \sec ax$.

3. Show that $y = x$ is an asymptote of $x^3 = (x^2 + 3a^2)y$.

4. $x + y = 2$ is an asymptote of $y^3 = 6x^2 - x^3$.

5. $x = 2a$ is an asymptote of $x^3 = (2a - x)y^2$.

6. $x^3 + y^3 = a^3$ has $y + x = 0$ for an asymptote.

7. The asymptotes of $(x - 2a)y^2 = x^3 - a^3$ are

$$x = 2a \quad \text{and} \quad a + x = \pm y.$$

$x = 2a$ is readily seen to be an asymptote. For the others express Dy in terms of x and make $x = \infty$; the result is ± 1. Find the intercept in same way.

8. Find the asymptote of the Folium of Descartes

$$x^3 + y^3 = 3axy.$$

See Fig. 49. The asymptote is $x + y + a = 0$. Put $y = mx$ in the equation to determine slope and intercept.

II. We can sometimes find the asymptotes to curves by expansion in a series of powers. Thus, if

$$y = a_0x + a_1 + \frac{a_2}{x} + \frac{a_3}{x^2} + \ldots,$$

then $y = a_0x + a_1$ is an asymptote. For, evaluating as in I, we have $m = a_0$, $Y_i = a_1$.

Observe also, if we have

$$y = \phi(x) + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots,$$

then when $x = \infty$ the difference between the ordinate to this curve and that of the curve $y = \phi(x)$ continually decreases as x increases. We say the two curves are asymptotic to each other.

EXAMPLES.

9. In Ex. 1, I, we have

$$y = \pm \frac{b}{a}x\left(1 - \frac{a^2}{x^2}\right)^{\frac{1}{2}} = \pm \frac{b}{a}x\left(1 - \frac{1}{2}\frac{a^2}{x^2} + \ldots\right).$$

As x increases indefinitely, the point x, y converges to the straight-line asymptote $ay = \pm bx$.

10. Solve Ex. 3, I, by expansion.

11. Solve Ex. 6, I, by expansion. Here we find that the given curve and the hyperbola

$$y^2 = x^2 + 2ax + 4a^2$$

have the same asymptotes.

III. We pass now to the most convenient method of determining the asymptotes to algebraic curves.

If the given curve is a polynomial, $f(x, y) = 0$, in x and y, or can be reduced to that form, we can always find its asymptotes as follows:

Rule 1. Equate to 0 the coefficients of the two highest powers of x in

$$f(x, mx + b) = 0.$$

These two equations solved for m and b furnish the asymptotes *oblique* to the axes.

Rule 2. Equate to 0 the coefficients of the highest powers of x and of y in $f(x, y) = 0$. The first furnishes all the asymptotes *parallel* to the x-axis, the second those *parallel* to the y-axis.

Proof: (A). The straight line

$$y = mx + b \tag{1}$$

cuts the curve

$$f(x, y) = 0 \tag{2}$$

in points whose abscissæ are the values of x obtained from the solution of the equation in x,

$$f(x, mx + b) = 0. \tag{3}$$

If (2) is of the nth degree in x and y, then (3) is of the nth degree in x, and will furnish, in general, n values of x (real or imaginary).

Let (3), when arranged according to powers of x, be

$$A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0 = 0. \tag{3}$$

If one of the points of section of (1) and (2) moves off to an infinite distance from the origin, then one root of (3) is infinite, and the coefficient, A_n, of the highest power of x must be 0, or $A_n = 0$.

This is readily seen to be true by substituting $1/z$ for x in (3), and arranging according to powers of z. Then when $z = 0$, we have $x = \infty$, and $A_n = 0$.

In like manner if a second point of intersection of (1) and (2) moves off to an infinite distance on the curve, a second root of (3) is infinite and we must have the coefficient of x^{n-1} equal to 0, or $A_{n-1} = 0$.

When (1) and (2) intersect in two infinitely distant points, then (1) is an asymptote of (2), and we have for determining the asymptotes the two equations

$$A_n = 0, \quad A_{n-1} = 0.$$

These two equations when solved for m and b give the slopes and intercepts on the y-axis of the oblique asymptotes of (2).

EXAMPLES.

12. Consider $\quad x^3 = (x^2 + 3a^2)y, \quad$ see Ex. 3.

Here
$$x^3 - x^2y - 3a^2y = 0$$
becomes
$$(1 - m)x^3 - bx^2 - 3a^2mx - 3a^2b = 0,$$
when $mx + b$ is substituted for y. Hence
$$1 - m = 0 \quad \text{and} \quad -b = 0$$
give $y = x$ as the oblique asymptote.

13. In $\quad x^3 + y^3 = 3axy$, Ex. 8, put $\quad y = mx + b$.
$$\therefore \quad (1 + m^3)x^3 + 3m(mb - a)x^2 + \ldots = 0,$$
it being unnecessary to write the other terms.

Hence $\quad m = -1, \quad b = -a$. Therefore the oblique asymptote is
$$y = -x - a.$$

14. Show that $\quad y = x + \frac{1}{3}a \quad$ is an asymptote of $\quad y^3 = ax^2 + x^3$.

15. The asymptotes of $\quad y^4 - x^4 + 2ax^2y = b^2x$

are
$$y = x - \tfrac{1}{2}a \quad \text{and} \quad y + x + \tfrac{1}{2}a = 0.$$

16. $x^3 + 3x^2y - xy^2 - 3y^3 + x^2 - 2xy + 3y^2 + 4x + 5 = 0$

has for asymptotes
$$y + \tfrac{1}{3}x + \tfrac{3}{4} = 0, \quad y = x + \tfrac{1}{4}, \quad y + x = \tfrac{3}{2}.$$

(B). If the term $A_{n-1}x^{n-1}$ is missing in (3), or if the value of m obtained from $A_n = 0$ makes A_{n-1} vanish, then (3) has three infinite roots when

$$A_n = 0 \quad \text{and} \quad A_{n-2} = 0,$$

which equations give the values of m and b which furnish the asymptotes. A_{n-2} will be of the second degree in b, furnishing two b's for each m, and there will be for each m two parallel oblique asymptotes, which we say meet the curve in three points at ∞.

If also the term $A_{n-2}x^{n-2}$ is missing, or if A_{n-2} vanishes for the value of m obtained from $A_n = 0$, then the equations

$$A_n = 0, \quad A_{n-3} = 0$$

furnish three parallel oblique asymptotes, in general, for each m.

EXAMPLES.

17. If $\quad (x + y)^2(x^2 + y^2 + xy) = a^2y^2 + a^3(x - y)$,

then
$$A_{n-1} = (1 + m)^2(1 + m + m^2),$$
$$A_{n-1} = 0,$$
$$A_{n-2} = b^2 - a^2.$$
$$\therefore \quad m = -1, \quad b = \pm a \quad \text{give asymptotes} \quad y = -x \pm a.$$

18. In $x^2(y + x)^2 + 2ay^2(x + y) + 8a^2xy + a^3y = 0$,
the asymptotes are $y + x = 2a, \quad y + x + 4a = 0.$

19. Find the asymptotes to the curves:

(*a*). $xy^2 - x^2y = a^2(x + y) + b^3.$ $\quad x = 0, \quad y = 0, \quad x = y.$
(*b*). $y^3 - x^3 = a^2x.$ $\quad y = x.$
(*c*). $x^4 - y^4 = a^2xy + b^2y^2.$ $\quad x + y = 0, \quad x = y.$

(C). For the asymptotes parallel to the coordinate axes, the following simple process determines them:

Arrange $f(x, y) = 0$ according to powers of y, thus:

$$Ay^n + (Bx + C)y^{n-1} + (Fx^2 + Gx + H)y^{n-2} + \ldots = 0. \qquad (4)$$

If the highest power of y is n, the degree of the curve, there will be no asymptote parallel to Oy, since then $A \neq 0$. If, however, the term Ay^n is missing, or $A = 0$, then for any assigned x one root in the equation (4) in y will be ∞. If, now, $Bx + C = 0$, a second root of (4), in y, is ∞ at $x = -C/B$, and this will be an asymptote to the curve, since D_xy is ∞ for the same value of x which makes $y = \infty$.

If the terms involving the two highest powers of y in (4) are missing, then

$$Fx^2 + Gx + H = 0$$

makes three roots of (4), in y, infinite, and this is the equation to two asymptotes parallel to Oy, and so on generally.

In like manner, arranging $f(x, y)$ according to powers of x, we find the asymptotes parallel to Ox by equating to 0 the coefficient of the highest power of x.

Therefore the coefficients of the highest powers of x and y in the equation to the curve, equated to zero, give all the asymptotes parallel to the axes. Of course, if these coefficients do not involve x or y they cannot be 0 and there are no asymptotes parallel to the axes.

EXAMPLES.

20. Find the asymptotes to the following curves:

(*a*). $y^2x - ay^2 = x^3 + ax^2 + b^3.$ $\quad x = a, \quad y = x + a, \quad y + x + a = 0.$
(*b*). $y(x^2 - 3bx + 2b^2) = x^3 - 3ax^2 + a^3.$ $\quad x = b, \quad x = 2b, \quad y + 3a = x + 3b.$
(*c*). $x^2y^2 = a^2(x^2 + y^2).$ $\quad x = \pm a, \quad y = \pm a.$
(*d*). $x^2y^2 = a^2(x^2 - y^2).$ $\quad y + a = 0, \quad y - a = 0.$
(*e*) $y^2a = y^2x + x^3.$ $\quad x = a.$
(*f*). $(x^2 - y^2)^2 - 4y^2 + y = 0.$
(*g*). $x^2(x - y)^2 - a^2(x^2 + y^2) = 0.$
(*h*). $x^2(x^2 + a^2) = (a^2 - x^2)y^2.$
(*i*). $x^2y^2 = x^3 + x + y.$
(*j*). $x^2y^2 = (a + y)^2(b^2 - y^2).$
(*k*). $y(x - y)^3 = y(x - y) + 2.$

95. Asymptotes to Polar Curves.—If $f(\rho, \theta) = 0$ is the equation to a curve in polar coordinates, then, when it has an asymptote, that asymptote must be parallel to the radius vector to the point at ∞ on the curve, if the asymptote passes within a finite distance of the origin.

The distance of the asymptote from the origin is the limiting value of the polar subtangent when the point of contact is infinitely distant.

To determine the polar asymptotes to $f(\rho, \theta) = 0$, determine the values of θ which make $\rho = \infty$. These values of θ give the directions of the asymptotes.

If the equation can be written as a polynomial in ρ, the values of θ are furnished by equating to 0 the coefficient of the highest power of ρ.

To construct the asymptote when, $\theta = \alpha$, the direction has been found; evaluate for $\theta (=) \alpha$ and $\rho = \infty$ the subtangent

$$p = \mathcal{L}\, S_t = \mathcal{L}\, \rho^2 \frac{d\theta}{d\rho} = - \mathop{\mathcal{L}}_{u(=)0} \frac{d\theta}{du},$$

where $\rho u = 1$. The perpendicular on the asymptote is to be laid off from the origin to the right or left of an observer at the origin looking toward the point of contact, according as p is + or − respectively.

EXAMPLES.

21. Let $\rho = a \sec \theta + b \tan \theta$.

$$\rho = \infty \quad \text{when} \quad \theta = \tfrac{1}{2}\pi; \quad \text{also,}$$

$$\rho^2 \frac{d\theta}{d\rho} = \frac{(a + b \sin \theta)^2}{a \sin \theta + b},$$

the limit of which is $a + b$. The asymptote is then perpendicular to the initial line at a distance $a + b$ to the right of O. Also, when $\theta = \frac{3}{2}\pi$, $\rho = \infty$, and the corresponding value of the subtangent gives $a - b$ and another asymptote.

22. Show that $\rho^2 \sin (\theta - \alpha) + a\rho \sin (\theta - 2\alpha) + a^2 = 0$ has the asymptotes $\rho \sin (\theta - \alpha) = \pm a \sin \alpha$.

23. Find the asymptotes of $\rho \sin \theta = a\theta$.

24. Find the straight asymptotes of $\rho \sin 4\theta = a \sin 3\theta$.

25. Show that $\rho \cos \theta = - a$ is an asymptote of $\rho \cos \theta = a \cos 2\theta$.

26. $b = (\rho - a) \sin \theta$ has $\rho \sin \theta = b$ for asymptote.

27. Determine the asymptotes of $\rho \cos 2\theta = a$.

Polar curves may have asymptotic circles or asymptotic points.

EXAMPLES.

28. Find the asymptotes of $\rho\theta = a$, for $\theta = 0$, $\theta = \infty$. Fig. 57.

29. Find the circular asymptotes of $\rho(\theta + a) = b\theta$, and of

$$\rho = \frac{a\theta^2}{\theta^2 \pm b^2}; \quad \rho = \frac{a\theta^2}{\theta + \sin \theta}; \quad \rho = \frac{\theta + \cos \theta}{\theta + \sin \theta}.$$

CHAPTER XI.

CONCAVITY, CONVEXITY, AND INFLEXION.

96. On the Contact of a Curve and a Straight Line.—Let $y = f(x)$ be the equation to a curve. The equation to the tangent, § 87, at $x = a$ (Y being the ordinate corresponding to x) is

$$Y = f(a) + (x - a)f'(a).$$

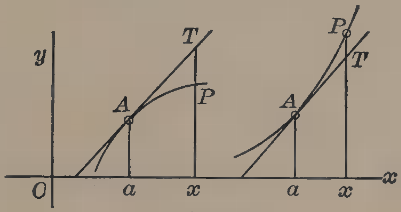

FIG. 21.

The difference between the ordinates of the curve and tangent at any point (by the theorem of mean value) is [illegible]

$$f(x) - Y = \tfrac{1}{2}(x - a)^2 f''(\xi).$$

If $f''(a) \neq 0$, this difference will retain its sign unchanged for all values of x in the neighborhood of a. Therefore throughout this neighborhood the curve will lie wholly on one side of the tangent. It will lie below the tangent when $f''(a)$ is $-$, and above it when $f''(a)$ is $+$.

The curve $y = f(x)$ is said to be *concave* at a when $f''(a)$ is *negative*, or the curve lies below the tangent there; and is said to be *convex* at a when $f''(a)$ is *positive*, or the curve lies above the tangent there.

EXAMPLES.

1. The curve $y = e^x$ is always convex, since $D^2 e^x = e^x$ is always positive.

2. The curve $y = \log x$ is always concave, since $D^2 \log y = -x^{-2}$ is always negative.

3. The curve $y = x^3 + ax$ is convex when x is positive and concave when x is negative, since $D^2 y = 6x$.

POINTS OF INFLEXION.

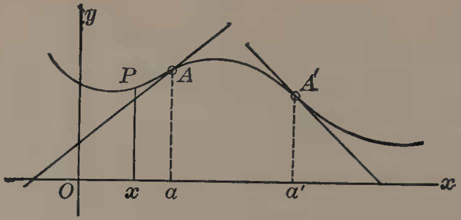

FIG. 22.

Suppose, at $x = a$, we have $f''(a) = 0$, but $f'''(a) \neq 0$. Then the difference between the ordinates of the curve and tangent at a is

$$f(x) - Y = \tfrac{1}{6}(x - a)^3 f'''(\xi).$$

Since $f'''(a) \neq 0$, then throughout the neighborhood of a, $f'''(\xi)$ keeps the same sign as its limit $f'''(a)$. But $(x - a)^3$ changes from $-$ to $+$ as x increases through a. Consequently the corresponding point P on the curve crosses over from one side of the tangent to the other as P passes through A.

The curve is convex on one side of A and concave on the other. The curve is said to have a point of *inflexion* at x, y when at this point we have $D_x y$ determinate and $D^2 y = 0$, $D^3 y \neq 0$.

At a point of inflexion $x = a$ a curve is said to be *convexo-concave* when it changes from convex to concave as x increases through a, and to be *concavo-convex* when it changes from concave to convex as x increases through a. See the points A and A' in Fig. 22.

EXAMPLES.

4. If $$y = 2(x - a)^3 + 4x - 1,$$

$$\therefore \quad y'' = 12(x - a) = 0, \qquad \text{when } x = a,$$

and $$y''' = 12.$$

The curve has a concavo-convex inflexion at $x = a$.

5. Show that every cubic

$$f(x) \equiv ax^3 + bx^2 + cx + d$$

has an inflexion and classify it.

Again, suppose at $x = a$ we have

$$f''(a) = 0, \qquad f'''(a) = 0, \qquad f^{\text{iv}}(a) \neq 0.$$

Then

$$f(x) - Y = \frac{(x - a)^4}{4!} f^{\text{iv}}(\xi).$$

In the neighborhood of a, $f^{\text{iv}}(\xi)$ keeps its sign unchanged, as also does $(x - a)^4$. Consequently the curve lies wholly on one side of the tangent, and is convex or concave according as $f^{\text{iv}}(a)$ is $+$ or $-$.

In general, if $f''(a) = \ldots\ldots = f^m(a) = 0$, $f^{m+1}(a) \neq 0$, then

$$f(x) - Y = \frac{(x - a)^{m+1}}{(m + 1)!} f^{m+1}(\xi).$$

If $m + 1$ is *even*, the curve is *concave* or *convex* at a according as $f^{m+1}(a)$ is *negative* or *positive*.

If $m + 1$ is *odd*, the curve has an inflexion at $x = a$, and is *concavo-convex* if $f^{m+1}(a)$ is $+$, and is *convexo-concave* if $f^{m+1}(a)$ is $-$.

The tangent at a point of inflexion is sometimes called a *stationary* tangent, since $D_x\theta = 0$ there. For, θ being the angle which the tangent makes with Ox, we have $\tan\theta = D_xy$, etc.

The conditions for a point of inflexion given above, for $f(x, y) = 0$, are exactly those which have been previously given for a maximum or a minimum of D_xy. For $y = f(x)$ has a *convexo-concave* inflexion whenever $f'(x)$ is a *maximum*, and a *concavo-convex* inflexion whenever $f'(x)$ is a *minimum*. The investigation of $y = f(x)$ for points of inflexion amounts to the same thing as investigating the maximum and minimum values of $y = f'(x)$.

It is not necessary to give many examples of finding points of inflexion, since it would be but repeating the work of finding the maximum and minimum values of functions.

EXAMPLES.

6. Show that $x^3 = (a^2 + x^2)y$ has an inflexion at the origin. What kind of inflexion?

7. Show that $a^3y = bxy + cx^3 + dx^4$ inflects at 0, 0.

8. The origin is an inflexion on $a^{m-1}y = x^m$, if $m > 2$ is an odd integer.

9. When is the origin an inflexion on $y^n = kx^m$?

10. Find the point of inflexion on $x^3 - 3bx^2 + a^2y = 0$, and classify it. [$x = b$, $y = 2b^3/a^2$.]

11. Show that the inflexions on $f(\rho, \theta) = 0$ are to be determined from

$$\rho^2 + 2\left(\frac{d\rho}{d\theta}\right)^2 - \rho\frac{d^2\rho}{d\theta^2} = 0.$$

See § 56. If we put $\rho = 1/u$, this takes the simpler form

$$u + u''_\theta = 0.$$

The polar curve is concave or convex with respect to the pole according as $u + u''_\theta$ is $+$ or $-$. The curve in the neighborhood of the point of contact is concave or convex with respect to the pole according as it does or does not lie on the same side of the tangent as the pole.

12. Find the inflexion on $\rho \sin\theta = a\theta$.

13. In $\rho\theta^m = a$ there is an inflexion when $\theta = \sqrt{m(1 - m)}$.

14. Find the points of inflexion on the curves:

(*a*). $\tan ax = y$.
(*b*). $y = \sin ax$.
(*c*). $y = \cot ax$.
(*d*). $y = e^{-a^2x^2}$.
(*e*). $y = (x - 1)(x - 2)(x - 3)$.
(*f*). $\rho(\theta^2 - 1) = a\theta^2$.

15. Show that the curve $x(x^2 - ay) = a^3$ has an inflexion where it cuts Ox. Find the equation to the tangent there.

16. Show that $x^3 + y^3 = a^3$ has inflexions on O_x and O_y.

17. The inflexions of $x^2y = a^2(x - y)$ are at $x = 0$, $x = \pm a\sqrt{3}$.

18. $x = \log(y/x)$ inflects at $x = -2$, $y = -2e^{-2}$.

19. $\rho\theta^{\frac{1}{2}} = a$ has an inflexion at $\rho = a\sqrt{2}$.

CHAPTER XII.

CONTACT AND CURVATURE.

97. In the preceding chapter we have studied the character of the contact of a curve with its straight-line tangent. Now we propose to study the nature of the contact of two curves which have a common tangent at a point.

98. Contact of Two Curves.

I. Let $y = \phi(x)$ and $y = \psi(x)$ be two curves, the functions ϕ and ψ having determinate derivatives at a.

If we solve $y = \phi(x)$ and $y = \psi(x)$ for x and y, we find the points of intersection of the curves.

(1). If $\phi(a) = \psi(a)$ and $\phi'(a) \neq \psi'(a)$, the curves *cut* at a, and cross there. For, by the law of the mean applied to the function

$$F(x) \equiv \phi(x) - \psi(x),$$

we have

$$\phi(x) - \psi(x) = (x - a)[\phi'(\xi) - \psi'(\xi)]. \qquad (\S\ 62)$$

The derivatives $\phi'(\xi)$, $\psi'(\xi)$ are arbitrarily nearly equal to $\phi'(a)$ and $\psi'(a)$ for x in the neighborhood of a. Therefore, since $\phi'(a) \neq \psi'(a)$, the difference $\phi'(\xi) - \psi'(\xi)$ keeps its sign unchanged in the neighborhood of a, and $x - a$ changes sign as x passes through a.

(2). If we have $\phi(a) = \psi(a)$, $\phi'(a) = \psi'(a)$, but $\phi''(a) \neq \psi''(a)$, then the curves have a common tangent at a, and are said to be tangent to each other, and to have a contact of the first order at a.

By the law of the mean, the difference

$$\phi(x) - \psi(x) = \tfrac{1}{2}(x - a)^2[\phi''(\xi) - \psi''(\xi)]$$

shows that this difference does not change sign as x increases through a, and therefore the curves do not cross at a.

(3). If $\phi(a) = \psi(a)$, $\phi'(a) = \psi'(a)$, $\phi''(a) = \psi''(a)$, but $\phi'''(a) \neq \psi'''(a)$, then the curves have a contact of the second order at a, and we have

$$\phi(x) - \psi(x) = \tfrac{1}{6}(x - a)^3[\phi'''(\xi) - \psi'''(\xi)].$$

This shows that the curves do cross at a, since the difference of their ordinates changes sign as x increases through a.

(4). In general, if $\phi(x)$ and $\psi(x)$ and their first n derivatives at a are equal, but their $(n + 1)$th derivatives are unequal, then the curves are said to have an nth contact at a, or a contact of the nth order. They *do* or *do not* cross at the point of contact according as $n + 1$ is *odd* or *even*. 1st n [illegible]

For we have, by the law of the mean,

$$\phi(x) - \psi(x) = \frac{(x-a)^{n+1}}{(n+1)!}[\phi^{n+1}(\xi) - \psi^{n+1}(\xi)],$$

which changes sign or does not according as $n + 1$ is odd or even when x increases through a.

Two functions are said to have a contact of order n at a value of the variable when for that value of the variable the corresponding values of the functions and their first n derivatives are equal.

II. The character of the contact of two curves is made clear by the following theorem:

If two curves $y = \phi(x)$ and $y = \psi(x)$ intersect in n distinct points at $a_1, a_2, \ldots, a_n$, then when these n points of intersection converge to one point, the curves have a contact of order $n - 1$.

To prove this the following lemma* will be established:

If $F(x)$ vanishes at $a_1, a_2, \ldots, a_n$, then

$$F(x) = \frac{(x-a_1)\ldots(x-a_n)}{n!}F^n(\xi),$$

where ξ is some number between the greatest and least of the numbers $x, a_1, \ldots, a_n$.

Consider the function of z,

$$J(z) = (x-a_1)\ldots(x-a_n)F(z) - (z-a_1)\ldots(z-a_n)F(x).$$

7 x [illegible]

We have $J(z) = 0$ at the $n + 1$ values of z equal to $x, a_1, \ldots, a_n$. By Rolle's theorem, $J'(z)$ vanishes n times, once between each consecutive pair of these numbers. Also by the same theorem $J''(z)$ vanishes $n - 1$ times, once between each consecutive pair of numbers at which $J'(z)$ vanishes; and so on, until finally $J^n(z)$ vanishes once between the pair of values for which $J^{n-1}(z)$ vanishes. This value, say ξ, at which $J^n(z)$ vanishes is certainly between the greatest and least of $x, a_1, \ldots, a_n$. Hence

$$J^n(\xi) = (x-a_1)\ldots(x-a_n)F^n(\xi) - n!\ F(x) = 0,$$

and the lemma is proved.

Now let $F(x) \equiv \phi(x) - \psi(x)$. Then

$$\phi(x) - \psi(x) = \frac{(x-a_1)\ldots(x-a_n)}{n!}[\phi^n(\xi) - \psi^n(\xi)].$$

This shows that when $a_1 = a_2 = \ldots = a_n = a$, we have

$$\phi(x) - \psi(x) = \frac{(x-a)^n}{n!}[\phi^n(\xi) - \psi^n(\xi)],$$

where ξ lies between x and a.

---

* Due to Ossian Bonnet.

This last equation shows that $\phi(x)$ and $\psi(x)$ and their first $n - 1$ derivatives at a are equal, or the two curves have a contact at a of order $n - 1$. Therefore, when two curves have a contact of the nth order, it means that they have $n + 1$ coincident points in common at a; or, as we sometimes say, they intersect in $n + 1$ consecutive points. A curve which cuts another n times in the neighborhood of a point, leaves that curve on the same side it approaches it when n is even and leaves on the opposite side when n is odd. Thus we see why it is that curves having even contact cross, while those having odd contact do not cross, at the point of contact.

99. To find the order of contact of two given curves, we must solve their equations for the points of intersection, and compare their corresponding ordinate derivatives at these points.

EXAMPLES.

1. Find the order of contact of the curves

$$y = x^3 \quad \text{and} \quad y = 3x^2 - 3x + 1.$$

Solving the equations, we find that $x = 1$, $y = 1$ is a point common to both curves. Also, their first derivatives, Dy, are equal to 3 there, and their second derivatives, D^2y, are equal to 6; while their third derivatives, D^3y, are not equal to each other. Therefore, at the point 1, 1 the curves have a contact of the second order.

2. Show that the straight line $y = x - 1$ and the parabola $4y = x^2$ have a first-order contact.

3. Find the order of contact of

$$9y = x^3 - 3x^2 + 27 \quad \text{and} \quad 9y + 3x = 28. \qquad \text{[Second.]}$$

4. Find the orders of contact of the curves:

(*a*). $y = \log(x - 1)$ and $x^2 - 6x + 2y + 8 = 0$. [Second.]
(*b*). $4y = x^2 - 4$ and $x^2 + y^2 - 2y = 3$. [Third.]
(*c*). $xy = a^2$ and $(x - 2a)^2 + (y - 2a)^2 = 2xy$. [Third.]

5. Find the value of a in order that the hyperbola $xy = 3x - 1$ and parabola $y = x + 1 + a(x - 1)^2$ may have contact of the second order.

100. Osculation.—(1). We can always find a straight line which has a contact of the first order with a given curve $y = \phi(x)$ at a given arbitrary point. In general, at any point of ordinary position, a straight line cannot have a contact with a curve of order higher than the first.

For, let $y = mx + b$ be the equation to a straight line, in which m and b are arbitrary and are to be so determined that the straight line shall have the closest possible contact with the curve $y = \phi(x)$ at $x = a$.

Then we must have

$$ma + b = \phi(a),$$
$$m = \phi'(a).$$

These two conditions completely determine m and b, and give

$$y = \phi(a) + (x - a)\phi'(a)$$

as the equation to the required line, which has contact of the first order with the curve at a. This is the familiar equation to the tangent to the curve at a.

The line can have no higher contact with the curve at a unless we have $\phi''(a) = 0$, and so on, see § 98. At an ordinary point of inflexion the tangent has a contact of the second order, and cuts the curve there in three coincident points crossing the curve.

(2). Consider the equation to the circle

$$(X - \alpha)^2 + (Y - \beta)^2 = R^2. \tag{1}$$

This is the most general form of the equation to the circle, and can be made to represent any circle whatever, by assigning proper values to the arbitrary constants α, β, R, the coordinates of the centre and the radius.

Let us determine α, β, and R, so that the circle shall have the closest possible contact with a given curve $y = \phi(x)$ at a given point x, y of general position on the curve.

Differentiating (1) twice with respect to X,

$$X - \alpha + (Y - \beta)DY = 0, \tag{2}$$

$$1 + (Y - \beta)D^2Y + (DY)^2 = 0. \tag{3}$$

The conditions for the contact are

$$Y = y, \qquad DY = \phi'(x), \qquad D^2Y = \phi''(x).$$

The values of α, β, R determined from the three equations

$$(x - \alpha)^2 + (y - \beta)^2 = R^2, \tag{4}$$

$$x - \alpha + (y - \beta)\phi'(x) = 0, \tag{5}$$

$$1 + (y - \beta)\phi''(x) + [\phi''(x)] = 0, \tag{6}$$

determine the circle of closest contact, of the second order, at x, y on the curve. Solving these equations and writing y', y'' for ϕ', ϕ'', we have for the coordinates of the centre of curvature

$$\beta = y + \frac{1 + y'^2}{y''}, \quad \alpha = x - y'\frac{1 + y'^2}{y''}, \tag{7}$$

and for the radius of curvature

$$R = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''}. \tag{8}$$

Whenever the coordinates x, y are given, we can substitute in these formulæ and compute α, β, and R, and write out the equation to the circle.

Observe that the three equations completely determine the circle, and the circle at a point of ordinary position on the curve can have no closer contact with the curve than that of second order. Observe that this is the same circle obtained in § 79, Ill. (3), where we considered the circle which was the limiting position of a circle through three points on the curve when these three points converge to x, y

as a limit. Having a contact of the second order with the curve, the circle of curvature crosses over the curve at the point of contact.

This circle is called the circle of curvature of the curve at the point x, y, and R is called the radius of curvature, the point α, β is called the centre of curvature of the curve at x, y.

(3). In general, when the equation of a curve $y = \psi(x)$ contains a number, $n + 1$, of arbitrary constants, we can determine the values of these constants so that the curve shall have a contact of the nth order with a given curve $y = \phi(x)$, at a given point of arbitrary position and no higher contact. For, if we equate the values of the function ψ and its first n derivatives to the corresponding values of ϕ and its first n derivatives, we shall have $n + 1$ equations between the $n + 1$ arbitrary constants in ψ. These equations serve to determine the values of these constants which will make $y = \psi(x)$ have an nth contact with $y = \phi(x)$ at the point under consideration. This is the highest contact such a curve $y = \psi$ can have with a given curve $y = \phi$ at a point of ordinary position. Then $y = \psi$ is said to *osculate* the curve $y = \phi$ at x, y.

At certain singular points an osculating curve can have a contact of higher order with a given curve than that which it has at a point of ordinary position—as, for example, the tangent line to a curve at an inflexion.

101. Construction of the Circle of Curvature.—Since Dy is the same, at the point of contact, for the circle and the curve, they have a common tangent and normal there; also, the centre of curvature is on the normal to the curve. They have the same convexity or concavity at the point of contact. The radius of curvature, involving the radical sign, is ambiguous; we remove the ambiguity by taking R as positive when y'' is positive, or when the curve is convex; and negative when y'' is negative or the curve is concave. Consequently the value of R is

$$R = \frac{|1 + y'^2|^{\frac{3}{2}}}{y''}.$$

The center of curvature is to be constructed by measuring off R from the point of contact along the normal, upward or downward according as R or y'' is $+$ or $-$.

EXAMPLES.

1. Find the radius of curvature at any point on the parabola $x^2 = 4my$.

Here $\quad 2my' = x, \quad 2my'' = 1, \quad 1 + y'^2 = 1 + y/m;$

$$\therefore \quad \rho = + \frac{2(m + y)^{\frac{3}{2}}}{\sqrt{m}}.$$

2. Find the radius of curvature in the catenary

$$y = \tfrac{1}{2}a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

Here $y' = \frac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right), \quad y'' = y/a^2; \qquad \therefore \quad \rho = +y^2/a.$

Show that the radius of curvature is equal and opposite to the normal-length.

3. In the cubical parabola $3a^2y = x^3$ we have

$$a^2y' = x^2, \quad a^2y'' = 2x, \quad (1 + y'^2)^{\frac{3}{2}} = (a^4 + x^4)^{\frac{3}{2}}/a^6;$$

$$\therefore \quad \rho = \frac{(a^4 + x^4)^{\frac{3}{2}}}{2a^4x}.$$

4. Newton's Rule for the Radius of Curvature. At any point P on a given curve draw a circle tangent to the curve and cutting it in a third point Q at distances p and q from the common normal and tangent respectively.

Let r be the radius of the circle. Then, by elementary geometry, the products of the segments of the secants are equal, and we have

$$p^2 = q(2r - q),$$

or

$$r = \frac{p^2}{2q} + \frac{q}{2}.$$

When $Q(=)P$, the circle becomes the circle of curvature at P and $\pounds r = R$.

$$\therefore \quad R = \pounds \frac{p^2}{2q},$$

when $p(=)0$, $q(=)0$.

5. If Q, P, R are three points on any curve, such that V is the middle point of the chord QR, and P is the mid-point of the arc QR, show that

$$R = \pounds \frac{QV^2}{2PV},$$

when $Q(=)P$, $R(=)P$.

EXERCISES.

1. Find the parabola $y = Ax^2 + Bx + C$ which has the same curvature as a given curve $y = f(x)$ at a given point x, y.

$$Y = f(x) + (X - x)f'(x) + \tfrac{1}{2}(X - x)^2 f''(x).$$

2. Show that a straight line has contact of second order with a curve at a point of ordinary inflexion.

3. Show that the radius of curvature is ∞ at a point of inflexion, and explain geometrically.

4. Show that the circle of curvature has a contact of third order at a maximum or a minimum value of R, and therefore does not cross the curve at such a point.

At a max. or min. value of R we have $D_xR^2 = 0$. Differentiating (8), § 100, and solving, we find for the curve

$$D_x^3y = \frac{3y'y''^2}{1 + y'^2}.$$

Computing D_x^3y, for the circle, from (5) and (6), we find it has the same value.

5. Show from (5), § 100, that the normal passes through the center of curvature.

6. Find the radius of curvature for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$R = \frac{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{\frac{3}{2}}}{ab} = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^{\frac{3}{2}} a^2b^2,$$

ϕ being the eccentric angle.

7. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is satisfied by $x = a\cos^3\phi$, $y = a\sin^3\phi$. Show that

$$R = -3(axy)^{\frac{1}{3}}.$$

8. Show that the radius of curvature of $e^{\frac{y}{a}} = \sec(x/a)$ is

$$R = a\sec(x/a).$$

9. The coordinates of a point on a curve are

$$x = c\sin 2t(1 + \cos 2t), \quad y = c\cos 2t(1 - \cos 2t);$$

show that

$$R = 4c\cos 3t.$$

10. Find R for $x^3 = ay^2$.

11. Show that, when $y = \sin x$ is a maximum, $R = 1$.

12. Find the center and radius of curvature of $xy = a^2$.

$$\alpha = (3x^2 + y^2)/2x, \quad \beta = (x^2 + 3y^2)/2y, \quad R = (x^2 + y^2)^{\frac{3}{2}}/2a^2.$$

13. Show that if a variable normal converges to a fixed normal as a limit, their intersection converges to the center of curvature as a limit.

The equations to the normals at x_1, y_1 and x, y are

$$(Y - y_1)\frac{dy_1}{dx_1} + X - x_1 = 0, \quad (Y - y)\frac{dy}{dx} + X - x = 0.$$

The ordinate of their intersection is

$$Y = -\frac{y_1\dfrac{dy_1}{dx_1} - y\dfrac{dy}{dx} + x_1 - x}{\dfrac{dy}{dx} - \dfrac{dy_1}{dx_1}},$$

which takes the illusory form 0/0 for $x_1 = x$.

When evaluated in the usual way, we have, when $x_1 (=) x$,

$$Y = y + \frac{1 + y'^2}{y''^2},$$

which is the ordinate of the center of curvature.

Substitution of $Y - y$ in the equation of the normal gives X as the abscissa of the center of curvature.

14. Find the radius of curvature at the origin for

$$2x^2 + 3xy - 4y^2 + x^3 - 6y = 0.$$

Using Newton's method,

$$R = £\frac{x^2}{2y} = \frac{3}{2}.$$

15. Find the radius of curvature at the maximum ordinate of $y = e^{-a^2x^2}$. What is the order of contact of the circle of curvature?

16. If $f(\rho, \theta) = 0$ is the polar equation to any curve, show that at any point ρ, θ the radius of curvature is

$$R = \frac{(\rho^2 + \rho'^2)^{\frac{3}{2}}}{\rho^2 + 2\rho'^2 - \rho\rho''},$$

where for brevity we write $\rho' \equiv D_\theta\rho$, $\rho'' \equiv D_\theta^2\rho$.

This follows immediately from substituting for Dy and D^2y, (1) and (2), § 56, in (8), § 100.

17. Show that if $\rho u = 1$, $u' = D_\theta u$, $u'' = D_\theta^2 u$, the value of the radius in Ex 16 becomes

$$R = \frac{(u^2 + u'^2)^{\frac{3}{2}}}{u^3(u + u'')}.$$

18. Since at a point of inflexion $y'' = 0$, we have there $R = \infty$. Therefore the inflexion condition for a polar curve is, as found before, $u + u'' = 0$.

19. If $\rho = a\theta$, show that $R = a(1 + \theta^2)^{\frac{3}{2}}/(2 + \theta^2)$.

20. If $\rho = a^\theta$, then $R = \rho[1 + (\log a)^2]^{\frac{1}{2}}$.

21. If $\rho = 29 - 11 \cos 2\theta$, $R = \infty$ at $\cos 2\theta = \frac{9}{11}$.

22. Show that $R = \frac{1}{2}ab$, for $\rho = a \sin b\theta$, at the origin.

23. Find the radius of curvature for the hyperbola

$$x^2/a^2 - y^2/b^2 = 1.$$

24. Find the radius of curvature of:

The circle $\rho = a \sin \theta$; the lemniscate $\rho^2 = a^2 \cos 2\theta$; the logarithmic spiral $\rho = e^{a\theta}$; the trisectrix $\rho = 2a \cos \theta - a$.

25. If R is the radius of curvature of $f(x, y) = 0$, show that

$$R = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{d^2y\,dx - dy\,d^2x},$$

regardless of the independent variable.

Differentiate the equation of the circle of curvature,

$$R^2 = (x - a)^2 + (y - b)^2.$$

$$\therefore \quad 0 = (x - a)dx + (y - b)dy,$$

$$0 = dx^2 + (x - a)d^2x + dy^2 + (y - b)d^2y.$$

The elimination of $x - a$ and $y - b$ gives the result.

CHAPTER XIII.

ENVELOPES.

102. If $f(x, y) = 0$ is the equation of a certain line containing a constant α, then we can implicitly indicate that the position of this curve depends on the value of α by including it in the functional symbol, thus:

$$f(x, y, \alpha) = 0.$$

If we change α by substituting for it another number α_1, we get another curve,

$$f(x, y, \alpha_1) = 0,$$

which will, in general, intersect the first curve.

The arbitrary constant α in $f(x, y, \alpha) = 0$ is called a *parameter*. All the curves obtained by assigning different values to α are said to belong to the same *family* of curves, of which α is the variable parameter. Thus

$$f(x, y, \alpha) = 0 \tag{1}$$

is the equation of a *family* of curves when we regard α as a variable, and any curve obtained by assigning a particular value to α is a particular member of that family.

Thus, in the figure, let the curves 1, 2, 3, . . . be the particular curves of the family (1), obtained by assigning to α the particular values α_1, α_2, . . . taken in order.

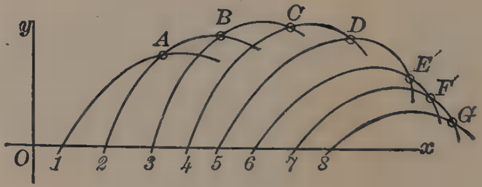

FIG. 23.

Two curves of this family are said to be consecutive when they correspond to consecutive values of α. The sequence of curves corresponding to α_1, α_2, . . . , as drawn in the figure, intersect in points A, B, C, . . .

ILLUSTRATIONS.

The arbitrary constant or parameter being α:

(*a*). $y = mx + \alpha$ is the family of parallel straight lines sloped m to the axis of x. Consecutive members of this family do not intersect in the finite part of the plane.

(*b*). $y = \alpha x + b$ is the family of straight lines passing through the point $0, b$.

(*c*). $x \cos \alpha + y \sin \alpha = p$ is the family of straight lines tangent to the circle $x^2 + y^2 = p^2$.

(*d*). $y = \alpha x + b/\alpha$ is a family of straight lines tangent to a parabola $y^2 = 4bx$, and

$$y = \alpha x - 2b\alpha - b\alpha^3$$

is the family of normals to the same curve.

(*e*). $(x - a)^2 + (y - b)^2 = \alpha^2$ is the family of circles with center a, b and variable radii. The curves of the family do not intersect.

(*f*). $x^2 + y^2 - 2\alpha x + r^2 = 0$ is the family of circles with radius r having their centers on Ox. Two curves of the family do intersect, provided we take their centers near enough together.

103. The Envelope of a Family of Curves.—If

$$f(x, y, \alpha) = 0 \tag{1}$$

and

$$f(x, y, \alpha_1) = 0 \tag{2}$$

are two curves of the same family which intersect at a point x, y, let us seek to determine the limiting position of the point of intersection x, y when $\alpha_1 (=) \alpha$. When $\alpha_1 (=) \alpha$ all points on curve (2) converge to corresponding points on (1), and in the limit curve (2) passes over into curve (1) and they have an infinite number of points in common. Therefore the attempt to determine the limiting position of the point x, y of intersection of (1) and (2), by solving (1) and (2) for the coordinates and then making $\alpha_1 (=) \alpha$, leads to indeterminate forms.

We shall proceed to find the limit to which converges the point x, y of intersection of (1) and (2), by finding a third line which also passes through their intersection, and which does not coincide with (1) when $\alpha_1 (=) \alpha$.

Assign to x and y the numbers a and b, the coordinates of the intersection of (1) and (2), and let α be a variable number. Then $f(a, b, \alpha)$ is a function of the single variable α, and we have, by the law of the mean,

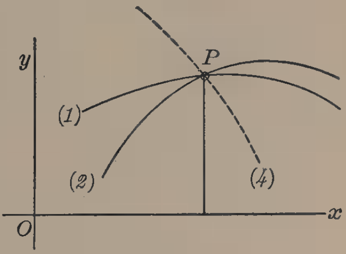

FIG. 24.

$$f(a, b, \alpha_1) - f(a, b, \alpha) = (\alpha_1 - \alpha) f'_\mu(a, b, \mu), \tag{3}$$

where μ is some number between α_1 and α.

But, a, b being on (1) and (2), we have

$$f(a, b, \alpha_1) = 0 \quad \text{and} \quad f(a, b, \alpha) = 0.$$

Therefore

$$f'_\mu(a, b, \mu) = 0. \tag{4}$$

For the particular value μ assigned in (3) we have $f'_\mu(x, y, \mu) = 0$ as the equation to some curve passing through the intersection of (1) and (2), in virtue of equation (4). We do not know the number μ in (3) and (4), since all we know about it is that it lies between α and α_1.

But, whatever be the number μ satisfying (4), we know that the curve

$$f'_\mu(x, y, \mu) = 0 \tag{5}$$

passes through the intersection of (1) and (2). Now, when $\alpha_1 (=) \alpha$, then $\mu (=) \alpha$. If, therefore, when $\alpha_1 (=) \alpha$, the two curves (1) and (2) intersect in a point which converges to a fixed point as a limit, then (5) becomes

$$f'_\alpha(x, y, \alpha) = 0, \tag{6}$$

the equation to a curve which passes through the limit of the intersection of (1) and (2) as (2) converges to (1). Moreover, (6), being a curve distinct from (1), has in general a definite intersection with (1).

If, between the equations

$$f(x, y, \alpha) = 0, \tag{1}$$

$$f'_\alpha(x, y, \alpha) = 0, \tag{6}$$

the variable parameter α be eliminated, we obtain the locus

$$E(x, y) = 0 \tag{7}$$

of *all points* in which the consecutive curves of the family $f(x, y, \alpha) = 0$ intersect as α varies continuously.

The curve (7) is called the *envelope** of the family (1).

ILLUSTRATION OF THE ENVELOPE.

As the parameter α varies continuously, the curve $f(x, y, \alpha) = 0$ sweeps over or generates a certain portion of the surface of the plane xOy, and leaves unswept a certain portion. The envelope may be regarded as the line which is the boundary between these two portions of the plane xOy.

104. The envelope, $E(x, y) = 0$, is tangent to each member of the family $f(x, y, \alpha) = 0$ which it envelops.

We are not prepared to give a rigorous proof of this statement now. This proof requires a knowledge of functions of several variables. We can, however, give a geometrical picture which will illustrate the general truth of the statement. For this proof see § 227.

Let (A), (B), (C) be three contiguous curves of the family, (A)

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* Strictly speaking, the equation of the envelope is the equation gotten by equating to 0 that factor of $E(x, y)$ which occurs only once in $E(x, y)$. See Chapter XXXIX.

and (C) intersecting the fixed curve (B) in points P and Q respectively. When (A) and (C) converge to coincidence with (B),

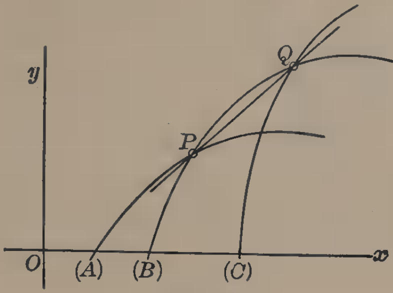


FIG. 25.

the points P and Q converge to each other and to two coincident points on the envelope. The straight line PQ converges to a common tangent to (B) and the envelope.

EXAMPLES.

The variable parameter being α, find the envelopes of the following curve families:

1. $x \cos \alpha + y \sin \alpha - p = 0 = f(x, y, \alpha)$.

$f'_\alpha = - x \sin \alpha + y \cos \alpha$. Square and add. Hence

$x^2 + y^2 = p^2$, a circle with radius p.

2. Envelope the family $f \equiv y - \alpha x - b/\alpha = 0$.

$f'_\alpha = - x + b/\alpha^2$. $\therefore\ \alpha = \sqrt{b/x}$. Hence $y^2 = 4bx$.

3. Envelope the family $f \equiv y - \alpha x + 2b\alpha + b\alpha^3$.

$f'_\alpha = - x + 2b + 3b\alpha^2$. $\therefore\ \alpha^2 = (x - 2b)/3b$. Hence

$$27y^2b = 4(x - 2b)^3.$$

4. Find the envelope of $(x \cos \alpha)/a + (y \sin \alpha)/b = 1$.

5. Find the envelopes of $y = ax + \sqrt{a^2\alpha^2 \pm b^2}$. $[x^2/a^2 \pm y^2/b^2 = 1.]$

6. Envelope the family $x^2 + y^2 - 2\alpha x = r^2$.

105. Envelopes when there are Two Connected Parameters.

Let $$\phi(x, y, \alpha, \beta) = 0 \qquad (1)$$

be the equation to a curve, involving two arbitrary parameters α and β which are related by the condition

$$\psi(\alpha, \beta) = 0. \qquad (2)$$

I. When we can solve (2) with respect to α or β and substitute in (1), we reduce that equation to that of a family with one parameter. The envelope is then found as before.

EXAMPLE.

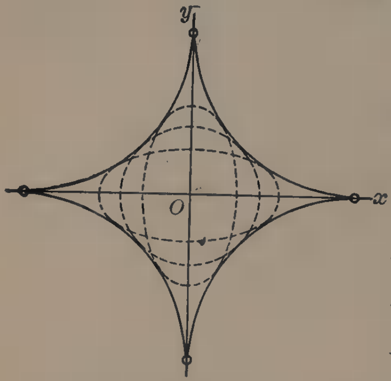


FIG. 26.

Find the envelope of the ellipses

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1, \qquad (1)$$

when $\alpha + \beta = c$.

We have $\beta = c - \alpha$. Therefore

$$\frac{x^2}{\alpha^2} + \frac{y^2}{(c - \alpha)^2} = 1.$$

Differentiating with respect to α, and solving for α,

$$\therefore \quad \alpha = \frac{cx^{\frac{2}{3}}}{x^{\frac{2}{3}} + y^{\frac{2}{3}}} \quad \text{and} \quad \beta = \frac{cy^{\frac{2}{3}}}{x^{\frac{2}{3}} + y^{\frac{2}{3}}},$$

which substituted in (1) give

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

II. Otherwise, when it is inconvenient to solve (2), it is generally simpler to proceed as follows:

Let x, y be constant, and differentiate (1) and (2) with respect to any one of the parameters, say β. Eliminate α, β and $\alpha' \equiv d\alpha/d\beta$, between the four equations.

$$\phi(x, y, \alpha, \beta) = 0, \qquad (1)$$

$$\phi'_\beta(x, y, \alpha, \beta) = 0, \qquad (2)$$

$$\psi(\alpha, \beta) = 0, \qquad (3)$$

$$\psi'_\beta(\alpha, \beta) = 0. \qquad (4)$$

The result is the envelope $E(x, y) = 0$.

For example, take the same question proposed in I. We have for (1), (2), (3), (4),

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1, \qquad (1)$$

$$\frac{x^2}{\alpha^3}\alpha' + \frac{y^2}{\beta^3} = 0, \qquad (2)$$

$$\alpha + \beta = c, \qquad (3)$$

$$\alpha' + 1 = 0. \qquad (4)$$

The elimination gives the same result as before.

EXERCISES.

1. Find the envelope of a straight line of given constant length, whose ends move on fixed rectangular axes. [$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$.]

2. Find the envelope of the ellipses

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$$

when the area is constant. $\alpha\beta = c.$ [$2xy = c^2$.]

3. Find the envelope of a straight line when the sum of its intercepts is constant. [$x^{\frac{1}{2}} + y^{\frac{1}{2}} = c^{\frac{1}{2}}$.]

4. One angle of a triangle is fixed; find the envelope of the opposite side when the area is given. [Hyperbola.]

5. Find the envelope of $x/\alpha + y/\beta = 1$ when $\alpha^m + \beta^m = c^m$.

$$\left[x^{\frac{m}{m+1}} + y^{\frac{m}{m+1}} = c^{\frac{m}{m+1}}.\right]$$

6. Show that the envelope of $x/l + y/m = 1$, where $l/a + m/b = 1$ is the parabola $(x/a)^{\frac{1}{2}} + (y/b)^{\frac{1}{2}} = 1$.

7. From a point P on the hypothenuse of a right-angled triangle, perpendiculars PM, PN are drawn to the sides; find the envelope of the line MN.

8. Find the envelope of the circles on the central radii of an ellipse as diameters.

9. Find the envelope of $y = 2\alpha x + \alpha^4$. [$16y^3 + 27x^4 = 0$.]

10. Find the envelope of the parabola $y^2 = \alpha(x - \alpha)$. [$4y^2 = x^2$.]

11. Find the envelope of a series of circles whose centers are on Ox and radii proportional to their distances from O.

12. The envelope of the lines $x \cos 3\alpha + y \sin 3\alpha = a(\cos 2\alpha)^{\frac{3}{2}}$ is the lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

13. Find the envelope of the circles whose diameters are the double ordinates of the parabola $y^2 = 4ax$. [$y^2 = 4a(a + x)$.]

14. Find the envelope of the circles passing through the origin, whose centers are on $y^2 = 4ax$. [$(x + 2a)y^2 + x^3 = 0$.]

15. Find the envelope of $x^2/\alpha^2 + y^2/\beta^2 = 1$, when $\alpha^2 + \beta^2 = k^2$. [$(x \pm y)^2 = k^2$.]

16. Circles through O with centers on $x^2 - y^2 = a^2$ are enveloped by the lemniscate $(x^2 + y^2)^2 = 4a^2(x^2 - y^2)$.

17. Show that the envelope of

$$L\alpha^2 + 2M\alpha + N = 0,$$

in which L, M, N are functions of x and y, and α is a variable parameter, is $LN = M^2$.

18. In Ex. 17 show that if L, M, N are linear functions of x and y, the envelope is a conic tangent to $L = 0$, $N = 0$ and having $M = 0$ for chord of contact.

Differentiate $LN - M^2 = 0$ with respect to x,

$$\therefore\ L'N + N'L = 2MM'.$$

At the intersection of $L = 0$ and $M = 0$ we have $L'N = 0$; and since there $N \neq 0$, we have $L' = 0$. The D_xy from this is the slope of the tangent to the envelope. Hence $L' = 0$ is the tangent at the intersection of $L = M = 0$ to the envelope, etc.

CHAPTER XIV.

INVOLUTE AND EVOLUTE.

106. Definition.—When the point of contact, P, of the circle of curvature of a given curve moves along the curve, the center of curvature, C, describes a curve called the *evolute* of the given curve.

The evolute of a given curve is the locus of its center of curvature. The given curve is called an *involute* of the evolute.

107. There are two common methods of finding the evolute of a given curve.

I. If $\phi(x, y) = 0$ is the equation of the given curve, and α, β are the coordinates of the center of curvature, then we have, § 100, (2),

$$\alpha - x = -y'\frac{1 + y'^2}{y''}, \quad \beta - y = \frac{1 + y'^2}{y''}. \qquad (1)$$

If we eliminate x and y from these two equations and the equation to the curve, $\phi(x, y) = 0$, we leave α and β tied up with constants in the equation to the evolute.

Eliminations are, in general, difficult and no general rule can be given for effecting them. Another method of finding the evolute will be given in II, which frequently simplifies the problem.

EXAMPLES.

1. Find the evolute of the parabola $y^2 = 4px$.

We have $y' = p^{\frac{1}{2}}x^{-\frac{1}{2}}$; $y'' = -\frac{1}{2}p^{\frac{1}{2}}x^{-\frac{3}{2}}$. Hence

$$\alpha - x = 2(x + p), \quad \beta - y = -2\left(p^{-\frac{1}{2}}x^{\frac{3}{2}} + p^{\frac{1}{2}}x^{\frac{1}{2}}\right).$$

$$\therefore \quad \alpha = 2p + 3x, \quad \beta = -2p^{-\frac{1}{2}}x^{\frac{3}{2}}.$$

Eliminating x, we have for the equation to the evolute (§ 112, Ex. 17, Fig. 44)

$$4(\alpha - 2p)^3 = 27p\beta^2.$$

2. To find the evolute of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

We can differentiate directly, or use the eccentric angle

$$x = a\cos\phi, \quad y = b\sin\phi, \quad \text{and find}$$

$$y' = -b^2x/a^2y, \quad y'' = -b^4/a^2y^3.$$

$$\therefore \quad \alpha = \frac{a^2 - b^2}{a^4}x^3, \qquad \beta = -\frac{a^2 - b^2}{b^4}y^3.$$

Hence $$(a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}. \qquad \text{(Fig. 43.)}$$

II. The evolute of a given curve $f(x, y) = 0$ is the envelope of the normals to the curve.

The equation to the normal to $f = 0$ at x, y is

$$X - x + (Y - y)y' = 0. \tag{1}$$

But y and y' are functions of x, from the equation $f = 0$ to the curve. Therefore x is a parameter in (1), by varying which we get the system or family of normals. Hence the required locus is to be found by differentiating (1) with respect to x, keeping X, Y constant. Thus

$$-1 + (Y - y)y'' - y'^2 = 0. \tag{2}$$

Eliminating x between (1) and (2), we have

$$Y - y = \frac{1 + y'^2}{y''} \quad \text{and} \quad X - x = -y'\frac{1 + y'^2}{y''},$$

in which X and Y are the coordinates of the center of curvature, § 100, (2). This proves the statement.

EXAMPLES.

1. Find the evolute of the parabola $y^2 = 4px$.

The equation to the normal is

$$y = \alpha x - 2p\alpha - p\alpha^3. \tag{1}$$

$$\therefore \quad 0 = x - 2p - 3p\alpha^2.$$

$$\therefore \quad \alpha = \left(\frac{x - 2p}{3p}\right)^{\frac{1}{2}},$$

which substituted in (1) gives as before in I, $4(x - 2p)^3 = 27py^2$.

2. Find the evolute of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Taking the equation to the normal

$$ax \sec\alpha - by \csc\alpha = a^2 - b^2,$$

$$\therefore \quad ax \sec\alpha \tan\alpha + by \csc\alpha \cot\alpha = 0.$$

Hence $\tan\alpha = -(by/ax)^{\frac{1}{3}}$, which leads to the same result as in I,

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

108. The normal to a curve is a tangent to the evolute.

Let

$$(x - \alpha)^2 + (y - \beta)^2 = R^2 \tag{1}$$

be the equation of the circle of curvature at x, y. Then, letting x, y vary on the circle, R remaining constant, we have, on differentiation with respect to x,

$$x - \alpha + (y - \beta)y' = 0, \tag{2}$$

$$1 + y'^2 + (y - \beta)y'' = 0. \tag{3}$$

Now let x, y vary along the curve, R being variable. The numbers α and β are also functions of x. Differentiate (2), which is the equation to the normal to the curve at x, y, with respect to x.

$$\therefore \quad 1 + y'^2 + (y - \beta)y'' - \alpha' - \beta' y' = 0, \tag{4}$$

Subtract (4) from (3).

$$\therefore \quad \frac{d\alpha}{dx} + \frac{d\beta}{dx}\frac{dy}{dx} = 0,$$

or

$$\frac{d\beta}{d\alpha} = -\frac{dx}{dy},$$

which proves the statement.

EXERCISES.

1. Find the centre of curvature of $y^3 = a^2x$.

$$\alpha = \frac{a^4 + 15y^4}{6a^2y}, \quad \beta = \frac{a^4y - 9y^5}{2a^4}.$$

These equations are the equations of the evolute, α and β being expressed in terms of y, a third variable.

2. Find the coordinates of the center of curvature of the catenary, Fig. 38,

$$y = \tfrac{1}{2}a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

$$\alpha = x - \frac{y}{a}\sqrt{y^2 - a^2}, \quad \beta = 2y.$$

3. Find the center of curvature and the evolute of the hypocycloid,

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

$$\alpha = x + 3x^{\frac{1}{3}}y^{\frac{2}{3}}, \quad \beta = y + 3x^{\frac{2}{3}}y^{\frac{1}{3}}; \quad (\alpha + \beta)^{\frac{2}{3}} + (\alpha - \beta)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$$

4. In the equilateral hyperbola $2xy = a^2$,

$$\alpha + \beta = \frac{(y + x)^3}{a^2}, \quad \alpha - \beta = \frac{(y - x)^3}{a^2};$$

$$(\alpha + \beta)^{\frac{2}{3}} - (\alpha - \beta)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$$

5. In the parabola $x^{\frac{1}{3}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}, \quad \alpha + \beta = 3(x + y).$

CHAPTER XV.

EXAMPLES OF CURVE TRACING.

109. Until functions of two variables have been studied we are not in position to consider the general problem of curve tracing in the most effective manner. Nevertheless it will be advantageous to apply the properties of curves which have been developed for functions of one variable to finding the forms of a few simple curves, whose figures will be useful in the sequel, before we study functions of more than one variable.

110. Principal Elements of a Curve at a Point.—We collect here for handy reference the principal elements of a curve at a point, as deduced in the preceding pages. The notations are the same as there used.

I. Rectangular Coordinates. $D_x y \equiv y'$, $D_x^2 y \equiv v''$.

1. Equation of the tangent:

$$(Y - y) = (X - x)y'.$$

2. Equation of the normal:

$$(Y - y)y' = -(X - x).$$

3. Subtangent and subnormal:

$$S_t = yy'^{-1}, \quad S_n = yy'.$$

4. Tangent-length and normal-length:

$$t = y\sqrt{1 + y'^{-2}}, \quad n = y\sqrt{1 + y'^2}.$$

5. Tangent intercepts on the axes:

$$X_i = x - yy'^{-1}, \quad Y_i = y - xy'.$$

6. Perpendicular from origin on the tangent:

$$p = \frac{y - xy'}{\sqrt{1 + y'^2}}.$$

7. Radius of curvature:

$$R = \frac{|1 + y'^2|^{\frac{3}{2}}}{y''}.$$

8. Coordinates of center of curvature:

$$\alpha = x - y'\frac{1 + y'^2}{y''}, \quad \beta = y + \frac{1 + y'^2}{y''}.$$

II. Polar Coordinates. $D_\theta \rho \equiv \rho'$, $D^2_\theta \rho \equiv \rho''$. $u\rho = 1$.

1. Angle between the tangent and radius vector:

$$\tan \psi = \frac{\rho}{\rho'}.$$

2. Angle between the tangent and the initial line:

$$\tan \phi = \frac{\rho + \rho' \tan \theta}{\rho' - \rho \tan \theta}.$$

3. Subtangent and subnormal:

$$S_t = \frac{\rho^2}{\rho'} = -\frac{d\theta}{du}, \quad S_n = \rho' = \frac{d\rho}{d\theta}.$$

4. Tangent-length and normal-length:

$$t = \frac{\rho}{\rho'}\sqrt{\rho^2 + \rho'^2}, \quad n = \sqrt{\rho^2 + \rho'^2}.$$

5. Perpendicular from the origin on the tangent:

$$p = \frac{\rho^2}{\sqrt{\rho^2 + \rho'^2}}, \quad \frac{1}{p^2} = u^2 + u'^2.$$

6. Radius of curvature:

$$R = \frac{(\rho^2 + \rho'^2)^{\frac{3}{2}}}{\rho^2 + 2\rho'^2 - \rho\rho''},$$

$$= \frac{(u^2 + u'^2)^{\frac{3}{2}}}{u^3(u + u'')}.$$

111. Explicit One-valued Functions.—If the equation to a curve can be solved with respect to the ordinate or the abscissa so as to give

$$y = \phi(x) \quad \text{or} \quad x = \psi(y)$$

as its equation, in which either $\phi(x)$ or $\psi(y)$ is a one-valued function, or if more than one-valued the branches can be separated, we have the simplest class of curves for tracing.

Given any value of the variable, we can compute the value of the function. We thus obtain the coordinates of a point on the curve. By finding the first and second derivatives, y', y'', we can compute all the elements of the curve at this point. y' gives the direction and y'' the curvature at the point.

A regular method of procedure for tracing a curve is:

1. Examine the equation for symmetry.

If the equation is unchanged when the sign of y is changed, the curve is symmetrical with respect to Ox.

If the equation is unchanged when the sign of x is changed, the curve is symmetrical with respect to Oy.

If the equation is unchanged when the signs of x and y are changed, the curve is symmetrical with respect to the origin which is a center of the curve.

If the equation is unchanged when x and y are interchanged, the curve is symmetrical with respect to the line $y = x$.

If the equation is unchanged when x and y are interchanged and the signs of both x and y changed, the curve is symmetrical with respect to $x + y = 0$. $y = -x$

2. Examine for important points.

These are: the origin, the points where the curve cuts the axes, maximum and minimum points, and points of inflexion.

If $x = 0$, $y = 0$ satisfy the equation, the curve passes through the origin. Put $x = 0$ and solve for y to find the intercepts on Oy; put $y = 0$ and solve for x to find the intercepts on Ox.

Find the maximum and minimum and inflexion points by the regular methods of the text.

3. Determine the asymptotes, if any.

4. Compute a sufficient number of points on the curve to give a fair idea of the locus, and sketch the curve through the points.

(In the following examples all details, omitted in the hints, must be supplied.)

EXAMPLES.

1. Trace the *common parabola* $y = x^2$. The curve is symmetrical with respect to Oy. It passes through O and cuts neither axis elsewhere. Since $y' = 2x$ is 0 at O, Ox is tangent. Also, y' is positive as x continually increases from 0, and y, the ordinate, continually increases. Since $y'' = 2$ is always $+$, the curve is everywhere convex. Investigation shows that the curve has no asymptote. The form of the curve is as in the figure. (Fig. 27.)

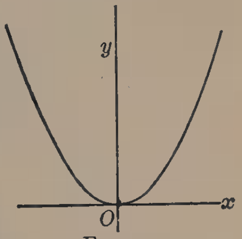


FIG. 27.

2. Trace

$$y = 2x^2 - 3x + 4.$$
$$y' = 4x - 3, \quad y'' = 4.$$

The curve is always convex. It has a minimum ordinate, $y = 2\frac{7}{8}$, at $x = \frac{3}{4}$. Its slope $\pm$ according as $x \lesseqgtr \frac{3}{4}$. It cuts Oy at $y = +4$, and neither axis elsewhere. It is symmetrical with respect to the line $x = \frac{3}{4}$. The curve is the common parabola. (Fig. 28.)

[illegible]

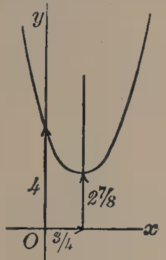


FIG. 28.

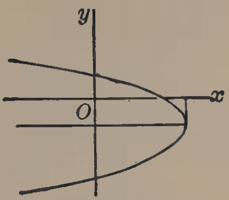


FIG. 29.

3. Trace $x = 3 - 2y - 3y^2$. Here x is a one-valued function of y. $D_y x = -2 - 6y$, $D_y^2 x = -6$. x is a maximum at $y = -\frac{1}{3}$, when $x = 3\frac{1}{3}$. The curve cuts Ox at $y = 0$, $x = 3$, and Oy at $y = +0.78$, $y = -1.44$. It is everywhere concave to Oy. x continually diminishes from its maximum value, and the curve has no asymptote at a finite distance. It is as before the common parabola. (Fig. 29.)

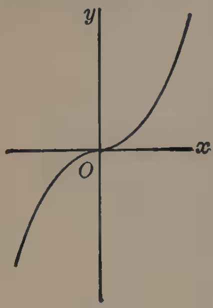


FIG. 30.

4. Trace the *cubical parabola* $y = x^3$. Here $y' = 3x^2$, $y'' = 6x$. The curve passes through O. It is symmetrical with respect to O, since the equation is unchanged when the signs of x and y are changed. The ordinate is + when x is +, and y is − when x is −. The curve lies in the first and third quadrants. In the first quadrant it is everywhere convex, in the third everywhere concave to Ox. It changes its curvature at the origin where there is concavo-convex inflexion. There are no asymptotes and the absolute value of y is ∞ when that of x is ∞. (Fig. 30.) (Read foot-note, p. 164.)

5. Trace the semi-cubical parabola $y^2 = x^3$. Ox is an axis of symmetry. The origin is on the curve. When x is −, y is imaginary and the curve does not exist in the plane to the left of Oy. $x = y^{\frac{2}{3}}$ is a one-valued function of y. $D_y x = \frac{2}{3} y^{-\frac{1}{3}}$ shows the slope ∞ at O with respect to Oy, and this slope is ± for y ± respectively. $D_y^2 x = -\frac{2}{9} y^{-\frac{4}{3}}$ is always negative, or the curve is concave to Oy. There are no asymptotes and x, y become ∞ together. (Fig 31.)

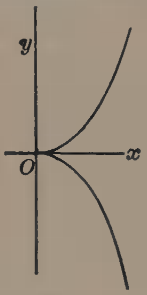


FIG. 31.

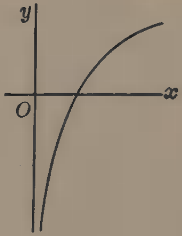


FIG. 32.

6. Trace the *logarithmic curve* $y = \log x$. We adopt the convention that the logarithm of a negative number is imaginary. Then the curve does not exist as a continuous function to the left of Oy. The ordinate is negative and infinite for $x = 0$, positive and infinite for $x = +\infty$, and is 0 when $x = 1$ where it cuts the axis Ox. The derivative $y' = 1/x$ is infinite for $x = 0$, which line is an asymptote. y' is always positive and decreases as x increases. The ordinate continually increases. $y'' = -x^{-2}$ is always −, hence the curve is everywhere concave and as in the figure. (Fig. 32.)

7. Trace the *exponential curve* $y = e^x$. y is always +, by convention e^x is +. $y = +\infty$ when $x = +\infty$; $y(=)0$ when $x = -\infty$. Also $y' = y'' = e^x$. The curve is always convex and increasing, and since $y'(=)0$ when $x = -\infty$, Ox is an asymptote. When $x = 0$, $y = 1$, where the curve cuts Oy. If we agree with some authors that y has negative values for $x = (2n + 1)/2m$, m and n being integers, then there will be a corresponding series of points representing the function lying below Ox on a curve represented by a dotted

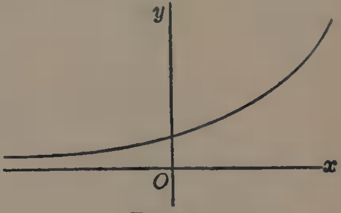


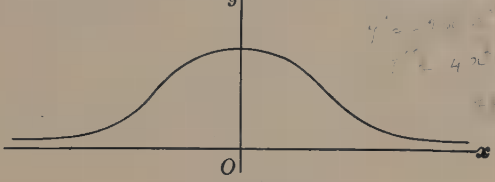
FIG. 33.

line symmetrical with that above Ox. The exponential curve, however, is conventionally taken to be the locus of the equation

$$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \equiv e^x.$$

The curve $y = e^x$ is identically the same as the curve in Ex. 6 if we interchange x and y. (Fig. 33.)

8. Trace the probability curve $y = e^{-x^2}$. The ordinate is always $+$; it has a maximum at 0, 1; it is 0 when x is $\pm\infty$. There is a concavo-convex inflexion at



$x = \pm\frac{1}{\sqrt{2}}$

FIG. 34.

$x = +1/\sqrt{2}$ and a convexo-concave inflexion at $x = -1/\sqrt{2}$. Ox is an asymptote in both directions, and Oy an axis of symmetry. Show that the curve is as in the figure. (Fig. 34.)

9. Trace the *cissoid* of Diocles, $(2a - y)x^2 = y^3$. The curve has Oy as an axis of symmetry, and passes through O, and cuts the axes nowhere else. Since

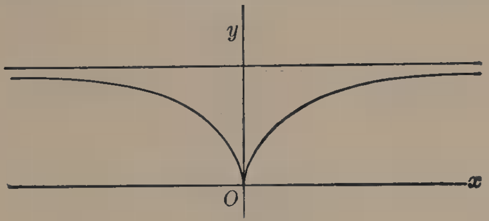


FIG. 35.

$y^3 + x^2y = 2ax^2$, y cannot be negative if a is positive. We find that $y = 2a$ is an asymptote in both directions, since $x = \pm\infty$ when $y = 2a$.

Again, corresponding to an assigned y, there are only two equal and opposite values of x. Therefore, for an assigned x, there is only one value of y. Also,

$$y' = 2x\,\frac{2a - y}{3y^2 + x^2}$$

is $\pm$ according as x is $\pm$. The curve is decreasing for x negative and increasing for x positive. To find y' at the origin, the above value of y' is indeterminate. But we have directly from the equation to the curve

$$\underset{x(=)0}{\mathcal{L}} \left(\frac{y}{x}\right)^2 = \mathcal{L}\,\frac{2a - y}{y} = \infty,$$

which is the slope of the curve at O. Therefore Oy is tangent to the curve. The origin, like that in Ex. 5, is a singular point which we call a *cusp*. By plotting a sufficient number of points, we find the curve to have the form as drawn in the figure. (Fig. 35.)

10. Trace the *witch* of Agnesi, $y = 8a^3/(x^2 + 4a^2)$. The ordinate is always $+$, and has a maximum $y = 2a$, at $x = 0$. Oy is an axis of symmetry, and Ox an asymptote. There are inflexions at $x = \pm 2a/\sqrt{3}$. (Fig. 36.)

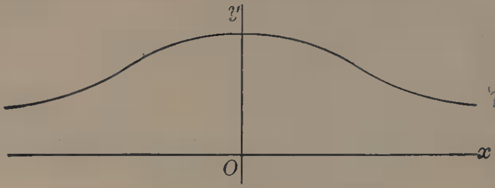


FIG. 36.

$\frac{16a^3x}{(x^2+4a^2)^2}$

$y'' = (x^2+4a^2)(-16a^3) + 16a^3x \cdot 4x$

$48a^3x^2 - 64a^5 = 0$

$3x^2 - 4a^2 = 0$

$x^2 - 2ax$

$2x - 2a$

11. Trace the *cubic* $a^2y = \frac{1}{3}x^3 - ax^2 + 2a^3$, in which a is positive. There is a maximum $y = 2a$ at $x = 0$; and a minimum $y = \frac{2}{3}a$, at $x = 2a$. An inflexion occurs at $x = a$. For $x < a$ the curve is concave, and for $x > a$ convex. There are no asymptotes. The curve crosses Ox between $x = -a$ and $x = -2a$. Also, $y = \pm\infty$ when $x = \pm\infty$. (Fig. 37.)

$-\frac{1}{3}a^3 \pm a^3 + 2a^3 +$

$- 4a^3 + 2a^3$

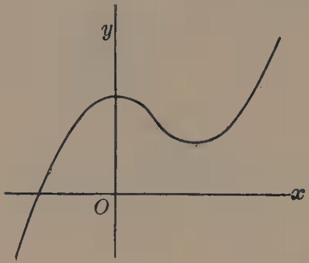


FIG. 37.

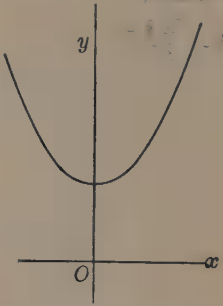


FIG. 38.

12. Trace the *catenary*, $y = \frac{1}{2}a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$, in which a is a positive constant. The curve is the form of a heavy flexible inextensible chain hung by its ends. The ordinate y is a minimum and equal to a when $x = 0$, and is $+$ for all values of x. The curve is convex everywhere. $y = +\infty$ when $x = \pm\infty$, and there are no asymptotes. The slope continually increases with x. (Fig. 38.)

13. Trace the *cubical parabola* $x^2 = y^2(y - a)$, where a is positive.

Since $x = \pm y\sqrt{y - a}$, the point 0, 0 is on the curve. But *no other point in the neighborhood of the origin is on the curve*, since for such values of y, x is imaginary. The origin is therefore a remarkable point, it is an *isolated* point of the curve, and such points are called *conjugate* points. For each value of y greater than a there are two equal and opposite values of x. The curve is symmetrical with respect to Oy. The ordinate y is a minimum at $x = 0$, where the tangent is horizontal. $y'' = 0$ gives inflexions at $y = \frac{4}{3}a$, $x = \pm\frac{4}{3\sqrt{3}}a^{\frac{3}{2}}$, which for $x +$ is convexo-concave and for $x -$ is concavo-convex. There is no asymptote, and $y = +\infty$ for $x = \pm\infty$. (Fig. 39.)

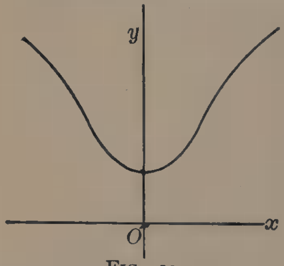


FIG. 39.

14. Trace $y = (x^2 - 1)^2$. The curve lies above Ox and has Oy for an axis of symmetry. y has a maximum at $x = 0$, and minima at $x = \pm 1$. There are inflexions at $x = \pm 1/\sqrt{3}$.

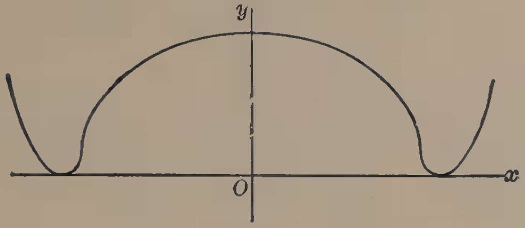


FIG. 40.

The infinite branches have no asymptote. (Fig. 40.)

15. Trace the curve $y = \left(1 + \frac{1}{x}\right)^x$

The ordinate has the limit e when $x = \pm \infty$. This is the important limit on which differentiation was founded. y has the limit 1 when $x = 0$ and continually increases with x. For $-1 < x < 0$ the curve does not exist. The point 0, 1 is what is called a *stop* point, the branch ending abruptly there. For $x < -1$, and converging to -1, y is greater than e and is ∞. As x decreases to $-\infty$, the curve decreases continually and becomes asymptotic to $y = e$. (Fig. 41.)

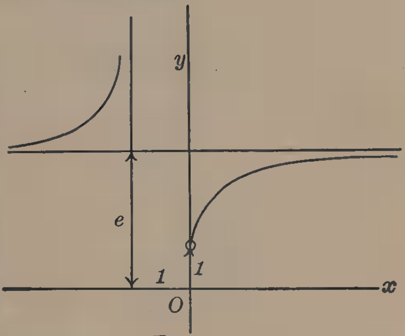


FIG. 41.

EXERCISES.

1. Trace the curves $y = \sin x$, $y = \cos x$.

2. Trace $y = \tan x$, $y = \cot x$.

3. Trace $y = \sec x$, $y = \csc x$.

4. Trace $y = \text{vers}\, x = 1 - \cos x$.

5. Trace $y = e^{-\frac{1}{x}}$.

6. Trace the curves

$$xy = 1, \quad (x - 1)(y - 2) = 3, \quad y(x - 1)(x - 2) = 1.$$

7. Trace the curve $y(x - 1)(x - 2) = (x - 3)(x - 4)$.

8. Trace $y(a^2 + x^2) = a^2(a - x)$.

9. Trace $x^2(y - a) = a^3 - xy^2$.

10. Trace $a^2x = y(x - a)^2$.

11. Trace $y^3 = x^2(2a - x)$.

12. Trace $(x^2 + 4) = yx^3$.

13. Trace $3x(1-x)y = 1 - 5x$.

14. Trace the *quadratrix* $y = x \tan \dfrac{\pi(a-x)}{2a}$.

15. Trace the curve $y = \sin(\pi \sin x)$.

16. Trace $y = (2x-a)^{\frac{1}{3}}(x-a)^{\frac{2}{3}}$.

112. Implicit Functions.—In general, when the equation to a curve is given in the implicit form $f(x, y) = 0$, and we cannot solve for either variable, the investigation requires more advanced treatment than we are prepared to give here. This subject will be taken up again in Book II. The ordinates to such curves are, in general, several-valued functions of the variable.

We give here simple examples of important curves. The student will do well to study the hints given in tracing such curves.

15. Trace the hypocycloid of four cusps,

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

The curve is symmetrical with respect to O, Ox, and Oy. There are two equal and opposite values of y to each x, and two of x to each y, for either variable less than a. The curve does not exist for values of x or y greater than a. We have in the first quadrant

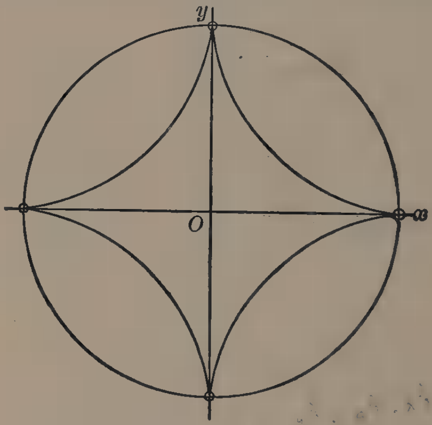


FIG. 42.

$$y' = -\left|\frac{y}{x}\right|^{\frac{1}{3}},$$

and the curve is tangent to Ox at $x = a$, and to Oy at $x = 0$, $y = a$. y'' being positive in the first quadrant, the curve is convex at any point on it. The curve is sometimes called the asteroid. It is the locus of a fixed point on the circumference of a circle as that circle rolls inside the circumference of another circle whose radius is four times that of the rolling circle. (Fig. 42.)

16. Trace the evolute of the ellipse

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$$

in the same way as above. (Fig. 43.)

Show by inspection that four normals can be drawn to the ellipse from any point inside the evolute.

From what points can 1, 2, or 3 normals be drawn?

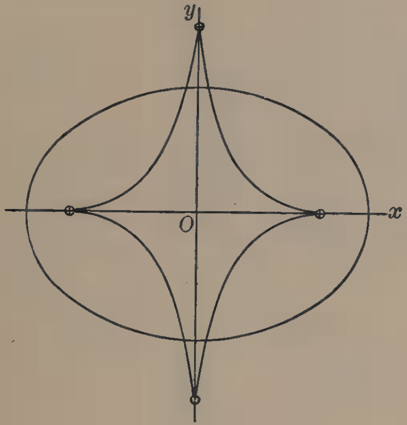


FIG. 43.

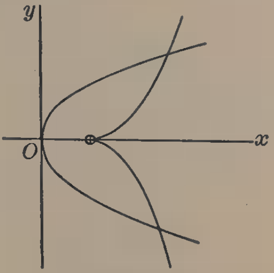


FIG. 44.

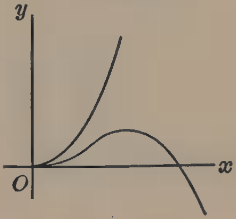


FIG. 45.

17. Trace the parabola $y^2 = 4px$, and its evolute, $4(x - 2p)^3 = 27py^2$. Show that the curves are as drawn. Find the angle at which they intersect. Show from which points in the plane can be drawn 1, 2, or 3 normals to the parabola. (Fig. 44.)

18. Trace the curve $(y - x^2)^2 = x^5$.

$$\therefore \quad y = x^2\left(1 \pm x^{\frac{1}{2}}\right).$$

There are two branches,

$$y = x^2\left(1 + x^{\frac{1}{2}}\right), \quad y = x^2\left(1 - x^{\frac{1}{2}}\right).$$

The first continually increases as x increases from 0. The second increases, attains a maximum, and then descends indefinitely, crossing Ox at $x = 1$. Both branches are tangent to Ox at O since

$$y' = 2x \pm \tfrac{5}{2}x^{\frac{3}{2}}$$

is 0 when $x = 0$. The curve does not exist in the plane to the left of Oy. Examine for asymptotes. Find the inflexion and the maximum ordinate. The origin is a *singular point* called a *cusp* of the *second species*. (Fig. 45.)

19. Trace in the same way the curve

$$x^4 - 2ax^2y - axy^2 + a^2y^2 = 0.$$

20. Trace the curve $y^2 = (x + 1)x^2$.

y is a two-valued function of x,

$$y = \pm\, x\sqrt{x + 1}.$$

Ox is an axis of symmetry.

The curve passes through the origin in two branches,

$$y = +\,x\sqrt{x + 1}, \quad y = -\,x\sqrt{x + 1}.$$

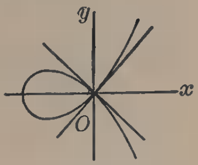


FIG. 46.

The curve does not exist in the plane to the left of $x = -1$. Between -1

and 0 the ordinate is finite, having a maximum and a minimum. We have for the slopes of the two branches passing through O at $x = 0$,

$$\underset{x(=)0}{\mathcal{L}} \frac{y}{x} = \pm \mathcal{L} \sqrt{x+1} = \pm 1.$$

As x increases positively, y increases without limit in absolute value. Are there asymptotes? (Fig. 46.)

The point in which two branches of the same curve cross each other, having two distinct tangents there, is called a *node*. In this curve the origin is a node.

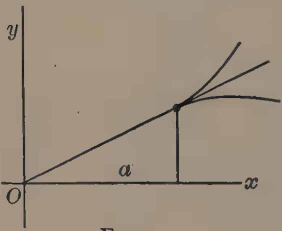


FIG. 47.

21. Trace the curve $(bx - cy)^2 = (x - a)^5$.

Clearly, $x = a, \quad y = ab/c,$

is on the curve. But these values make the derivative y' indeterminate. Differentiate the equation twice.

$$\therefore \quad (b - cy')^2 - (bx - cy)cy'' - 10(x - a)^3 = 0,$$

and at the point $x = a, \quad y = ab/c,$

$$(b - cy')^2 = 0,$$

gives $y' = b/c$. Since y is imaginary when $x < a$,

and $$y = \frac{b}{c}x \pm \frac{1}{c}\sqrt{(x - a)^5},$$

the curve is as in the figure. The point a, ab/c, is a *cusp* of the *first species*. (Fig. 47.)

22. Trace the curve $4y^2 = 4x^3 + 12x^2 + 9x$.

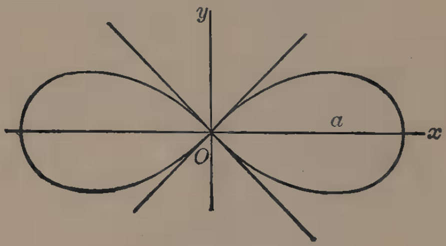


FIG. 48.

23. Trace the lemniscate,

$$(x^2 + y^2)^2 = a^2(x^2 - y^2).$$

$$x^2 + y^2 = + a\sqrt{x^2 - y^2}$$

shows that y cannot be greater than x and only equal to x when they are both 0 at the origin. The curve is symmetrical with respect to O, Ox, Oy. Also,

$$x + \frac{y}{x}y = a\sqrt{1 - \left(\frac{y}{x}\right)^2},$$

and since $y \leqq x$, we have, when $x = 0$, $y = 0$,

$$\mathcal{L}\frac{y}{x} = \pm 1, \qquad \text{(See Ill. (2), § 79.)}$$

which are the slopes of the two branches of the curve passing through the origin. Again,

$$y' = \frac{x}{y} \cdot \frac{a^2 - 2(x^2 + y^2)}{a^2 + 2(x^2 + y^2)}.$$

In the first quadrant y' is $+$ from $x = 0$ to the point determined by

$$2(x^2 + y^2) = a^2, \quad 4(x^2 - y^2) = a^2,$$

where it changes sign, giving y a maximum, and y' decreases until $y' = \infty$ at $x = a$, $y = 0$.

Being symmetrical with respect to the axes the curve is as in the figure. No part of the curve exists for $x > a$, since the equation is of the fourth degree and a straight line cannot cut the curve in more than four points.

Put $y = mx$, and plot points on the curve by assigning different values to m. Thus, in terms of the third variable m, we have

$$x = \pm a \frac{\sqrt{1 - m^2}}{1 + m^2}, \quad y = \pm am \frac{\sqrt{1 - m^2}}{1 + m^2}. \qquad \text{(Fig. 48.)}$$

113. General Considerations in Tracing Algebraic Curves.— The equation of any algebraic curve when rationalized is of the form of a polynomial of the nth degree in x and y. It can always be written

$$0 = u_0 + u_1 + \ldots + u_n \equiv U, \qquad (1)$$

where u_0 is the constant term (independent of x and y), u_1, u_2, etc., are homogeneous functions or polynomials in x, y of respective degrees 1, 2, etc.

If $u_0 = 0$, the origin is a point on the curve.

(1). To find the tangent at the origin when $u_1 \neq 0$.

When $u_0 = 0$, the line $y = mx$ intersects the curve at O.

Substitute mx for y in the equation to the curve. Then, if $u_1 \equiv px + qy$, the equation (1) becomes

$$(p + qm)x + T_2 + \ldots = 0, \qquad (2)$$

where the terms T_2, etc., contain higher powers of x than the first. Divide the equation (2) by x, which factor accounts for one 0 root. Then let $x = 0$, and (2) becomes

$$p + qm = 0, \quad \text{or} \quad m = -p/q.$$

This value of m is the slope of the curve at the origin, since now the line $y = mx$ cuts the curve in two coincident points at the origin, and

$$u_1 \equiv px + qy = 0$$

is the equation of the tangent at the origin.

If $u_0 = 0$, $u_1 = 0$, and $u_2 \equiv rx^2 + sxy + ty^2$.

Then, as before, put mx for y and the equation becomes

$$(r + sm + tm^2)x^2 + T_3 + \ldots = 0, \qquad (3)$$

where the terms T_3, etc., contain higher powers of x than 2.

Divide by x^2, which accounts for two zero roots of (3); in the result put $x = 0$.

$$\therefore \; tm^2 + sm + r = 0 \qquad (4)$$

is a quadratic giving two values of m, the two slopes of the curve at O. The equation to the two tangents at O is

$$u_2 \equiv rx^2 + sxy + ty^2 = 0.$$

These are real and different, real and coincident, or imaginary, according as the roots of the quadratic (4) in m are real and unequal, equal, or imaginary. The origin being a double point called a node, cusp, or conjugate point accordingly.

In like manner if also $u_2 = 0$, the equation of the three tangents at O is

$$u_3 = 0,$$

and the origin is a triple point.

Hence, when the origin is on the curve, the homogeneous part of the equation of lowest degree equated to 0 is the equation of the tangents at O.

Further discussion of singular points and method of tracing the curve at a singular point will be given in Book II.

(2). A straight line cannot meet a curve of the nth degree in more than n points. For, if we put $mx + b$ for y in $U = 0$, we have an equation of the nth degree in x for finding the abscissæ of the points of intersection of $y = mx + b$ and $U = 0$.

If now u_r is the term of lowest degree in U, and we put mx for y in U, then x^r is a factor and represents r roots equal to 0. The line $y = mx$ cuts the curve $U = 0$, r times at the origin, and therefore cannot cut it in more than $n - r$ other points. This will frequently enable us to construct a curve by points, when otherwise the computations would be quite difficult.

(3). Singular Points. A point through which two or more branches of a curve pass is called a *singular* point. Illustrations have been given of nodes, cusps, and conjugate points.

At a singular point on a curve $D_x y$ is indeterminate. Points at which $D_x y$ is determinate and unique are called points of ordinary position, or ordinary points.

To find a singular point on a curve $\phi(x, y) = 0$, differentiate with respect to x. The result will be

$$M + Ny' = 0, \tag{1}$$

where M and N are functions of x and y. At a singular point y' is indeterminate and $M = 0$, $N = 0$. Any pair of values of x, y satisfying the equations

$$\phi = 0, \quad M = 0, \quad N = 0$$

is a singular point. If (1) be differentiated again, we have

$$P + Qy' + Ry'^2 + Ny'' = 0.$$

At the singular point $N = 0$, leaving a quadratic in y' for determining the slopes of the curve, if the point is a double point. If a triple point, another differentiation will give a cubic in y' for determining the slopes, etc.

If the curve has a singular point whose coordinates are α, β, and we transform the origin to the singular point by writing $x + \alpha$, $y + \beta$ for x and y in the equation to the curve, the construction of the curve will be simplified as in (1), (2).

EXAMPLES.

24. Trace the *lemniscate*, Ex. 23.

$$(x^2 + y^2)^2 - a^2(x^2 - y^2) = 0.$$

Here $u_2 = x^2 - y^2 = 0$ is the equation to the tangents at 0, or $y = \pm x$, as before in Ex. 23.

Put $y = mx$ in the equation and compute a number of points. Clearly m cannot be greater than 1.

25. Trace the *folium* of Descartes,

$$x^3 + y^3 - 3axy = 0.$$

The equation of the tangents at the origin is $3xy = 0$, or $x = 0$, $y = 0$. We find that

$$x + y + a = 0$$

is the only asymptote. Put $y = mx$, then

$$x = \frac{3am}{1 + m^3}, \quad y = \frac{3am^2}{1 + m^3}.$$

x, y are finite for $0 < m < +\infty$. Compute a number of points corresponding to assigned values of m. Observe that if we change m into $1/m$, x and y interchange values. The curve is symmetrical with respect to the line $y = x$. In the first quadrant there is a loop, the farthest point from the origin being $x = y = \frac{3}{2}a$. Determine the maximum values of x and y for this loop. For negative values of m we construct the infinite branches above the asymptote, since $y = mx$ cuts the curve before it does the asymptote. (Fig. 49.)

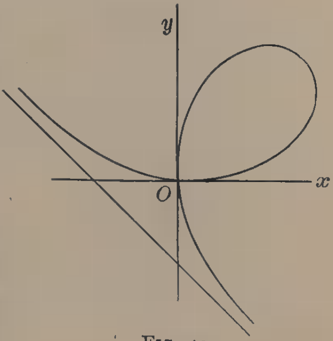


FIG. 49.

26. Trace the curve $(y - 2)^2(x - 2)x = (x - 1)^2(x^2 - 2x - 3)$.
Examining for singular points, we find

$$y' = \left(1 + \frac{3}{x^2(x - 2)^2}\right)\frac{x - 1}{y - 2}.$$

Therefore $x = 1$, $y = 2$ is a singular point. Transform the origin to this point by writing $x + 1$ for x, $y + 2$ for y.

Then the equation becomes

$$y^2(x^2 - 1) = x^2(x^2 - 4).$$

Examining for asymptotes, we find the asymptotes $x = \pm 1$, $y = \pm x$. The equation to the tangents at O is $y^2 = 4x^2$, $\therefore y = \pm 2x$. When $y = 0$; $x = \pm 2$, $x = 0$. The curve is symmetrical with respect to Ox, Oy, and O. We need therefore trace it only in the first quadrant, in order to draw the whole curve.

The line $y = mx$ cuts the curve in points whose coordinates are

$$x = \sqrt{\frac{4 - m^2}{1 - m^2}}, \quad y = m\sqrt{\frac{4 - m^2}{1 - m^2}}.$$

These increase continually as m increases from 0 to 1, and the branch approaches the asymptote as drawn. The coordinates are imaginary for $1 < m < 2$, and when $m = 2$: $x = 0$, $y = 0$. As m increases from 2 to $+\infty$, x and y are real and increasing, and $m = \infty$ gives $x = \pm 1$, $y = \infty$, the curve approaches the asymptote as drawn. The origin is an inflexional node. (Fig. 50.)

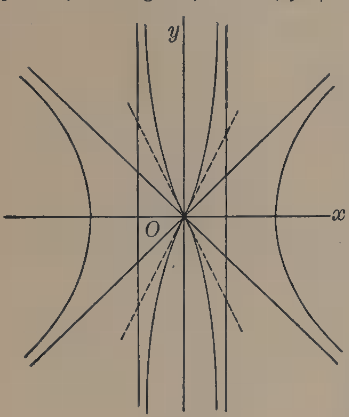


FIG. 50.

27. Trace the curve $(x+3)y^2 = x(x-1)(x-2)$.

28. Trace the curve $a^3y^2 = bx^4 + x^5$.

29. Trace the *dumb-bell* $a^4y^2 = a^2x^4 - x^6$.

30. Show that $x^5 + y^5 = 5ax^2y^2$ has the form given in Fig. 51.

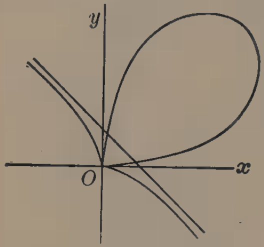


FIG. 51.

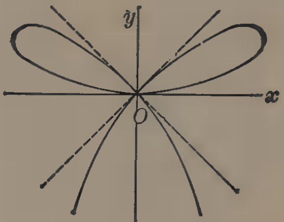


FIG. 52.

31. Trace $x^4 = (x^2 - y^2)y$.

The lowest terms are of third degree. The origin is a triple point. The tangents there being $y = 0$, $y = \pm x$. Oy is an axis of symmetry. There are no asymptotes. The line $y = mx$ cuts the curve in one point, besides the origin, whose coordinates are

$$x = m(1 - m^2), \quad y = m^2(1 - m^2).$$

This shows that there are two loops, in the first and fourth octants, and infinite branches in the sixth and seventh octants. The curve is a *double bow-knot* and has no asymptotes. (Fig. 52.)

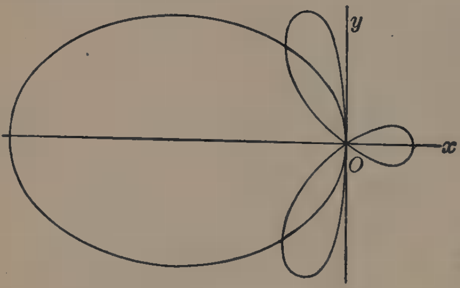


FIG. 53.

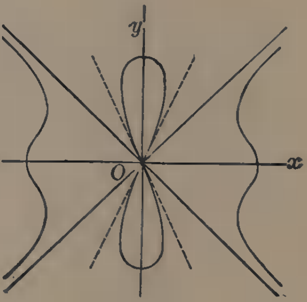


FIG. 54.

32. Trace the curves

$$y^3 = ax^2 - x^3, \quad y^3 = a^3 - x^3, \quad y^2(x-a) = (x-b)x^2.$$

33. Trace the *conchoid* of Nicomedes,

$$(x^2 + y^2)(b - y)^2 = a^2y^2, \quad \text{when} \quad b =, <, > a.$$

34. Trace the curves

$$y = (x-1)(x-2)(x-3), \quad a^2x = y(b^2 + x^2), \quad x^4 - y^4 + 2axy^2 = 0.$$

35. Show that $x^2y^2 + x^4 = a^2(x^2 - y^2)$ consists of two loops and find the form of the curve.

36. Show that the *scarabeus*

$$4(x^2 + y^2 + 2ax)^2(x^2 + y^2) = b^2(x^2 - y^2)^2$$

has the form given in Fig. 53.

37. Show that the *devil*

$$y^4 - x^4 + ay^2 + bx^2 = 0, \quad \text{where} \quad a = -24, \quad b = 25,$$

has the figure given (Fig. 54).

114. Tracing Polar Curves.—As in Cartesian coordinates, no fixed rule can be given for tracing these curves. The general directions are the same as before. The particular points are:

(1). Compute values of ρ corresponding to assigned values of θ, or vice versa, according to convenience. Plot a sufficient number of points to give a fair idea of the general position of the curve.

(2). Determine the asymptotes, by finding values of θ which make $\rho = \infty$ for the directions of the asymptotes. Place the asymptote in position by evaluating the limit of $\rho^2 D_\rho \theta = -D_u \theta$, for the perpendicular distance of the asymptote from the origin, as previously directed. Examine for asymptotic points and circles.

(3). The direction of a polar curve at any computed point is given by $\tan \psi = \rho/\rho'$.

(4). Examine for axes or points of symmetry.

(5). Examine for maximum and minimum values of ρ and for points of inflexion.

(6). Examine for periodicity.

115. Inverse Curves.—If $f(\rho, \theta) = 0$ is the polar equation to any curve, then $f(\rho^{-1}, \theta) = 0$ is the polar equation of the *inverse* curve.* We have been accustomed to put $\rho^{-1} = u$, so that $f(u, \theta) = 0$ is the equation of the inverse curve.

1. Show that if x, y are the rectangular coordinates of a point on a curve, the equation to the inverse curve is obtained by substituting

$$\frac{x}{x^2 + y^2}, \quad \frac{y}{x^2 + y^2}$$

for x and y in the equation to the given curve.

2. Show that the asymptotes of any curve are the tangents at the origin to the inverse curve.

3. Show that a straight line inverts into a circle and conversely. Note the case when it passes through the origin.

4. Show that the inverse of the hyperbola with respect to its centre is the lemniscate.

EXAMPLES.

38. Trace the *spiral of Archimedes*, $\rho = a\theta$. The distance from the pole is proportional to the angle described by the radius vector. $\tan \psi = \theta$. The curve is tangent to the initial line at O. The intercept PQ between two consecutive revolutions is constant and equal to $2\pi a$. Therefore we need only construct one turn directly. The dotted line shows the curve for negative values of θ, which is the same as the heavy line revolved about a perpendicular to the initial line through O. (Fig. 55.)

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* More generally two polar curves are the inverses of each other, when for the same θ their radii vectores are connected by $\rho_1\rho_2 = k^2$. $k =$ constant.

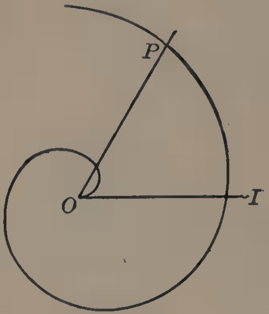

FIG. 55.

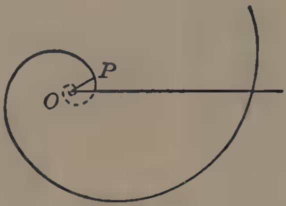

FIG. 56.

39. Trace the *equiangular* spiral $\rho = e^{\frac{\theta}{b}}$. We can write the equation

$$\theta = b \log \rho,$$ $\rho' = \frac{1}{b} e^{\theta/b}$. $\rho/\rho' = b$.

if we prefer. $\tan\psi = b$, or the angle between the radius and tangent is constant. $\rho = a$ for $\theta = 0$, and ρ increases as θ increases. $\rho (=) 0$ for $\theta = -\infty$. The pole O is an asymptotic point. (Fig. 56.)

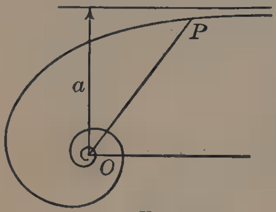

FIG. 57.

40. Trace the *hyperbolic* or *reciprocal spiral* $\rho\theta = a$. The pole O is an asymptotic point. A line parallel to the initial line at a distance a above it is an asymptote. For negative values of θ, rotate the curve through π about a normal to OA at O. (Fig. 57.)

41. Trace the *lemniscate* $\rho^2 = 2a^2 \cos 2\theta$.

42. Trace the conchoid $\rho = a \sec\theta \pm b$,

or $$(x^2 + y^2)(x - a)^2 = b^2x^2.$$

When $a < b$, there is a loop; when $a = b$, a cusp; when $a > b$, there are two points of inflexion. (Fig. 58.)

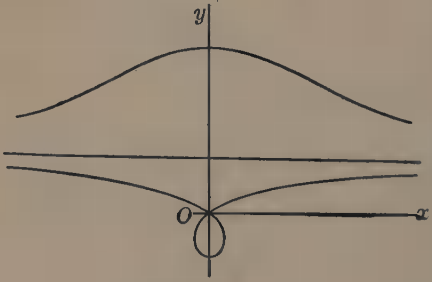

Fig. 58.

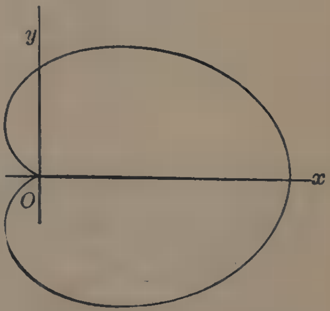

FIG. 59.

43. Trace the *cardioid* $\rho = a(1 + \cos\theta)$. The curve is finite and closed, symmetrical with respect to Ox. $\rho = 2a, a, 0$, for $\theta = 0, \frac{1}{2}\pi, \pi$, and diminishes continually as θ increases from 0 to π. Also, $\tan\psi = -\cot\frac{1}{2}\theta$. As $\theta (=) \pi$, $\psi (=) \pi$, or the curve is tangent to Ox at the pole, which point is a cusp. The rectangular equation is

$$x^2 + y^2 - ax = + a\sqrt{x^2 + y^2}. \qquad \text{(Fig. 59.)}$$

44. Trace the *three-leaved clover* $\rho = a\cos 3\theta$.

45. Trace the curves :

(1). $\rho = a\sin 2\theta$, $\rho = a\cos 2\theta$.
(2). $\rho = a\sin 3\theta$, $\rho = a\sin 4\theta$.
(3). $\rho = a\sec^2\frac{1}{2}\theta$, $\rho = a\sec\theta$.
(4). $\rho = a\sin\theta$, $\rho = a\sin^3\frac{1}{3}\theta$.

46. Trace the curve $\rho(\theta^2 - 1) = a\theta^2$.

47. Trace $\rho = a \operatorname{vers}\theta$ and $\rho = a(1 - \tan\theta)$.

48. Trace the evolutes of $y = \sin x$ and $y = \tan x$.

49. The *Cycloid.* The path described by a point on the circumference of a circle which rolls, without sliding, on a fixed straight line is called the cycloid.

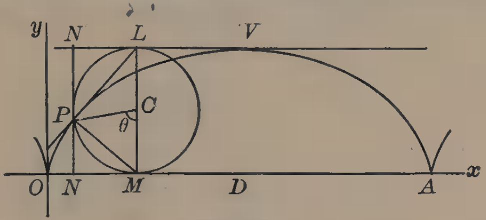

FIG. 60.

(1). Let the radius of the rolling circle MPL be a, the point P the generating point, M the point of contact with the fixed straight line Ox which is called the base. Take MO equal to the arc MP; then O is the position of the generating point when in contact with the base. Let O be the origin and x, y the coordinates of P, $\angle PCM = \theta$.

Then we have

$$x = OM - NM = a(\theta - \sin\theta), \quad y = PN = a(1 - \cos\theta).$$

The coordinates are then given in terms of the angle θ through which the rolling circle has turned. $OA = 2\pi a$ is called the base of one arch of the cycloid. The highest point V is called the vertex. Eliminating θ, we have the rectangular equation

$$x = a\cos^{-1}\frac{a - y}{a} - \sqrt{2ay - y^2}. \qquad \text{(Fig. 60.)}$$

(2). To find the equations to the cycloid when the vertex is the origin, the tangent and normal there are the axes of x and y, we have directly from the figure

$$x = a\theta + a\sin\theta, \quad y = a - a\cos\theta.$$

Eliminating θ for the rectangular equation,

$$x = a\cos^{-1}\frac{a - y}{a} + \sqrt{2ay - y^2}. \qquad \text{(Fig. 61)}$$

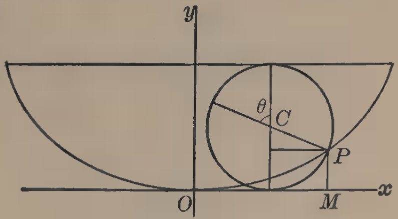

FIG. 61.

The cycloid is one of the most important curves.

50. The *Trochoids.* When a circle rolls on a fixed straight line, any point rigidly fixed to the rolling circle traces a curve called a trochoid. The curve is called the epitrochoid or hypotrochoid according as the tracing point is outside or inside the rolling circle.

Their equations are determined directly from the figure.

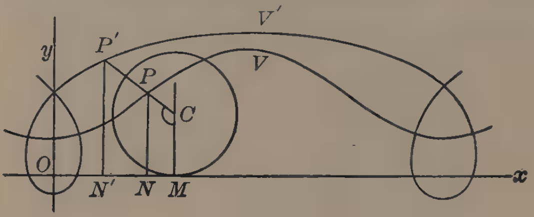

FIG. 62.

Let $CM = a,\quad CP = p,\quad CP' = p',\quad \angle MCP = \theta.$

Then $x = ON = a\theta - p\sin\theta,\quad y = PN = a - p\sin\theta,$

for a point P on the hypotrochoid PV. Replacing p by p', the same equations give the epitrochoid. (Fig. 62.)

51. Epicycloids and hypocycloids.

The curve traced by any point on a circle which rolls on a fixed circle is called an epicycloid or a hypocycloid, according as the circle rolls on the outside or on the inside of the fixed circle. (Fig. 63.)

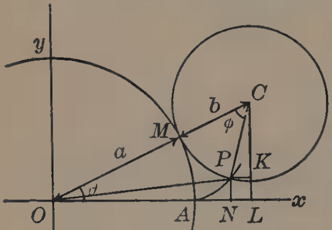

FIG. 63.

Let O be the center of the fixed circle of radius a, and C the center of the rolling circle of radius b, and P the tracing point. Then with the notations as figured, we have

arc AM = arc PM, or $a\theta = b\phi$. Hence

$$x = ON = OL - NL,$$
$$= (a+b)\cos\theta - b\cos(\theta+\phi),$$
$$= (a+b)\cos\theta - b\cos\frac{a+b}{b}\theta;$$
$$y = PN = CL - CK = (a+b)\sin\theta - b\sin(\theta+\phi),$$
$$= (a+b)\sin\theta - b\sin\frac{a+b}{b}\theta,$$

for the coordinates of the epicycloid. For the hypocycloid change the sign of b.

In this book convexity or concavity of a curve at a point is fixed by the sign of the second derivative of the ordinate representing the function. $D_x^2 y = +$ or $D_y^2 x = +$ means convexity with respect to O_x or O_y respectively. This is the equivalent of viewing the curve from the end of the ordinate at $-\infty$, instead of from the foot of the ordinate as is sometimes done.

PART III.

PRINCIPLES OF THE INTEGRAL CALCULUS.

CHAPTER XVI.

ON THE INTEGRAL OF A FUNCTION.

116. Definition.—The product of a *difference* of the variable $x_2 - x_1$ into the value of the function $f(x)$ taken anywhere in the interval (x_1, x_2) is called an *element.*

In symbols, if z is either of the numbers x_1 or x_2, or any assigned number between x_1 and x_2, the product

$$(x_2 - x_1) f(z)$$

is the element corresponding to the interval (x_1, x_2).

GEOMETRICAL ILLUSTRATION.

If $y = f(x)$ is represented by the curve AB in any interval (a, b), and x_1, x_2 are any two values of x in (a, b), then the *element* corresponding to (x_1, x_2) is represented by the area of any rectangle $x_1M_1Mx_2$, whose base is the interval $x_2 - x_1$, and altitude is the ordinate zZ to any point on the curve segment P_1P_2.

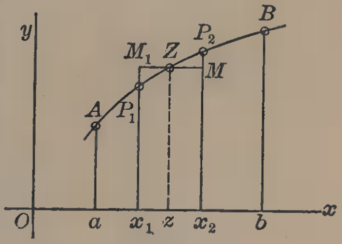

FIG. 64.

117. Definition.—The *integral* of a function $f(x)$ corresponding to an assigned interval (a, b) of the variable is defined as follows:

Divide (a, b) into n partial or sub-intervals (a, x_1), (x_1, x_2), . . . , (x_{n-2}, x_{n-1}), (x_{n-1}, b), by interpolating between a and b the numbers x_1, . . . , x_{n-1} taken in order from a to b. And for continuity of expression let $x_0 \equiv a$, $x_n \equiv b$.

The *integral* of a function is the *limit* of the sum of the *elements* corresponding to the n sub-intervals, when the *number* of these sub-intervals is increased indefinitely and at the same time *each* sub-interval converges to zero.

In symbols, we have for the integral of $f(x)$ corresponding to the interval (a, b),

$$\underset{n=\infty}{\overset{x_r(=)x_{r-1}}{\mathcal{L}}} \sum_{r=1}^{r=n} (x_r - x_{r-1})f(z_r).$$

In which z_r is either x_r, x_{r-1} or some number between x_r and x_{r-1}, or as we say, briefly, some number *of* the interval (x_{r-1}, x_r). At the same time that $n = \infty$ we must have $x_r - x_{r-1} (=) 0$.

GEOMETRICAL ILLUSTRATION.

If $y = f(x)$ is represented by a continuous and one-valued ordinate to a curve, then the integral of $f(x)$ for the interval (a, b) is represented by the area of the surface bounded by the curve, the x-axis, and the ordinates at a and b.

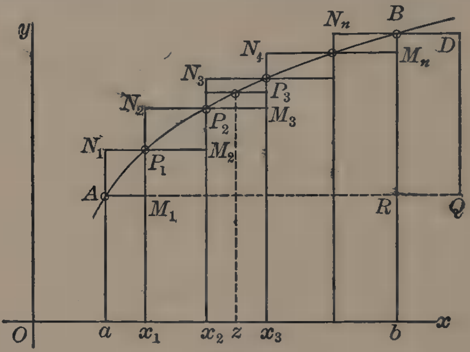

FIG. 65.

For, any *elementary* area, such as $(x_3 - x_2)f(z_3)$, lies between the areas of the rectangles $x_2P_2M_3x_3$ and $x_2N_3P_3x_3$ constructed on the subinterval (x_2, x_3), or is equal to one of them, according as $z_3 = x_2$, $z_3 = x_3$. Also, the corresponding area $x_2P_2P_3x_3$ bounded by the curve P_2P_3, Ox, and the ordinates at x_2 and x_3 lies between the areas of the same rectangles, in virtue of the continuity of $f(x)$, when $x_3 - x_2$ is made sufficiently small.

Hence the sum of the integral elements and the fixed area of the curve lie between the sum of the rectangular areas

$$N_1x_1 + N_2x_2 + \ldots + N_nb \qquad (1)$$

and

$$M_1a + M_2x_1 + \ldots + M_nx_{n-1}. \qquad (2)$$

Let RQ be not greater than the greatest of the subintervals into which (a, b) is divided. The difference between the areas (1) and (2) is not greater than the area of the rectangle $BDQR$, whose base is RQ and whose altitude BR is equal to the difference $f(b) - f(a)$ and to the sum of the altitudes of N_1M_1, N_2M_2, . . . , N_nM_n. This rectangle BQ has the limit 0, since each subinterval has the limit 0; and so also has RQ, while its altitude is finite and constant, or does not change with n.

Consequently the areas (1) and (2) converge to the constant area of the curve which lies between them, and so also must the area represented by the sum of the elements of the integral.

Hence the integral of $f(x)$ for (a, b) is equal to the area of the curve, as enunciated.

118. Evaluation of the Integral of a Function.[*]—In order that a function shall admit of the limit which we call the integral for a given interval, the function must, in general, be finite and continuous throughout the interval.

Should the function be finite and continuous everywhere in the interval (a, b) except at certain isolated values of the variable, at which singular points it is discontinuous, either infinite or indeterminately finite, then special investigation is necessary for such singular values, and we omit the consideration of them.

We shall assume that the functions considered are uniform, finite, and continuous throughout the interval, unless specially mentioned otherwise.

The process of evaluating the *limit* defined as the integral, in § 117, is called *integration.*

In evaluating the limit

$$\mathop{\pounds}_{n=\infty} \sum_{1}^{n}(x_r - x_{r-1})f(z_r), \qquad x_r - x_{r-1}(=)0.$$

we are said to integrate the function f from $a \equiv x_0$ to $b \equiv x_n$. The numbers a and b are called the boundaries or limits of the integration or integral. The lesser of the numbers a and b is called the *inferior,* the greater the *superior,* limit of the integration.[†]

In the differentiation of the elementary functions

$$x^a, \quad a^x, \quad \log x, \quad \sin x,$$

and like functions of them and their finite algebraic combinations, we have seen that the derivative could always be evaluated in terms of these same functions. Not so, however, is the case in evaluating the integrals of these functions. The integral cannot be always expressed in terms of these same functions, and when this is the case, the integral itself is a new function in analysis which takes us beyond the range of the elementary functions such as we have defined them to be.

We shall be interested, in this book, directly with only those functions whose integrals can be evaluated in terms of the elementary functions.

It can be stated in the beginning that there is no regular and systematic law known by which the integral of a given function can be determined as a function of its limits in general.

The process of integration is therefore a tentative one, dependent on special artifices.

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[*] For Riemann's Theorem: A one-valued and continuous function in a given interval is always integrable in that interval; see Appendix, Note 9.

[†] The word *limit* as here employed does not in any sense have the technical meaning *limit of a variable* as heretofore defined. It is an unfortunate use of the word, retained out of respect for ancient custom. It is contrary to the spirit of mathematical language to use the same word with different meanings, or in fact to use two words which have the same meaning.

The systematizing of the artifices of integration is the object of this part of the text.

119. Primitive and Derivative.—If we have two functions $F(x)$ and $f(x)$, so related that $f(x)$ is the *derivative* of $F(x)$, then $F(x)$ is called *a primitive* of $f(x)$. The indefinite article is used and $F(x)$ is called *a* primitive of $f(x)$, because if

$$DF(x) = f(x),$$

then also we have

$$D[F(x) + C] = f(x),$$

where C is any assigned constant.

Any one of the functions

$$F(x) + C,$$

obtained by assigning the constant C, is a primitive of $f(x)$. The primitive of $f(x)$ is the family of functions containing the arbitrary parameter C.

GEOMETRICAL ILLUSTRATION.

The two curves

$$y = F(x) + C_1, \tag{1}$$
$$y = F(x) + C_2, \tag{2}$$

are so related that at any point x their tangents at P_1 and P_2 are parallel, and each curve has for the same abscissa the same slope. Their ordinates differ by a constant. Each curve represents a primitive of $f(x)$. Any particular primitive is determined when we know or assign any point through which the curve must pass.

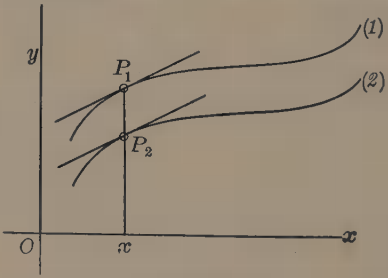

FIG. 66.

120. A General Theorem on Integration.—If a *primitive* of a given function can be found, then the integral of the given function from a to X can always be evaluated. The given function being continuous in (a, X).

Let $f(x)$ be a continuous function in (a, X), and let $F(x)$ be a primitive of $f(x)$.

Let $x_0 \equiv a$, $x_n \equiv X$. Interpolate the numbers $x_1, \ldots, x_{n-1}$ between a and X in the interval (a, X), in order from a to X, subdividing the interval (a, X) into the n subintervals

$$(x_0, x_1), (x_1, x_2), \ldots, (x_{n-2}, x_{n-1}), (x_{n-1}, x_n).$$

We have the sum

$$\sum_{r=1}^{n} (x_r - x_{r-1}) = X - a,$$

whatever be n.

Since $f(x)$ is the derivative of $F(x)$,

$$F'(x) = f(x).$$

By the law of the mean value applied to each of the subintervals, we have the n equations

$$\begin{aligned} F(X) - F(x_{n-1}) &= (x_n - x_{n-1})f(\xi_n), \\ F(x_{n-1}) - F(x_{n-2}) &= (x_{n-1} - x_{n-2})f(\xi_{n-1}), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ F(x_2) - F(x_1) &= (x_2 - x_1)f(\xi_2), \\ F(x_1) - F(a) &= (x_1 - x_0)f(\xi_1). \end{aligned}$$

Adding, we have

$$F(X) - F(a) = \sum_{r=1}^{n} (x_r - x_{r-1})f(\xi_r), \tag{1}$$

in which ξ_r is some particular number *in* the interval (x_r, x_{r-1}).

The sum on the right, in the above equation, is equal to the member on the left. The left side of the equation is independent of n. The equation is true whatever be the integer n, and when $n = \infty$. The sum on the right remains constant as we increase n, and being finite when $n = \infty$,

$$F(X) - F(a) = \underset{n=\infty}{\pounds} \sum_{r=1}^{r=n} (x_r - x_{r-1})f(\xi_r).$$

Now let z_r be any number whatever *of* the subinterval (x_r, x_{r-1}), for each subinterval. Then

$$f(\xi_r) = f(z_r) + \epsilon_r,$$

where $\epsilon_r (=) 0$, when $x_r - x_{r-1} (=) 0$, by reason of the continuity of $f(x)$.

Therefore

$$\begin{aligned} \sum_{r=1}^{n} (x_r - x_{r-1})f(\xi_r) &= \sum_{1}^{n} (x_r - x_{r-1})\,[f(z_r) + \epsilon_r], \\ &= \sum_{1}^{n} (x_r - x_{r-1})f(z_r) + \sum_{1}^{n} (x_r - x_{r-1})\epsilon_r. \end{aligned}$$

Let ϵ be the greatest, in absolute value, of the numbers $\epsilon_1, \ldots, \epsilon_n$. Then

$$\sum_{1}^{n} (x_r - x_{r-1})\epsilon_r \mid \leq \mid \epsilon \sum_{1}^{n} (x_r - x_{r-1}) = \epsilon(X - a),$$

the limit of which is 0, when $n = \infty$; provided each subinterval

$$x_r - x_{r-1} (=) 0$$

when $n = \infty$.

Therefore, when $n = \infty$, and at the same time *each* subinterval $x_r - x_{r-1} (=) 0$, we have

$$\underset{n=\infty}{\pounds}\sum_{r=1}^{n}(x_r - x_{r-1})f(\xi_r) = \underset{n=\infty}{\pounds}\sum_{r=1}^{n}(x_r - x_{r-1})f(z_r), \tag{2}$$

z_r being any number of the interval (x_r, x_{r-1}); that is, z_r may be x_r, or x_{r-1}, or any number we choose to assign between x_r and x_{r-1}.

The member on the right in (2) is, by definition, the integral of $f(x)$ from a to X, and we therefore have for that integral

$$\underset{n=\infty}{\pounds}\sum_{r=1}^{n}(x_r - x_{r-1})f(z_r) = F(X) - F(a),$$

which is evaluated whenever we know a primitive of $f(x)$, and can calculate its values at a and X.

Observe that it is not necessary that we should know the values of the primitive anywhere except at the limits a and X. The integral is therefore a function of its limits.

121. In the preceding articles of this chapter we have fixed no law by which the values $x_1, \ldots, x_{n-1}$ were interpolated between a and X. The integral has been defined and evaluated for any distribution of these numbers whatever, subject to the sole condition that the intervals between the consecutive numbers must converge to 0 at the same time that the number of the subintervals becomes indefinitely great.

Since it makes no difference how we subdivide the interval of integration, we shall generally in the future subdivide the interval of integration into n equal parts, so that

$$x_r - x_{r-1} \equiv \Delta x_r = h = (X - a)/n,$$

and we shall take the value of the function to be integrated at x_{r-1}, the lower end of each subinterval.

The integral of $f(x)$ from a to X is then

$$F(X) - F(a) = \overset{\Delta x(=)0}{\underset{n=\infty}{\pounds}}\sum_{r=0}^{n} f(x_r)\Delta x.$$

But observe that

$$f(x_r)\Delta x = F'(x_r)\Delta x = dF(x_r).$$

Hence the integral of $f(x)$ from $x = a$ to $x = X$ is the limit of the sum of the differentials of the primitive function.

122. Leibnitz's Notation.—The notation previously used to represent the integral, while valuable as indicative of the operation *ab initio* performed in evaluating this limit, is cumbersome, and when once clearly assimilated it can be replaced by a more convenient and abbreviated symbolism. We replace the limit-sum symbol by a more compact and serviceable symbol designed by Leibnitz. Thus, in future we shall write in the suggestive symbolism

$$\int_a^X f(x)dx \equiv \underset{n=\infty}{\overset{\Delta x(=)0}{\pounds}} \sum_{r=0}^{n} f(x_r)\Delta x,$$

as the symbol for the integral of $f(x)$ from a to X.

The characteristic symbol $\int$ is a modification of the letter S, the initial of *sum*, and is taken to mean *limit-sum*, or $\int \equiv \pounds\Sigma$. The symbol $f(x)\,dx$ represents the *type* of the elements whose sum is taken.

If $F(x)$ is a primitive of $f(x)$, then

$$\begin{aligned} F(X) - F(a) &= \int_a^X f(x)\,dx, \\ &= \int_a^X F'(x)\,dx, \\ &= \int_a^X dF(x). \end{aligned}$$

This, then, is the final reduction of the integral; and whenever the expression to be integrated, $f(x)\,dx$, can be reduced to the differential $dF(x)$, then $F(x)$ is recognized as a primitive of $f(x)$ and the integral can be evaluated when the limits are known.

123. Observations on the Integral.—Differentiation was founded on the exceptional case in the theorems in limits, wherein we sought the limit of the quotient of two variables when each converged to 0.

We found that the theorem stating: the limit of the quotient is equal to the quotient of the limits, did not hold, § 15, V (footnote) in the case when the limit of the numerator and of the denominator was 0, but that the limit sought or defined was the limit of the *quotient* of the variables.

Integration is founded on another exceptional case in the theorems in limits. Here we seek the limit of the sum of a number of terms when the number of terms increases indefinitely and also each term diminishes indefinitely. The limit we seek is the *limit of the sum*. The theorem which states: the limit of the sum of a number of variables is equal to the sum of their limits, was only enunciated and proved for a finite number of variables, and does not necessarily hold when that number is infinite. The *sum of the limits* of an infinite number of variables, each having the limit 0, is 0 and nothing else.

The important point in the definition of the integral which makes it a matter of indifference where in the subinterval of the integral element we take the value of the function, is an example of an important general theorem in summation, which can be stated thus:

Lemma. If the sum of n variables $u_1, \ldots, u_n$ has a determinate limit A when each converges to 0 for $n = \infty$, so that

$$\pounds(u_1 + \ldots + u_n) = A,$$

and there be any other n variables $v_1, \ldots, v_n$, such that each converges to 0 for $n = \infty$, and at the same time

$$\pounds\frac{u_1}{v_1} = 1, \ \ldots, \ \pounds\frac{u_n}{v_n} = 1,$$

then also

$$\pounds(v_1 + \ldots + v_n) = A.$$

For, whatever be r,

$$\frac{v_r}{u_r} = 1 + \epsilon_r,$$

where $\epsilon_r(=)0$, when $n = \infty$. Also,

$$\pounds\Sigma v_r = \pounds\Sigma(u_r + \epsilon_r u_r) = \pounds\Sigma u_r + \pounds\Sigma\epsilon_r u_r.$$

If ϵ is the greatest absolute value of $\epsilon_1, \ldots, \epsilon_n$, then

$$\Sigma\epsilon_r u_r| \leqq |\epsilon\Sigma u_r = \epsilon A,$$

the limit of which is 0, and, § 15, III,

$$\pounds\Sigma v_r = \pounds\Sigma u_r = A.$$

This principle is of far-reaching importance in integration, and will be frequently illustrated and applied in the applications of the Calculus.

GEOMETRICAL ILLUSTRATION.

Let $y = F(x)$ be represented by a curve, and let $F'(x) = f(x)$. Then $f(x)$ is the slope of the curve or of its tangent at x.

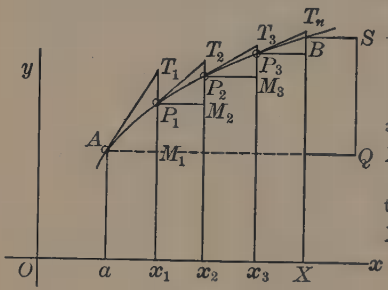

FIG. 67.

We have SQ equal to

$$F(X) - F(a) = M_1P_1 + M_2P_2 + \ldots + M_nB, \quad (1)$$
$$= \Sigma\varDelta F.$$

Also, the sum of the differentials of F at $a, x_1, \ldots,$ is

$$\Sigma dF = M_1T_1 + M_2T_2 + \ldots + M_nT_n. \quad (2)$$

The difference between this sum and that in (1) is

$$\Sigma dF - \Sigma\varDelta F = P_1T_1 + P_2T_2 + \ldots + BT_n.$$

But we know that the limit of

$$\frac{\varDelta F}{dF} = \frac{M_rP_r}{M_rT_r}$$

is 1 when $n = \infty$ and $\varDelta x(=)0$. Hence, by the lemma above, we have

$$F(X) - F(a) = \pounds\Sigma\,\varDelta F = \pounds\Sigma\,dF,$$
$$= \pounds\Sigma\,F'(x)\,dx,$$
$$= \pounds\Sigma\,f(x)\,dx,$$

which is another illustration of the integral.

124. The Indefinite Integral.—When we know a primitive of a given function we can integrate that function for given limits. It is therefore customary to call a primitive of a given function the *indefinite* integral of that function.

Indefinite integration is therefore a process by which we find a primitive of a given function. A primitive $F(x)$ of a given function $f(x)$ is called the *indefinite integral* of $f(x)$, and we write conventionally, omitting the limits,

$$\int f(x)\,dx = F(x).$$

This, of course, becomes the *definite* integral

$$\int_a^X f(x)\,dx = F(X) - F(a)$$

when the limits of integration a and X are assigned.

The indefinite symbol

$$\int f(x)\,dx$$

proposes the question: Find a function which differentiated results in $f(x)$; or, find a primitive of $f(x)$.

Before we can solve questions in the applications of the integral calculus, we must be able, when possible, to find the primitive of a proposed function. The next few chapters will be devoted to this object.

125. The Fundamental Integrals.—The two integrals

$$\int_a^X e^x\,dx \quad \text{and} \quad \int_a^X \sin x\,dx$$

are called the *fundamental* integrals. They can be determined directly by the *ab initio* process, and all other functions that can be integrated in terms of the elementary functions can be reduced to the standard form

$$\int du = u$$

by means of these fundamental integrals.

1. We have, where $(X - a)/n = h$,

$$\int_a^X e^x\,dx = \underset{h(=)0}{\pounds}\, h[e^a + e^{a+h} + \cdots + e^{a+(n-1)h}],$$

$$= \pounds\, h\frac{e^{nh} - 1}{e^h - 1}e^a = \pounds(e^X - e^a)\frac{h}{e^h - 1},$$

$$= e^X - e^a.$$

2. Also,

$$\int_a^X \sin x\,dx = \underset{h(=)0}{\pounds}\, h[\sin a + \sin(a+h) + \ldots + \sin(a+\overline{n-1}h)],$$

$$= \pounds\, h\frac{\sin[a+\frac{1}{2}(n-1)h]\sin\frac{1}{2}nh}{\sin\frac{1}{2}h},$$

by a well-known trigonometrical summation.*

But the expression under the limit sign is equal to

$$\{\cos(a-\tfrac{1}{2}h) - \cos[a+\tfrac{1}{2}(2n-1)h]\}\frac{\frac{1}{2}h}{\sin\frac{1}{2}h}$$

$$= \{\cos(a-\tfrac{1}{2}h) - \cos(X-\tfrac{1}{2}h)\}\frac{\frac{1}{2}h}{\sin\frac{1}{2}h},$$

which, when $h(=)0$, has the limit $\cos a - \cos X$.

$$\therefore \quad \int_a^X \sin x\,dx = -\cos X + \cos a.$$

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* See Loney's Trigonometry, Part I, § 241, p. 283.

CHAPTER XVII.

THE STANDARD INTEGRALS. METHODS OF INTEGRATION.

126. As stated in the preceding chapter: if $f(x)$ is the derivative of $F(x)$, then $F(x)$ is a primitive of $f(x)$, or an indefinite integral of $f(x)$. This and the next chapter will be devoted to finding primitives of given functions.* This process is nothing more than the *inverse* operation of differentiation. The word integrate, when used unqualified, for the present means "find a primitive."

If we choose to work in derivatives, then in the same sense that $Df(x)$ means, find the derivative of $f(x)$; the symbol $D^{-1}f(x)$ means, find a primitive of $f(x)$.

It is usually preferable to work with differentials and employ the symbol $\int f(x)\,dx$ to mean, find a primitive of $f(x)$, or simply, integrate $f(x)$.

If u is any function of x, then

$$u = \int du$$

and is the solution of the integral.

The solution of

$$\int f(x)\,dx$$

invariably consists in transforming $f(x)\,dx$ into the differential du of some function u of x, and when this is done the integral or primitive u is *recognized.*

But, inasmuch as every function that has been differentiated in the differential calculus furnishes a formula, which when inverted by integration gives the corresponding integral of a function, we do not consider it necessary that we should always reduce an integral completely to the irreducible form $\int du$. There are certain standard functions, such as those in the Derivative Catechism, which we select as the standard forms whose integrals we can recognize at once, and thus save the unnecessary labor of further and ultimate reduction to $\int du$.

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* This is the starting-point of the theory of differential equations, an extensive branch of the Calculus.

THE INTEGRAL CATECHISM.

1. $\int cu\,dx = c\int u\,dx.$

2. $\int (u+v)\,dx = \int u\,dx + \int v\,dx.$

3. $\int u\,dv = uv - \int v\,du.$

4. $\int u^a\,du = \dfrac{u^{a+1}}{a+1}.$ $\qquad a \neq = 1.$

5. $\int \dfrac{du}{u} = \log u.$

6. $\int e^u\,du = e^u.$

7. $\int a^u\,du = \dfrac{a^u}{\log a}.$

8. $\int \sin au\,du = -\dfrac{\cos au}{a}. \qquad \int \cos au\,du = \dfrac{\sin au}{a}.$

9. $\int \sec^2 au\,du = \dfrac{\tan au}{a}. \qquad \int \csc^2 au\,du = -\dfrac{\cot au}{a}.$

10. $\int \sec u \tan u\,du = \sec u. \qquad \int \csc u \cot u\,du = -\csc u.$

11. $\int \dfrac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\dfrac{u}{a} = -\cos^{-1}\dfrac{u}{a}.$

12. $\int \dfrac{du}{\sqrt{u^2 \pm a^2}} = \log\left(u + \sqrt{u^2 \pm a^2}\right).$

13. $\int \dfrac{du}{u^2+a^2} = \dfrac{1}{a}\tan^{-1}\dfrac{u}{a}, \quad \text{or} \quad -\dfrac{1}{a}\cot^{-1}\dfrac{u}{a}.$

14. $\int \dfrac{du}{u^2-a^2} = \dfrac{1}{2a}\log\dfrac{u-a}{u+a}, \quad \text{or} \quad \dfrac{1}{2a}\log\dfrac{a-u}{a+u} = \int \dfrac{du}{a^2-u^2}.$

15. $\int \tan u\,du = \log \overset{co}{\sec}\, u. \qquad \int \cot u\,du = \log \sin u.$

16. $\int \sec u\,du = \log \tan\left(\tfrac{1}{2}u + \tfrac{1}{4}\pi\right). \qquad \int \csc u\,du = \log \tan \tfrac{1}{2}u.$

17. $\int \sqrt{a^2-u^2}\,du = \tfrac{1}{2}u\sqrt{a^2-u^2} + \tfrac{1}{2}a^2 \sin^{-1}\dfrac{u}{a}.$

18. $\int \sqrt{u^2 \pm a^2}\,du = \tfrac{1}{2}u\sqrt{u^2 \pm a^2} \pm \tfrac{1}{2}a^2 \log\left(u + \sqrt{u^2 \pm a^2}\right).$

19. $\int \sin^2 u\,du = \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u. \qquad \int \cos^2 u\,du = \tfrac{1}{2}u + \tfrac{1}{4}\sin 2u.$

20. $\int \log u\,du = u(\log u - 1).$

21. $\int \dfrac{du}{u\sqrt{u^2-a^2}} = \dfrac{1}{a}\sec^{-1}\dfrac{u}{a} = -\dfrac{1}{a}\csc^{-1}\dfrac{u}{a}.$

22. $\int \dfrac{du}{\sqrt{2u-u^2}} = \text{vers}^{-1} u = -\text{covers}^{-1} u.$

These standard forms are certain elementary functions of frequent occurrence, and they constitute the Integral Catechism, which should be memorized, and to which must be reduced all other functions proposed for integration.

In the formulæ, u, v, etc., are functions of x.

127. Principles of Integration.—The first two formulæ in the Catechism enunciate two fundamental principles of integration.

I. Since $c\,du = d(cu)$, where c is any constant, we have

$$\int c\,du = \int d(cu) = cu = c\int du,$$

or the integral of the product of a constant and a variable is equal to the product of the constant into the integral of the variable. Therefore a constant factor may be transposed from one side of $\int$ to the other without changing the integral.

EXAMPLES.

1. $\int x^3\,dx = \frac{4}{4}\int x^3\,dx = \frac{1}{4}\int (4x^3)dx = \frac{1}{4}\int d(x^4) = \frac{1}{4}x^4.$

2. $\int \sin ax\,dx = -\frac{1}{a}\int(-a\sin ax)dx = -\frac{1}{a}\int d(\cos ax) = -\frac{\cos ax}{a}.$

II. Since

$$\begin{aligned} d(u+v+w) &= du+dv+dw, \\ \therefore \quad \int(du+dv+dw) &= \int d(u+v+w), \\ &= u+v+w, \\ &= \int du + \int dv + \int dw. \end{aligned}$$

It follows, therefore, that the integral of the sum of a *finite* number of functions is equal to the sum of the integrals of the functions, and conversely.

EXAMPLES.

1. $$\begin{aligned} \int (ax+cx^3)dx &= \int ax\,dx + \int cx^3\,dx, \\ &= a\int x\,dx + c\int x^3\,dx, \\ &= a\int d(\tfrac{1}{2}x^2) + c\int d(\tfrac{1}{4}x^4), \\ &= \tfrac{1}{2}ax^2 + \tfrac{1}{4}cx^4. \end{aligned}$$

2. $$\begin{aligned} \int(\cos x - \sin ax)\,dx &= \int \cos x\,dx - \int \sin ax\,dx, \\ &= \int d(\sin x) + \int d\left(\frac{\cos ax}{a}\right), \\ &= \sin x + \frac{1}{a}\cos ax. \end{aligned}$$

128. Methods of Integration.—The first and simplest method of integrating a given function is, when possible, to

Complete the Differential.

This means, to transform the integral into $\int du$ by *inspection*, and thus recognize u. Except for the simplest functions this cannot be done directly, and we have recourse to the following.

The methods employed by which we reduce a proposed function to be integrated to the irreducible fundamental form $\int du$, or to the recognized form of one of the standard tabulated functions in the Catechism, are

I. Substitution.

(1) *Transformation.* (2) *Rationalization.*

II. Decomposition.

(3) *Parts.* (4) *Partial Fractions.*

129. While nearly all the standard integrals in the catechism are immediately obvious by the inversion of corresponding familiar formulæ in the derivative catechism, we shall deduce them by aid of the principles of § 127 and the methods of § 128, and the two fundamental integrals

$$\int e^x\,dx = e^x, \quad \int \sin x = -\cos x,$$

established in § 125, in order to illustrate the methods of integration laid down in § 128, and to fix the standard integrals in the memory.

130. Transformation (*Substitution*).—This is a method by which we transform the proposed integral into a new one by the substitution of a new variable for the old one. The object in view being to so choose the new variable that the new integral shall be of simpler form than the old one.

Thus, if the proposed integral is

$$\int f(x)\,dx,$$

and we put $x = \phi(z)$, then $dx = \phi'(z)\,dz$. The integral is transformed after substitution into the new integral

$$\int f[\phi(z)]\,\phi'(z)\,dz.$$

This when integrated appears as a function of z, which is retransformed into a function of x by solving $x = \phi(z)$ for z and substituting this value $z = \psi(x)$. The final result is the proposed integral

$$\int f(x)\,dx.$$

EXAMPLES.

1. Use a substitution to find $\int \frac{du}{u}$.

Put $u = e^v$, then $du = e^v\,dv$.

$$\therefore \quad \int \frac{du}{u} = \int dv = v = \log u.$$

2. Make use of $\int e^u\,du = e^u$ to find $\int u^a\,du$.

Put $u^a = e^v$. $\therefore$ $au^{a-1}du = e^v dv$. Hence

$$u^a\,du = \frac{1}{a}\,e^v u\,dv = \frac{1}{a}\,e^{\frac{a+1}{a}v}\,dv = \frac{1}{a+1}\,e^{\frac{a+1}{a}v}\,d\left(\frac{a+1}{a}v\right).$$

$$\therefore \quad \int u^a\,du = \frac{e^{\frac{a+1}{a}v}}{a+1} = \frac{u^{a+1}}{a+1}.$$

3. Integrate $\int \cos x\,dx$, given $\int \sin u\,du = -\cos u$.

We have $\cos x\,dx = -\sin(\frac{1}{2}\pi - x)d(\frac{1}{2}\pi - x)$.
Hence, if $u = \frac{1}{2}\pi - x$,

$$\int \cos x\,dx = -\int \sin u\,du = \cos u = \sin x.$$

4. Integrate $\tan x\,dx$. We have, by Ex. 1,

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{d(\cos x)}{\cos x} = -\log\cos x.$$

5. Integrate $\int \cot x\,dx$.

$$\int \cot x\,dx = \int \frac{d(\sin x)}{\sin x} = \log \sin x.$$

6. Show that

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax; \quad \int \cos ax\,dx = \frac{1}{a}\sin ax.$$

7. Show that $\int \tan ax\,dx = \frac{1}{a}\log\sec ax$.

8. Integrate $\int \frac{dx}{\sqrt{1 - x^2}}$.

Substitute $x = \sin z$. $\therefore$ $dx = \cos z\,dz$.

$$\therefore \int \frac{dx}{\sqrt{1 - x^2}} = \int dz = z = \sin^{-1}\cancel{z}.\ x$$

9. Integrate $\int (a + bx)^c\,dx$.

Put $a + bx = y$. $\therefore$ $dx = dy/b$.

$$\therefore \quad \int (a + bx)^c\,dx = \frac{1}{b}\int y^c\,dy = \frac{y^{c+1}}{b(c+1)} = \frac{(a + bx)^{c+1}}{b(c+1)}.$$

10. $\int \frac{dt}{a^2 + t^2}$. Put $t = a\tan\theta$. Then

$$dt = a\sec^2\theta\,d\theta. \quad \text{Hence}$$

$$\int \frac{dt}{a^2 + t^2} = \frac{1}{a}\int d\theta = \frac{\theta}{a} = \frac{1}{a}\tan^{-1}\frac{t}{a}.$$

11. $\int a^u\,du$. Put $y = a^u$. $\therefore\ dy = a^u \log a\ du$.

$$\therefore\ \int a^u\,du = \frac{1}{\log a}\int dy = \frac{y}{\log a} = \frac{a^u}{\log a}.$$

12. Integrate the functions

$$3^x,\quad x^2 - 2^x,\quad a + b^x + c^{2x},\quad \tfrac{1}{2}\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right).$$

13. Integate $\dfrac{x}{x+1}$, $\dfrac{x^2}{x^3+1}$, $\dfrac{x^{n-1}}{x^n+a^n}$.

14. $$\int \frac{\cos x\,dx}{1+\sin x} = \int \frac{d(1+\sin x)}{1+\sin x} = \log(1+\sin x).$$

15. $$\int \sin^2 x\,dx = \tfrac{1}{2}\int (1 - \cos 2x)dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x.$$

16. Find the integrals $\displaystyle\int \frac{\sin x\,dx}{1-\cos x}$, $\displaystyle\int \cos^2 x\,dx$.

17. Given the definite integral

$$\int_a^x t^c dt = \frac{t^{c+1}}{c+1}\Big]_a^x = \frac{x^{c+1} - a^{c+1}}{c+1},$$

deduce the integral $\displaystyle\int \frac{dt}{t} = \log t$.

In the value of the definite integral, let $c(=)-1$, then (see § 75, 0/0 form),

$$\underset{c(=)-1}{\mathcal{L}} \frac{x^{c+1} - a^{c+1}}{c+1} = \mathcal{L}(x^{c+1}\log x - a^{c+1}\log a),$$

$$= \log x - \log a.$$

$\log a$ is the constant of integration and we have

$$\int \frac{dx}{x} = \log x.$$

18. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}}$. Put $u = a\sec\theta$.

19. $\displaystyle\int \frac{ds}{\sqrt{2s - s^2}}$. This can be written

$$\int \frac{ds}{\sqrt{1-(1-s)^2}}.$$

Put $1 - s = \cos\theta$. $\therefore\ ds = \sin\theta\,d\theta$, and the integral becomes

$$\int d\theta = \theta = \cos^{-1}(1-s) = \operatorname{vers}^{-1}s.$$

20. $$\int \sec x \tan x\,dx = \int \frac{\sin x}{\cos^2 x}dx = -\int \frac{d(\cos x)}{\cos^2 x},$$

$$= \sec x.$$

21. $\displaystyle\int \csc x \cot x\,dx = ?$

22. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}}$. Put $\sqrt{x^2+a^2} = z - x$. $\therefore\ dx = \dfrac{z-x}{z}dz$.

$$\therefore\ \int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{dz}{z} = \log z = \log(x + \sqrt{x^2+a^2}).$$

23. Show by a like substitution that

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left(x + \sqrt{x^2 - a^2}\right).$$

24. Integrate $\displaystyle\int \frac{dx}{\sin x \cos x}$.

$$\int \frac{dx}{\sin x \cos x} = \int \frac{\sec^2 x \, dx}{\tan x} = \int \frac{d(\tan x)}{\tan x} = \log(\tan x).$$

25. $\displaystyle\int \frac{dx}{\sin x} = \int \frac{d(\frac{1}{2}x)}{\sin \frac{1}{2}x \cos \frac{1}{2}x} = \log(\tan \tfrac{1}{2}x)$, by Ex. 24.

26. To integrate $\displaystyle\int \frac{dx}{\cos x}$, put $x = \frac{1}{2}\pi - z$

$$\therefore \int \frac{dx}{\cos x} = -\int \frac{dz}{\sin z} = -\log(\tan \tfrac{1}{2}z) = \log(\cot \tfrac{1}{2}z),$$

$$= \log[\cot(\tfrac{1}{4}\pi - \tfrac{1}{2}x)] = \log\tan(\tfrac{1}{2}x + \tfrac{1}{4}\pi).$$

These results can be identified with 16 in the table.

27. Observing that we can, by inspection, write

$$\frac{1}{x^2 - a^2} = \frac{1}{2a}\left(\frac{1}{x - a} - \frac{1}{x + a}\right),$$

we have

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x - a}{x + a}.$$

This process is a particular case of the general method of decomposition into partial fractions.

Integrate this case, using the substitution $(x - a) = (x + a)z$.

Also, integrate the more general integral

$$\int \frac{dx}{(x - a)(x - b)},$$

by means of the transformation $x - a = (x - b)z$.

28. We can make use of Ex. 27 to obtain the integrals in Exs. 25, 26. For we have

$$\int \frac{dx}{\cos x} = \int \frac{\cos x \, dx}{\cos^2 x} = \int \frac{d(\sin x)}{1 - \sin^2 x} = \frac{1}{2} \log\left(\frac{1 + \sin x}{1 - \sin x}\right).$$

Show in like manner that

$$\int \frac{dx}{\sin x} = \frac{1}{2} \log\left(\frac{1 - \cos x}{1 + \cos x}\right).$$

29. Integrate $\displaystyle\int \frac{dx}{\sqrt{e^{2x} - 1}}$. Put $e^x = \sec\theta$.

Then $dx = \tan\theta \, d\theta$, and the integral becomes

$$\int d\theta = \theta = \cos^{-1}(e^{-x}).$$

30. Integrate $\displaystyle\int \frac{\sin\theta \, d\theta}{a - b\cos\theta}$.

We can complete the differential by inspection, for the integral becomes

$$\frac{1}{b} \int \frac{d(a - b\cos\theta)}{a - b\cos\theta} = \frac{1}{b} \log(a - b\cos\theta).$$

Otherwise, put $a - b\cos\theta = z$. $\therefore$ $b\sin\theta\, d\theta = dz$.
The integral is therefore

$$\frac{1}{b}\int\frac{dz}{z} = \frac{1}{b}\log z = \frac{1}{b}\log(a - b\cos\theta).$$

31. $\displaystyle\int\frac{dx}{\sqrt{x^2 \pm a^2}}$. Put $x^2 \pm a^2 = z^2$. Then $x\,dx = z\,dz$,

or

$$\frac{dx}{z} = \frac{dz}{x} = \frac{dx + dz}{z + x} = \frac{d(x + z)}{x + z}.$$

$$\therefore \quad \int\frac{dx}{\sqrt{x^2 \pm a^2}} = \int\frac{dx}{z} = \int\frac{d(x + z)}{x + z} = \log(x + z)$$

$$= \log(x + \sqrt{x^2 \pm a^2}).$$

131. Rationalization (*Substitution*).—The object of this process is to rationalize an irrational function proposed for integration, by the substitution of a new variable.

Rationalization by substitution is but a particular case of transformation by substitution. But, since the direct object in view in rationalization is not generally to reduce the function directly to a standard integral, but to first transform it into a rational function which can be subsequently integrated by decomposition into partial fractions, the process demands separate and distinct recognition.

Only a few simple examples will be given here in illustration. The subject will be considered more generally in the next chapter.

EXAMPLES.

1. Integrate $\displaystyle\int(a + bx^3)^{\frac{2}{3}}x^5\,dx$.

Put $a + bx^3 = z^3$. $\therefore$ $bx^2\,dx = z^2\,dz$. On substitution the integral becomes

$$\frac{1}{b^2}\int(z^7 - az^4)dz = \frac{1}{b^2}\left(\frac{z^8}{8} - a\frac{z^5}{5}\right),$$

$$= \frac{1}{40b^2}(a + bx^3)^{\frac{5}{3}}(5bx^3 - 3a).$$

2. Integrate $\displaystyle\int\frac{x\,dx}{(a + bx^2)^{\frac{2}{3}}}$.

Put $a + bx^2 = z^3$. $\therefore$ $x\,dx = 3z^2\,dz/2b$.

The integral is $\displaystyle\frac{3}{2b}\sqrt[3]{a + bx^2}$.

3. Put $a + bx = z^3$, and show that

$$\int\frac{x\,dx}{(a + bx)^{\frac{2}{3}}} = \frac{3}{4b^2}(bx - 3a)(a + bx)^{\frac{1}{3}}.$$

4. Put $a^2 - x^2 = z^3$, and show that

$$\int\frac{x^3\,dx}{(a^2 - x^2)^{\frac{1}{3}}} = -\frac{3}{20}(3a^2 + 2x^2)(a^2 - x^2)^{\frac{2}{3}}.$$

5. To integrate $\displaystyle\int\frac{dx}{x^4\sqrt{1 + x^2}}$.

Put $1 + 1/x^2 = z^2$. $\therefore$ $dx = -zx^3\,dz$.
The integral becomes

$$\int(1 - z^2)dz = z - \frac{1}{3}z^3 = \frac{2x^2 - 1}{3x^3}\sqrt{1 + x^2}.$$

6. $\displaystyle\int\frac{dx}{x^2\sqrt{1 - x^2}}$. Put $1/x^2 - 1 = z^2$. $\therefore$ $dx = -x^3 z\,dz$.

After substitution the integral becomes

$$-\int dz = -z = -\frac{1}{x}\sqrt{1 - x^2}.$$

7. $\displaystyle\int\frac{(1 + x^{\frac{1}{4}})}{1 - x^{\frac{1}{2}}}dx$. Put $x = z^4$. $\therefore$ $dx = 4z^3\,dz$.

The integral becomes

$$4\int\frac{z^3(1 + z)}{1 - z^2}dz = -4\int\frac{z^3\,dz}{z - 1} = -4\left\{\frac{z^3}{3} + \frac{z^2}{2} + z + \log(z - 1)\right\},$$

since $$\frac{z^3}{z - 1} = z^2 + z + 1 + \frac{1}{z - 1}.$$

$$\therefore\quad \int\frac{(1 + x^{\frac{1}{4}})dx}{1 - x^{\frac{1}{2}}} = -\frac{4}{3}x^{\frac{3}{4}} - 2x^{\frac{1}{2}} - 4x^{\frac{1}{4}} - 4\log\left(x^{\frac{1}{4}} - 1\right).$$

8. $\displaystyle\int\sqrt{a^2 - x^2}\,dx$. Put $x = a\sin\theta$. $\therefore$ $dx = a\cos\theta\,d\theta$.

$$\begin{aligned}\therefore\quad \int\sqrt{a^2 - x^2}\,dx &= a^2\int\cos^2\theta\,d\theta = \tfrac{1}{2}a^2\int(1 + \cos 2\theta)d\theta,\\ &= \tfrac{1}{2}a^2(\theta + \tfrac{1}{2}\sin 2\theta),\\ &= \tfrac{1}{2}a^2\sin^{-1}\frac{x}{a} + \tfrac{1}{2}x\sqrt{a^2 - x^2}.\end{aligned}$$

Rationalization by trigonometrical substitutions will be considered more generally later.

132. Parts (*Decomposition*).—This important method of decomposing an integral into two parts, one of which is immediately integrable by definition and the other is an integral of more simple form than the original integral, is one of the most powerful methods of integration we possess. It is based on the formula for the differentiation of the product of two functions,

$$d(uv) = u\,dv + v\,du.$$
$$\therefore\quad u\,dv = d(uv) - v\,du.$$

Integrating, we have the formula for integration by parts,

$$\int u\,dv = uv - \int v\,du.$$

EXAMPLES.

1. Integrate $\displaystyle\int\log x\,dx$.

Decompose the differential $\log x\,dx$, so that

$$u = \log x \quad\text{and}\quad dv = dx.$$
$$\therefore\quad du = \frac{dx}{x} \quad\text{and}\quad v = x.$$

Hence

$$\int \log x\,dx = x\log x - \int dx = x\log x - x.$$

2. Integrate $\int \tan^{-1}x\,dx.$

Put $u = \tan^{-1}x, \quad dv = dx.$

Then $du = \dfrac{dx}{1+x^2}, \quad v = x.$

$$\therefore \int \tan^{-1}x\,dx = x\tan^{-1}x - \int \frac{x\,dx}{1+x^2},$$

$$= x\tan^{-1}x - \log\sqrt{1+x^2}.$$

3. Integrate $\int x\,e^x\,dx.$

Put $u = x, \quad dv = e^x\,dx.$

Then $du = dx, \quad v = e^x.$

$$\therefore \int x\,e^x\,dx = x\,e^x - \int e^x\,dx = e^x(x-1).$$

4. Integrate $\int x^a \log x\,dx.$

Put $u = \log x, \quad dv = x^a.$

$$\therefore \quad du = \frac{dx}{x}, \qquad v = \frac{x^{a+1}}{a+1}.$$

$$\therefore \int x^a \log x\,dx = \frac{x^{a+1}}{a+1}\log x - \int \frac{x^a\,dx}{a+1},$$

$$= \frac{x^{a+1}}{a+1}\log x - \frac{x^{a+1}}{(a+1)^2}.$$

5. Integrate $\int \sqrt{x^2+a^2}\,dx.$

Put $u = \sqrt{x^2+a^2}, \quad dv = dx.$

$$\therefore \quad du = \frac{x\,dx}{\sqrt{x^2+a^2}}, \qquad v = x.$$

Hence

$$\int \sqrt{x^2+a^2}\,dx = x\sqrt{x^2+a^2} - \int \frac{x^2\,dx}{\sqrt{x^2+a^2}}.$$

But

$$\int \sqrt{x^2+a^2}\,dx = \int \frac{x^2+a^2}{\sqrt{x^2+a^2}}\,dx = \int \frac{a^2\,dx}{\sqrt{x^2+a^2}} + \int \frac{x^2\,dx}{\sqrt{x^2+a^2}}.$$

Adding, we have

$$2\int \sqrt{x^2+a^2}\,dx = x\sqrt{x^2+a^2} + a^2\int \frac{dx}{\sqrt{x^2+a^2}}.$$

$$\therefore \int \sqrt{x^2+a^2}\,dx = \tfrac{1}{2}x\sqrt{x^2+a^2} + \tfrac{1}{2}a^2\log(x+\sqrt{x^2+a^2}),$$

by Ex. 22, § 130, or Ex. 31, § 137.

6. Show, in like manner, that

$$\int \sqrt{x^2 - a^2}\, dx = \tfrac{1}{2}x\sqrt{x^2 - a^2} - \tfrac{1}{2}a^2 \log\left(x + \sqrt{x^2 - a^2}\right).$$

7. We can frequently determine the value of an integral by repeating the process of integrating by parts. Thus, integrate

$$\int e^{ax} \sin bx\, dx.$$

Put $u = \sin bx, \qquad dv = e^{ax}\, dx.$

$$\therefore \quad du = b \cos bx\, dx, \qquad v = \frac{1}{a}e^{ax}.$$

$$\therefore \quad \int e^{ax} \sin bx\, dx = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a}\int e^{ax} \cos bx\, dx.$$

But, in the same way, we have

$$\int e^{ax} \cos bx\, dx = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a}\int e^{ax} \sin bx\, dx.$$

Substituting and solving, we get the integrals

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx),$$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx).$$

Put $b/a = \tan \alpha$, then these integrals can be written

$$\frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin(bx - \alpha) \quad \text{and} \quad \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos(bx - \alpha).$$

respectively.

8. Use Exs. 5, 6 to integrate

$$\int \frac{x^2\, dx}{\sqrt{x^2 + a^2}} \quad \text{and} \quad \int \frac{x^2\, dx}{\sqrt{x^2 - a^2}}.$$

9. Show that $\int \sin^{-1}x\, dx = x \sin^{-1}x + \sqrt{1 - x^2}$ by putting $u = \sin^{-1}x$, $dv = dx$.

10. Use the method of Ex. 5 to show that

$$\int \sqrt{a^2 - x^2}\, dx = \tfrac{1}{2}x\sqrt{a^2 - x^2} + \tfrac{1}{2}a^2 \sin^{-1}\frac{x}{a}.$$

11. Use the work of Ex. 10 to get

$$\int \frac{x^2\, dx}{\sqrt{a^2 - x^2}} = -\tfrac{1}{2}x\sqrt{a^2 - x^2} + \tfrac{1}{2}a^2 \sin^{-1}\frac{x}{a}.$$

133. Rational Fractions (*Decomposition*).—Whenever the function to be integrated is a rational algebraic function, we know from algebra (see C. Smith's Algebra, § 297) that it can always be decomposed into the sum of a number of partial fractions, each of which is simpler than the proposed function. (See Chapter XVIII.)

We do not propose to consider here the general process of integrating rational fractions, but merely consider a few elementary examples illustrating the process.

If the function to be integrated is the rational fraction

$$\frac{\phi(x)}{\psi(x)},$$

and the degree of ϕ is higher than that of ψ, we can always divide ϕ by ψ, so as to get

$$\frac{\phi(x)}{\psi(x)} = f(x) + \frac{F(x)}{\psi(x)},$$

in which the quotient $f(x)$ is a polynomial in x and can be integrated immediately. The remainder $F(x)/\psi(x)$ is a rational function in which $F(x)$ is a polynomial of one lower degree than $\psi(x)$, the general integration of which will be considered later.

EXAMPLES.

1. $\displaystyle\int \frac{x^3\,dx}{1+x} = \int \left(x^2 - x + 1 - \frac{1}{1+x}\right) dx,$

$$= \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - \log(1+x).$$

2. $\displaystyle\int \frac{2-3x}{x^2-4}\,dx = 2\int \frac{dx}{x^2-4} - \frac{3}{2}\int \frac{d(x^2-4)}{x^2-4},$

$$= \frac{1}{2}\log\frac{x-2}{x+2} - \frac{3}{2}\log(x^2-4).$$

3. $\displaystyle\int \frac{x^3-3x+1}{x^2+4}\,dx.$

$$\frac{x^3-3x+1}{x^2+4} = x - \frac{7x-1}{x^2+4} = x + \frac{1}{x^2+4} - \frac{7x}{x^2+4}.$$

$$\therefore \quad \int \frac{x^3-3x+1}{x^2+4}\,dx = \int x\,dx + \int \frac{dx}{x^2+4} - \frac{7}{2}\int \frac{d(x^2+4)}{x^2+4},$$

$$= \frac{1}{2}x^2 + \frac{1}{2}\tan^{-1}\frac{x}{2} - \frac{7}{2}\log(x^2+4).$$

4. To integrate $\displaystyle\int \frac{dx}{(x-a)(x-b)}.$

We can always write

$$\frac{1}{(x-a)(x-b)} = \frac{1}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right)$$

by inspection. Therefore

$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{a-b}\log\frac{x-a}{x-b}.$$

134. Observations on Integration.—The processes of *Substitution* and *Decomposition*, in their four subdivisions:

1. *Substitution*,
2. *Rationalization*,
3. *Parts*,
4. *Partial Fractions*,

constitute the methods of finding a primitive of a given function by reduction to a recognized or tabular form. These may be regarded as the rules of integration in general form corresponding to the rules of differentiation. With this difference, however, that in integration there are no regular methods of applying these rules to all functions as is the case in differentiation.

The successful treatment of a given function depends on practice and familiarity with the processes of the operation.

Sometimes different processes of reduction lead to apparently different results. It must be remembered, in this connection, that the indefinite integral found is but *a* primitive of the function proposed, and both results may be correct. They must, however, differ only by a constant.

Frequently, in reducing an integral to a standard form, we shall have to use all four of the methods of reduction. Experience soon teaches the best methods of attack.

In the next chapter we shall consider the subject more generally and make more systematic the methods of reduction to the standard forms.

EXERCISES.

Integrate Exs. 1 to 10 by the primary method of completing the differential by inspection.

1. $\int x^4\,dx, \quad \int ax^{-3}dx, \quad \int 2x^{-\frac{1}{2}}dx.$

2. $\int (x^2 + 1)^{\frac{1}{2}}\,x\,dx = \frac{1}{3}(x^2 + 1)^{\frac{3}{2}}.$

3. $\int \frac{(x^2 - a^2)dx}{x^3 - 3a^2x} = \log\,(x^3 - 3a^2x)^{\frac{1}{3}}.$

4. $\int \left(10t^{\frac{2}{3}} - t^{-4}\right)dt = 6t^{\frac{5}{3}} + \frac{1}{3}t^{-3}.$

5. $\int \left(x^{-\frac{1}{2}} + x^{-\frac{3}{2}}\right)dx, \quad \int (s^2 - 1)ds/s, \quad \int v\,dv/(v^2 - 1).$

6. $\int \frac{u + 1}{u^2 + 2u}du = \log\sqrt{u^2 + 2u}.$

7. $\int (t^2 - 2)^3 t^{-5}\,dt = 2t^{-4} - 6t^{-2} + \frac{1}{2}t^2 - \log t^6.$

8. $\int (a^2 - x^2)^3\sqrt{x}\,dx, \quad \int \left(\sqrt{a} - \sqrt{x}\right)^3 dx, \quad \int (x + 1)^2 dx.$

9. $\int \frac{2ax + b}{ax^2 + bx + c}dx, \quad \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx, \quad \int \frac{2ax + b}{(ax^2 + bx + c)^2}\,dx.$

$\int (e^x + a)^n e^x\,dx, \quad \int \frac{e^x}{e^x + a}dx, \quad \int \frac{\sec^2 x}{\tan x}\,dx.$

$\int \frac{(1 + x^2)^{-1}}{\tan^{-1}x}\,dx, \quad \int \frac{(1 - x^2)^{-\frac{1}{2}}}{\sin^{-1}x}\,dx, \quad \int \frac{dx}{\log x^x}.$

10. Write immediately the integrals of

$\frac{1}{x + 1}, \quad \frac{x}{x + 1}, \quad \frac{x}{x^2 + 1}, \quad \frac{x^2}{x^3 + 1}, \quad \frac{x^{n-1}}{x^n + a^n}.$

$\cos^2 \frac{1}{2}x, \quad \cos^n x \sin x, \quad \tan^n x \sec^2 x.$

11. $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx = 2 \sin \sqrt{x}.$ Put $x = z^2$.

12. $\int e^x \cos e^x \, dx = ?$ Put $e^x = z$.

13. $\int nx^{n-1} \cos x^n \, dx = ?$ Put $x^n = z$.

14. $\int \frac{\cos (\log x)}{x} dx = ?$ Put $\log x = z$.

15. $\int \frac{2x}{1 + x^4} dx = ?$ Put $x^2 = z$.

16. $\int \frac{3x^2}{1 + x^6} dx = ?$ Put $x^3 = z$.

17. $\int \frac{dx}{\sqrt{1 - 4x^2}}, \quad \int \frac{dx}{\sqrt{1 - 2x^2}}, \quad \int \frac{u\,dv + v\,du}{\sqrt{1 - u^2v^2}}$

18. $\int \frac{dx}{1 + 4x^2}, \quad \int \frac{dx}{9x^2 + 4}, \quad \int \frac{dx}{x\sqrt{16x^2 - 1}}.$

19. $\int \sin 3x \, dx, \quad \int \sec^2 4\theta \, d\theta, \quad \int \cos \tfrac{1}{2}\phi \, d\phi.$

20. $\int \frac{x^{n-1}\,dx}{a + bx^n} = \frac{1}{nb} \log (a + bx^n).$

21. $\int \frac{dx}{a + bx^2} = \frac{1}{\sqrt{ab}} \tan^{-1}\left(x\sqrt{\frac{b}{a}}\right).$

22. $\int \frac{(1 + x^2)^2\,dx}{x} = \log x + x^2 + \tfrac{1}{4}x^4.$

23. $\int \frac{(x - 2)\,dx}{x\sqrt{x}} = 2\sqrt{x} + \frac{4}{\sqrt{x}}.$

24. $\int \tan^2 \phi \, d\phi = \tan \phi - \phi. \qquad \int \cot^2 \phi \, d\phi = ?$

25. $\int \sin 2\theta \, d\theta = ? \qquad \int \cos 2\theta \, d\theta = ?$

26. $\int \cos mx \cos nx \, dx = \frac{\sin (m + n)x}{2(m + n)} + \frac{\sin (m - n)x}{2(m - n)}.$

$\int \sin mx \sin nx \, dx = \frac{\sin (m - n)x}{2(m - n)} - \frac{\sin (m + n)x}{2(m + n)}.$

Use $\cos \alpha \cos \beta = \tfrac{1}{2} \cos (\alpha + \beta) + \tfrac{1}{2} \cos (\alpha - \beta)$, etc.

27. $-\int \sin mx \cos nx \, dx = \frac{\cos (m + n)x}{2(m + n)} + \frac{\cos (m - n)x}{2(m - n)}.$

28. $\int \sin \tfrac{1}{2}x \cos \tfrac{1}{2}x \, dx = ? \qquad \int \cos 3x \cos 5x \, dx = ?$

29. $\int \frac{\log x}{x} dx = \tfrac{1}{2}(\log x)^2.$

30. $\int \sqrt{\frac{a + x}{a - x}} \, dx = a \sin^{-1}\frac{x}{a} - \sqrt{a^2 - x^2}.$

Multiply the numerator and denominator by $\sqrt{a+x}$.

31. $\int x\sqrt{x+a}\,dx = \frac{2}{5}(x+a)^{\frac{5}{2}} - \frac{2}{3}a(x+a)^{\frac{3}{2}}$. Put $x+a=z^2$.

32. $\int x^2e^x\,dx = e^x(x^2-2x+2)$. Parts.

33. $\int x^3e^x\,dx = e^x(x^3-3x^2+6x-6)$. Parts.

34. $\int \frac{dx}{x\sqrt{2ax-a^2}} = \frac{2}{a}\tan^{-1}\sqrt{\frac{2x-a}{a}}$. Put $2ax-a^2=z^2$.

35. $\int \cot^{-1}x\,dx = x\cot^{-1}x + \frac{1}{2}\log(1+x^2)$.

36. $\int x\tan^{-1}x\,dx = \frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x$.

37. $\int x^2\sin x\,dx = 2\cos x + 2x\sin x - x^2\cos x$.

38. $\int x^2\cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x$.

39. $\int \cos x\log\sin x\,dx = \sin x(\log\sin x - 1)$.

40. $\int xe^{ax}\,dx = e^{ax}\frac{ax-1}{a^2}$.

41. $\int\frac{dx}{(x-1)(x+3)}$, $\int\frac{dx}{x^2+3x-10}$, $\int\frac{dx}{3x^2-3x-6}$.

Hint. Complete the square.

42. $\int\frac{dx}{\sqrt{(x-\alpha)(x-\beta)}} = 2\log\left(\sqrt{x-\alpha}+\sqrt{x-\beta}\right)$.

Put $x-\alpha=z^2$, then $dx=2z\,dz$.

$$\therefore\quad \int\frac{dx}{\sqrt{(x-\alpha)(x-\beta)}} = 2\int\frac{dz}{\sqrt{z^2+\alpha-\beta}}$$
$$= 2\log\left(z+\sqrt{z^2+\alpha-\beta}\right).$$

43. $\int\frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} = 2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}}$.

Put $x-\alpha=z^2$, as above, and the integral becomes

$$2\int\frac{dz}{\sqrt{\beta-\alpha-z^2}}.$$

44. $\int\sqrt{a+2bx+cx^2}\,dx$.

$$a+2bx+cx^2 = c^{-1}(ac+2bcx+c^2x^2),$$
$$= c^{-1}[(cx+b)^2-(b^2-ac)].$$

Put $cx+b=z$. $\therefore$ $dx=dz/c$, and the integral becomes

$$\frac{1}{c^{\frac{3}{2}}}\int\sqrt{z^2-(b^2-ac)}\,dz,$$

the standard form 18, § 126.

45. Integrate $\int \frac{dx}{(x-a)^m(x-b)^n}$, where m and n are positive integers, or $m+n$ is a positive integer greater than 1.

Put $x - a = (x - b)z$, then

$$x = \frac{a - bz}{1 - z}. \quad \therefore \; x - a = \frac{(a-b)z}{1-z}, \quad x - b = \frac{a-b}{1-z}, \quad dx = \frac{a-b}{(1-z)^2}dz;$$

and the expression transforms into

$$\frac{(1-z)^{m+n-2}\,dz}{(a-b)^{m+n-1}\,z^m}.$$

Expand the numerator by the binomial formula and integrate directly.

46. Integrate $\int \sin^p x \cos^q x \, dx$, whenever $p + q$ is an *even negative integer*.

Let $p + q = -2n$. Then

$$\sin^p x \cos^q x = \sin^p x \cos^{-p-2n} x = \tan^p x \sec^{2n} x,$$
$$= \tan^p x (1 + \tan^2 x)^{n-1} \sec^2 x.$$

Put $\tan x = t$. Then

$$\int \sin^p x \cos^q x \, dx = \int t^p (1 + t^2)^{n-1} dt.$$

Expand by the binomial formula and integrate directly.

47. Integrate $\sin^p x \cos^q x \, dx$, whenever p or q is an *odd positive integer*.

Let $p = 2r + 1$, then

$$\int \sin^{2r+1} x \cos^q x \, dx = -\int (\sin^2 x)^r \cos^q x \, d(\cos x),$$
$$= -\int (1 - \cos^2 x)^r \cos^q x \, d(\cos x),$$
$$= -\int (1 - c^2)^r c^q \, dc.$$

Expand by the binomial formula and integrate.

48. $\int \sin^3\theta \, d\theta = \frac{1}{3}\cos^3\theta - \cos\theta.$

49. $\int \cos^3\theta \, d\theta = ?$ Check by putting $\frac{1}{2}\pi - x$ for x.

50. $\int \cos^5\theta \, d\theta = \sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta.$

51. $\int \sin^3\theta \cos^7\theta \, d\theta = \frac{1}{10}\cos^{10}\theta - \frac{1}{8}\cos^8\theta.$

52. $\int \sin^5 x \cos^{-2} x \, dx = \sec x + 2\cos x - \frac{1}{3}\cos^3 x.$

53. $\int \sqrt{\sin x}\cos^3 x \, dx = \frac{2}{3}\sin^{\frac{3}{2}} x - \frac{2}{7}\sin^{\frac{7}{2}} x.$

54. $\int \cos^3 x \csc^{\frac{2}{3}} x \, dx = 3\sin^{\frac{1}{3}} x - \frac{3}{7}\sin^{\frac{7}{3}} x.$

55. $\int \csc^{\frac{3}{2}} x \sec^{\frac{5}{2}} x \, dx = \frac{2}{3}\tan^{\frac{3}{2}} x - 2\cot^{\frac{1}{2}} x.$

56. $\int \sin^2 x \sec^6 x \, dx = \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x.$

57. $\int \sin^{\frac{1}{2}} x \sec^{\frac{5}{2}} x \, dx = \frac{2}{3} \tan^{\frac{3}{2}} x.$

58. $\int \csc^{\frac{1}{2}} x \sec^{\frac{7}{2}} x \, dx = 2 \tan^{\frac{1}{2}} x \, (1 + \frac{1}{5} \tan^2 x).$

59. $\int \tan^n\theta \, d\theta = \int \tan^{n-2}\theta \, (\sec^2\theta - 1) \, d\theta,$

$$= \frac{\tan^{n-1}\theta}{n-1} - \int \tan^{n-2}\theta \, d\theta.$$

60. Show that $\int \cot^n\theta \, d\theta = -\frac{\cot^{n-1}\theta}{n-1} - \int \cot^{n-2}\theta \, d\theta.$

61. $\int \tan^4\theta \, d\theta = \frac{1}{3}\tan^3\theta - \tan\theta + \theta.$

62. $\int \cot^3\theta \, d\theta = -\frac{1}{2}\cot^2\theta - \log(\sin\theta).$

63. $\int \cot^4\theta \, d\theta = -\frac{1}{3}\cot^3\theta + \cot\theta + \theta.$

64. $\int \cot^5\theta \, d\theta = -\frac{1}{4}\cot^4\theta + \frac{1}{2}\cot^2\theta + \log(\sin\theta).$

65. $\int \sin x \cos x \, (a^2 \sin^2 x + b^2 \cos^2 x)^{\frac{1}{2}} dx.$

Note, $d(a^2 \sin^2 x + b^2 \cos^2 x) = 2(a^2 - b^2) \sin x \cos x \, dx.$ Hence the integral is

$$\frac{1}{3(a^2 - b^2)} (a^2 \sin^2 x + b^2 \cos^2 x)^{\frac{3}{2}}.$$

66. $\int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{1}{ab} \tan^{-1}\left(\frac{b}{a} \tan x\right).$

Divide the numerator and denominator by $\cos^2 x$.

67. $\int \frac{dx}{a \sin x + b \cos x}.$ Divide the numerator and denominator by $\sqrt{a^2 + b^2}$, and put $\tan \alpha = a/b$. Then we have

$$\frac{1}{\sqrt{a^2 + b^2}} \int \frac{dx}{\cos(x - \alpha)} = \frac{1}{\sqrt{a^2 + b^2}} \log \tan \left(\tfrac{1}{2}x - \tfrac{1}{2}\alpha + \tfrac{1}{4}\pi\right).$$

68. $\int \frac{dx}{a + b \cos x}.$ We have

$$a + b \cos x = a\,(\sin^2 \tfrac{1}{2}x + \cos^2 \tfrac{1}{2}x) + b\,(\cos^2 \tfrac{1}{2}x - \sin^2 \tfrac{1}{2}x)$$
$$= (a + b) \cos^2 \tfrac{1}{2}x + (a - b) \sin^2 \tfrac{1}{2}x,$$

which reduces the integral to the form of Ex. 66.

Divide the numerator and denominator by $\cos^2 \frac{1}{2}x$, and put $z = \tan \frac{1}{2}x$. Then the integral becomes

$$2 \int \frac{dz}{(a + b) + (a - b) z^2},$$

which is standardized. Hence

$$\int \frac{dx}{a + b \cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left\{ \sqrt{\frac{a - b}{a + b}} \tan \frac{x}{2} \right\}, \qquad a > b;$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log \frac{\sqrt{b + a} + \sqrt{b - a} \tan \frac{1}{2}x}{\sqrt{b + a} - \sqrt{b - a} \tan \frac{1}{2}x}, \qquad a < b.$$

69. $\displaystyle\int \frac{dx}{5 + 4 \sin x} = \frac{2}{3} \tan^{-1} \frac{4 + 5 \tan \frac{1}{2}x}{3}$

70. Integrate $\displaystyle\int \frac{1 + \cos x}{(x + \sin x)^3}\, dx = -\frac{1}{2} \frac{1}{(x + \sin x)^2}.$

71. $\displaystyle\int x \sin x\, dx = \sin x - x \cos x.$

72. $\displaystyle\int \frac{1 - x}{1 + x}\, dx = \log (1 + x)^2 - x.$

73. $\displaystyle\int \frac{x^2\, dx}{(a^3 + x^3)^{\frac{3}{2}}} = -\frac{2}{3} \frac{1}{(a^3 + x^3)^{\frac{1}{2}}}.$

74. $\displaystyle\int \frac{dx}{(1 + x^2) \tan^{-1} x} = \log (\tan^{-1} x).$

75. $\displaystyle\int \frac{dx}{\sqrt{5 + 4x - x^2}} = 2 \sin^{-1} \sqrt{\frac{x + 1}{6}} = \cos^{-1} \frac{2 - x}{3}.$

76. $\displaystyle\int \frac{\cos (\log x)}{x}\, dx = \sin (\log x).$ Put $\overset{z}{x} = \log \overset{x}{z}.$

77. $\displaystyle\int \frac{dx}{4 - 5 \sin x} = \frac{1}{3} \log \frac{\tan \frac{1}{2}x - 2}{2 \tan \frac{1}{2}x - 1}.$

78. $\displaystyle\int \frac{dx}{5 - 4 \cos 2x} = \frac{1}{3} \tan^{-1} (3 \tan x).$

CHAPTER XVIII.

GENERAL INTEGRALS.

General Forms Directly Integrable.

135. The Binomial Differentials.—Expressions of the type

$$x^{\alpha}(a + bx^{\beta})^{\gamma}\, dx, \tag{A}$$

where α, β, γ are any rational numbers, are called *binomial differentials.*

This expression is directly integrable in *two* cases.

I. When $\dfrac{\alpha + 1}{\beta}$ is a *positive integer.*

The substitution is $a + bx^{\beta} = z$. Then

$$x = \left(\frac{z - a}{b}\right)^{\frac{1}{\beta}}, \quad dx = \frac{1}{b\beta}\left(\frac{z - a}{b}\right)^{\frac{1}{\beta} - 1} dz;$$

hence

$$x^{\alpha}(a + bx^{\beta})^{\gamma}\, dx = \frac{(z - a)^{\frac{\alpha+1}{\beta} - 1} z^{\gamma}}{\beta b^{\frac{\alpha+1}{\beta}}}\, dz.$$

Consequently, when $\dfrac{\alpha + 1}{\beta}$ is a positive integer, the transformed expression can be expanded by the binomial formula and immediately integrated.

II. When $\dfrac{\alpha + 1}{\beta} + \gamma$ is a *negative integer.*

The substitution is $a + bx^{\beta} = zx^{\beta}$.

For, if we substitute $x = 1/y$ in the differential $x^{\alpha}(a + bx^{\beta})^{\gamma}dx$, it becomes

$$-\, y^{-\beta\gamma - \alpha - 2}(ay^{\beta} + b)^{\gamma}\, dy,$$

which, by I, is integrable when $-\dfrac{\beta\gamma + \alpha + 1}{\beta}$ is a positive integer, or, what is the same thing, when

$$\frac{\alpha + 1}{\beta} + \gamma$$

is a negative integer. Also, the transformation $a + bx^\beta = z$ becomes $b + ay^\beta = zy^\beta$.

Hence, under the transformation,

$$x^\alpha(a + bx^\beta)^\gamma\,dx = \frac{1}{\beta}a^{\frac{\alpha+1}{\beta}+1}(b - z)^{-\left(\frac{\alpha+1}{\beta}+\gamma+1\right)}z^\gamma dz.$$

In working examples it is better to make the transformations than to use the transformed general formulæ, which are too complicated to be remembered.

When α, β, γ do not satisfy the conditions in I, II, the binomial differential must be reduced by parts.*

EXAMPLES.

1. $\int\frac{x^5\,dx}{(a^2 - x^2)^2}$. Ans. $\frac{a^4}{2(a^2 - x^2)} + \frac{x^2}{2} + a^2\log(a^2 - x^2)$.

2. $\int\frac{x^3\,dx}{(a + cx^2)^4}$. Ans. $-\frac{1}{4c^2(a + cx^2)^2} + \frac{a}{6c^2(a + cx^2)^3}$.

3. $\int\frac{x^5\,dx}{(1 + x^2)^3}$. Ans. $\frac{1}{x^2 + 1} - \frac{1}{4(x^2 + 1)^2} + \frac{1}{2}\log(x^2 + 1)$.

4. $\int\frac{dx}{(a + cx^2)^{\frac{5}{2}}}$. Ans. $\frac{x}{a^2(a + cx^2)^{\frac{1}{2}}}\left\{1 - \frac{cx^2}{3(a + cx^2)}\right\}$.

5. $\int\frac{x^2\,dx}{(a + cx^2)^{\frac{7}{2}}}$. Ans. $\frac{x^3}{a^2(a + cx^2)^{\frac{3}{2}}}\left\{\frac{1}{3} - \frac{cx^2}{5(a + cx^2)}\right\}$.

6. $\int\frac{x^3\,dx}{(a^2 + x^2)^{\frac{5}{2}}}$. Ans. $-\frac{2a^2 + 3x^2}{3(a^2 + x^2)^{\frac{3}{2}}}$.

7. $\int\frac{x^5\,dx}{(1 + x^3)^{\frac{1}{2}}}$. Ans. $\frac{2}{9}(1 + x^3)^{\frac{1}{2}}(x^3 - 2)$.

8. $\int\frac{dx}{(1 + x^3)^{\frac{4}{3}}}$. Ans. $\frac{x}{(1 + x^3)^{\frac{1}{3}}}$.

9. $\int\frac{dx}{x^2(1 + x^4)^{\frac{3}{4}}}$. Ans. $-\frac{1}{x}(1 + x^4)^{\frac{1}{4}}$.

10. $\int\frac{dx}{x^{\frac{1}{2}}(1 + x^2)^{\frac{5}{4}}}$. Ans. $\frac{2x^{\frac{1}{2}}}{(1 + x^2)^{\frac{1}{4}}}$.

136. Integration of $\frac{dx}{(A + Cx^2)(a + cx^2)^{\frac{1}{2}}}$. (B)

$x^2 = \frac{a}{z^2 - c} = -\frac{a}{c - z^2}$

The substitution is $a + cx^2 = x^2z^2$.

$$\therefore\quad c\,dx = z^2\,dx + zx\,dz, \quad\text{or}\quad \frac{dx}{xz} = \frac{dz}{c - z^2}.$$

$A + Cx^2 = Ac -$

$$\therefore\quad \frac{dx}{(A + Cx^2)(a + cx^2)^{\frac{1}{2}}} = \frac{dz}{(Ac - Ca) - Az^2},$$

which is standardized, being 13 or 14 (§ 126) according as $(Ac - Ca)/A$ is negative or positive.

---

* For formulæ of reduction see Appendix, Note 10.

If $(Ac - Ca)/A = -$, the integral is

$$\frac{1}{\sqrt{A(Ca - Ac)}} \tan^{-1} \frac{x\sqrt{Ca - Ac}}{\sqrt{A(a + cx^2)}}.$$

If $(Ac - Ca)/A = +$, the integral is

$$\frac{1}{2\sqrt{A(Ac - Ca)}} \log \frac{\sqrt{A(a + cx^2)} + x\sqrt{Ac - Ca}}{\sqrt{A(a + cx^2)} - x\sqrt{Ac - Ca}}.$$

EXAMPLES

1. $\displaystyle\int \frac{dx}{(1 + x^2)(1 - x^2)^{\frac{1}{2}}}.$ Ans. $\displaystyle\frac{1}{\sqrt{2}} \tan^{-1} \frac{x\sqrt{2}}{\sqrt{1 - x^2}}.$

2. $\displaystyle\int \frac{dx}{(3 + 4x^2)(4 - 3x^2)^{\frac{1}{2}}}.$ Ans. $\displaystyle\frac{1}{5\sqrt{3}} \tan^{-1} \frac{5x}{\sqrt{12 - 9x^2}}.$

3. $\displaystyle\int \frac{dx}{(4 - 3x^2)(3 + 4x^2)^{\frac{1}{2}}}.$ Ans. $\displaystyle\frac{1}{20} \log \frac{2(3 + 4x^2)^{\frac{1}{2}} + 5x}{2(3 + 4x^2)^{\frac{1}{2}} - 5x}.$

137. Integration of $\displaystyle\frac{p + qx}{a + 2bx + cx^2}dx.$ (C)

This is a particular and simple case of the rational fraction which will be treated generally in § 148. On account of its special importance we give it separate treatment here.

Let L represent the *linear* function $p + qx$.

Let Q represent the quadratic function $a + 2bx + cx^2$.

I. Consider $\displaystyle\int \frac{dx}{Q}.$

Completing the square in Q, we have

$$\int \frac{dx}{a + 2bx + cx^2} = \int \frac{c\,dx}{(cx + b)^2 - (b^2 - ac)}.$$

Put $cx + b = z$. Then the integral becomes

$$\int \frac{dz}{z^2 - (b^2 - ac)}.$$

This is standardized, and depends on whether $b^2 - ac$ is positive or negative. If *negative*, the roots of the denominator are *imaginary* and the integral is an *angle*, the standard 13. If *positive*, the roots of the denominator are *real* and the integral is a *logarithm*, the standard 14 (§ 126).

If $ac > b^2$,

$$\int \frac{dx}{Q} = \frac{1}{\sqrt{ac - b^2}} \tan^{-1} \frac{cx + b}{\sqrt{ac - b^2}}. \tag{1}$$

If $ac < b^2$,

$$\int \frac{dx}{Q} = \frac{1}{2\sqrt{b^2 - ac}} \log \frac{cx + b - \sqrt{b^2 - ac}}{cx + b + \sqrt{b^2 - ac}}. \tag{2}$$

II. Consider $\int \frac{L}{Q}\,dx$.

Since the derivative, Q', of Q is a linear function, we can always determine two constants A and B, such that

$$L \equiv A + BQ',$$

or

$$p + qx \equiv A + 2bB + 2cBx.$$

Equating the constant terms and coefficients of x,

$$B = q/2c, \qquad A = p - bq/c.$$

$$\therefore \quad \int \frac{L}{Q}dx = \frac{pc - qb}{c}\int \frac{dx}{Q} + \frac{q}{2c}\int \frac{dQ}{Q}.$$

The first integral has been reduced in (1), (2), and the second is $\log Q$.

In working examples, carry out the process and do not substitute in the general formula.

EXAMPLES.

1. $$\int \frac{x\,dx}{x^2 + 4x + 5} = \int \left\{ \frac{-2}{(x+2)^2 + 1} + \frac{1}{2}\,\frac{2x + 4}{x^2 + 4x + 5} \right\} dx,$$
$$= -2\tan^{-1}(x+2) + \tfrac{1}{2}\log(x^2 + 4x + 5).$$

2. $$\int \frac{x\,dx}{x^2 + 2x + 3} = -\frac{1}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}} + \frac{1}{2}\log(x^2 + 2x + 3).$$

3. $$\int \frac{x\,dx}{x^2 + 2x + 1} = \frac{1}{x+1} + \log(x+1).$$

4. $$\int \frac{(x+1)\,dx}{x^2 + 4x + 5} = \frac{1}{2}\log(x^2 + 4x + 5) - \tan^{-1}(x+2).$$

5. $$\int \frac{(x+1)\,dx}{3 + 2x - x^2} = -\log(3 - x).$$

6. $$\int \frac{2x^2 + 3x + 4}{x^2 + 6x + 10}dx = 2x - \log(x^2 + 6x + 10)^{\frac{9}{2}} + 11\tan^{-1}(x+3).$$

7. $$\int \frac{(x-1)^2\,dx}{x^2 + 2x + 2} = x - \log(x^2 + 2x + 2)^2 + 3\tan^{-1}(x+1).$$

8. To integrate $\int \frac{F(x)}{Q}dx$, where $F(x)$ is any polynomial in x, divide $F(x)$ by Q until the remainder is of the form L/Q, and integrate.

138. Integration of $\frac{(p + qx)^{\pm 1}}{\sqrt{a + 2bx + cx^2}}dx$. (D)

Let, as in § 137, L and Q represent the linear and quadratic functions respectively.

I. Consider $\int \frac{dx}{Q^{\frac{1}{2}}}$.

Complete the square in the quadratic, and then

$$\int \frac{dx}{Q^{\frac{1}{2}}} = \sqrt{c}\int \frac{dx}{\sqrt{(cx + b)^2 - (b^2 - ac)}},$$

which is the standard 11 or 12 according as b^2 is greater or less than ac. If a and c are both negative and $ac > b^2$, the function is imaginary.

We have, according as the roots of Q are real or imaginary,

$$\frac{1}{\sqrt{c}} \log [cx + b + \sqrt{c(a + 2bx + cx^2)}],$$

$$\frac{1}{\sqrt{c}} \sin^{-1} \frac{cx + b}{\sqrt{ac + b^2}},$$

as the corresponding values of the integral.

II. Consider $\int \frac{L}{Q^{\frac{1}{2}}}\, dx.$

Write, as in II, § 137, $L = A + BQ'$, and determine A and B. Then

$$\int \frac{L}{Q^{\frac{1}{2}}}\, dx = A \int \frac{dx}{Q^{\frac{1}{2}}} + B \int \frac{dQ}{Q^{\frac{1}{2}}}.$$

The first integral on the right was reduced in I, the second is $2Q^{\frac{1}{2}}$.

III. Consider $\int \frac{dx}{LQ^{\frac{1}{2}}}.$

Put $p + qx = 1/z. \quad \therefore \quad \frac{q\, dx}{p + qx} = -\frac{dz}{z}, \quad x = \frac{1 - pz}{qz}.$

Substitute in the integral and it transforms into

$$-\int \frac{dz}{\sqrt{a' + 2b'z + c'z^2}},$$

which can be integrated by I, then replace z by $1/(p + qx)$.

EXAMPLES.

1. $\int \frac{dx}{\sqrt{x^2 - ax}} = 2 \log (\sqrt{x} + \sqrt{x - a}).$

2. $\int \frac{dx}{\sqrt{ax - x^2}} = 2 \sin^{-1}\sqrt{\frac{x}{a}} = \sin^{-1}\left(\frac{2x}{a} - 1\right).$

3. $\int \frac{dx}{\sqrt{3x - x^2 - 2}} = 2 \sin^{-1}\sqrt{x - 1} = \sin^{-1}(2x - 3).$

4. $\int \frac{dx}{\sqrt{1 + x + x^2}} = \log (2x + 1 + 2\sqrt{1 + x + x^2}).$

5. $\int \sqrt{\frac{x + a}{x + b}}\, dx = \sqrt{(x + a)(x + b)} + (a - b) \log (\sqrt{x + a} + \sqrt{x + b}).$

6. $\int \frac{dx}{\sqrt{1 - x - x^2}} = \sin^{-1}\frac{2x + 1}{\sqrt{5}}.$

7. $\displaystyle\int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\cos^{-1}\frac{a}{x}.$

8. $\displaystyle\int\frac{dx}{(1+x)\sqrt{1-x^2}}=-\sqrt{\frac{1-x}{1+x}}.$

9. $\displaystyle\int\frac{dx}{x\sqrt{x^2+a^2}}=-\frac{1}{a}\log\frac{a+\sqrt{a^2+x^2}}{x}.$

10. $\displaystyle\int\frac{dx}{(1+x)\sqrt{1+2x-x^2}}=\frac{1}{\sqrt{2}}\sin^{-1}\frac{x\sqrt{2}}{1+x}.$

11. $\displaystyle\int\frac{dx}{(x-1)\sqrt{x^2-2x+3}}=\frac{1}{\sqrt{2}}\log\frac{\sqrt{x^2-2x+3}+\sqrt{2}}{x-1}.$

12. $\displaystyle\int\frac{(x+3)\,dx}{\sqrt{x^2+2x+3}}=\sqrt{x^2+2x+3}+\log\left(x+1+\sqrt{x^2+2x+3}\right)^2.$

REDUCTION BY PARTS.

139. Integration of Powers of Sine and Cosine.

$$\int\sin^n x\,dx=\int\sin^{n-1}x\sin x\,dx.$$

Put $u=\sin^{n-1}x,\qquad dv=\sin x\,dx;$

$\therefore\ du=(n-1)\sin^{n-2}x\cos x\,dx,\qquad v=-\cos x.$

Hence, applying the formula for parts,

$$\begin{aligned}\int\sin^n x\,dx&=-\sin^{n-1}x\cos x+(n-1)\int\sin^{n-2}x\cos^2 x\,dx,\\&=-\sin^{n-1}x\cos x+(n-1)\int\sin^{n-2}x(1-\sin^2 x)\,dx,\\&=-\sin^{n-1}x\cos x+(n-1)\int\sin^{n-2}x\,dx-(n-1)\int\sin^n x\,dx.\end{aligned}$$

$$\therefore\int\sin^n x\,dx=-\frac{\sin^{n-1}x\cos x}{n}+\frac{n-1}{n}\int\sin^{n-2}x\,dx.\qquad(1)$$

When n is a positive integer this reduces the exponent by 2, and leads to $\int dx$ or $\int\sin x\,dx$ according as n is even or odd.

Since integration by parts depends only on the differential equation $d(uv)=u\,dv+v\,du$, the formula is true when n is any positive or negative rational number.

Change n into $-n+2$ in (1), and we have

$$\int\frac{dx}{\sin^n x}=\frac{-\cos x}{(n-1)\sin^{n-1}x}+\frac{n-2}{n-1}\int\frac{dx}{\sin^{n-2}x}.\qquad(2)$$

In (1) and (2) change x into $\frac{1}{2}\pi-x$, then

$$\int\cos^n x\,dx=\frac{\cos^{n-1}x\sin x}{n}+\frac{n-1}{n}\int\cos^{n-2}x\,dx,\qquad(3)$$

$$\int\frac{dx}{\cos^n x}=\frac{\sin x}{(n-1)\cos^{n-1}x}+\frac{n-2}{n-1}\int\frac{dx}{\cos^{n-2}x}.\qquad(4)$$

These formulæ are important. They reduce the integrals to standard forms whenever n is an integer.

Formulæ (1), (2), (3), (4) can be obtained directly and independently by integration by parts. In practice this is the better method. The separation into the parts u and dv is indicated in each case in the formulæ below.

$$\int \begin{matrix}\sin^n\\ \cos^n\end{matrix} x\,dx = \int \begin{matrix}\sin^{n-1}\\ \cos^{n-1}\end{matrix} x \times \begin{matrix}\sin\\ \cos\end{matrix} x\,dx,$$

$$\int \begin{matrix}\sec^n\\ \csc^n\end{matrix} x\,dx = \int \begin{matrix}\sec^{n-2}\\ \csc^{n-2}\end{matrix} x \times \begin{matrix}\sec^2\\ \csc^2\end{matrix} x\,dx.$$

In the part $\int v\,du$ use $\sin^2 x + \cos^2 x = 1$, $\sec^2 x = 1 + \tan^2 x$, or $\csc^2 x = 1 + \cot^2 x$, as the case requires.

EXAMPLES.

1. $\int \sin^2 x\,dx = -\frac{\sin x \cos x}{2} + \frac{1}{2}x = \frac{1}{2}x - \frac{1}{4}\sin 2x.$

2. $\int \sin^3 x\,dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x = \frac{1}{3}\cos^3 x - \cos x.$

3. $\int \sin^4 x\,dx = -\frac{1}{4}\cos x \sin x\,(\sin^2 x + \frac{3}{2}) + \frac{3}{8}x.$

4. $\int \sin^5 x\,dx = -\frac{1}{5}\sin^4 x \cos x + \frac{4}{5}\int \sin^4 x\,dx.$

5. $\int \sin^6 x\,dx = -\frac{1}{2}\cos x\,(\frac{1}{3}\sin^5 x + \frac{5}{12}\sin^3 x + \frac{5}{8}\sin x) + \frac{5}{16}x.$

6. Find the corresponding values for cos x, integrating by parts. Check the result by putting $\frac{1}{2}\pi - x$ for x.

7. $\int \frac{dx}{\sin x} = \log \tan \frac{1}{2}x = \log(\csc x - \cot x).$

8. $\int \frac{dx}{\sin^2 x} = -\cot x.$

9. $\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\log \tan \frac{x}{2}.$

10. $\int \frac{dx}{\sin^4 x} = -\frac{1}{3}\frac{\cos x}{\sin^3 x} - \frac{2}{3}\cot x.$

11. $\int \frac{dx}{\sin^5 x} = -\frac{1}{4}\frac{\cos x}{\sin^4 x} - \frac{3}{8}\frac{\cos x}{\sin^2 x} + \frac{3}{8}\log \tan \frac{x}{2}.$

12. $\int \frac{dx}{\sin^6 x} = -\frac{1}{5}\frac{\cos x}{\sin^5 x} - \frac{4}{15}\frac{\cos x}{\sin^3 x} - \frac{8}{15}\cot x.$

13. Deduce the corresponding integrals of cos x, and check the result by putting $\frac{1}{2}\pi - x$ for x.

140. Integration of $\int \sin^m x \cos^n x \, dx$.

We have for all positive or negative rational values of m and n

$$\frac{d}{dx}\frac{\sin^{m-1}x}{\cos^{n-1}x} = (m-1)\frac{\sin^{m-2}x}{\cos^{n-2}x} + (n-1)\frac{\sin^m x}{\cos^n x}.$$

Therefore

$$\int \frac{\sin^m x}{\cos^n x} dx = \frac{1}{n-1}\frac{\sin^{m-1}x}{\cos^{n-1}x} - \frac{m-1}{n-1}\int \frac{\sin^{m-2}x}{\cos^{n-2}x} dx. \qquad (5)$$

In particular, when $m = n$,

$$\int \tan^n x \, dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x \, dx. \qquad (6)$$

Put $\frac{1}{2}\pi - x$, for x in (5) and (6). Then

$$\int \frac{\cos^m x}{\sin^n x} dx = \frac{-1}{n-1}\frac{\cos^{m-1}x}{\sin^{n-1}x} - \frac{m-1}{n-1}\int \frac{\cos^{m-2}x}{\sin^{n-2}x} dx, \qquad (7)$$

$$\int \cot^m x \, dx = -\frac{\cot^{n-1}x}{n-1} - \int \cot^{n-2}x \, dx. \qquad (8)$$

The same results are obtained immediately by changing the signs of m and n.

Change the sign of n in (5), then

$$\int \sin^m x \cos^n x \, dx = -\frac{\sin^{m-1}x \cos^{n+1}x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2}x \cos^{n+2}x \, dx.$$

But $\sin^{m-2}x \cos^{n+2}x = \sin^{m-2}x \cos^n x(1 - \sin^2 x)$,

$$= \sin^{m-2}x \cos^n x - \sin^m x \cos^n x.$$

Substituting and solving, we have

$$\int \sin^m x \cos^n x \, dx = \frac{m-1}{m+n}\int \sin^{m-2}x \cos^n x \, dx - \frac{\sin^{m-1}x \cos^{n+1}x}{m+n}. \qquad (9)$$

In like manner, change the sign of n in (7) and write $1 - \cos^2 x$ for $\sin^2 x$ in the last integral. Then

$$\int \cos^m x \sin^n x \, dx = \frac{m-1}{n+m}\int \cos^{m-2}x \sin^n x \, dx + \frac{\cos^{m-1}x \sin^{n+1}x}{m+n}. \qquad (10)$$

These formulæ serve to integrate $\sin^n x \cos^m x \, dx$ whatever be the integers m and n.

It is well to be able to integrate the functions of this article independently. The forms below show the separation into the parts u and dv which effect the integration directly when the trigonometrical relations $\sin^2 x + \cos^2 x = 1$, $\sec^2 x = 1 + \tan^2 x$, $\csc^2 x = 1 + \cot^2 x$ are used in the integral $\int v \, du$.

$$\int \sin^m x \cos^n x dx = \int \sin^{m-1} x \cos^n x \times \sin x dx = \int \sin^m x \cos^{n-1} x \times \cos x dx,$$

$$\int \begin{matrix}\tan^n \\ \cot^n\end{matrix} x\, dx = \int \begin{matrix}\tan^{n-2} \\ \cot^{n-2}\end{matrix} x \times \begin{matrix}\tan^2 \\ \cot^2\end{matrix} x\, dx.$$

EXAMPLES.

1. $\int \cos^2 x \sin^4 x\, dx = \frac{1}{2} \sin x \cos x (\frac{1}{3} \sin^4 x - \frac{1}{12} \sin^2 x - \frac{1}{8}) + \frac{1}{16} x.$

2. $\int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \log \tan \frac{1}{2} x.$

3. $\int \frac{dx}{\sin^3 x \cos^2 x} = \frac{1}{\cos x} - \frac{\cos x}{2 \sin^2 x} + \frac{3}{2} \log \tan \frac{x}{2}.$

4. $\int \tan^4 x\, dx = \frac{1}{3} \tan^3 x - \tan x + x.$

5. $\int \cot^4 x\, dx = -\frac{1}{3} \cot^3 x + \cot x + x.$

6. $\int \frac{dx}{\tan^3 x} = -\frac{1}{2 \tan^2 x} - \log (\sin x).$

7. $\int \frac{dx}{\tan^5 x} = \frac{-1}{4 \tan^4 x} + \frac{1}{2 \tan^2 x} + \log (\sin x).$

INTEGRATION OF RATIONAL FUNCTIONS.

141. General Statement.—Any rational function of x whose numerator is a polynomial N and denominator a polynomial D can by division be decomposed into

$$\frac{N}{D} = Q + \frac{R}{D},$$

where Q is a polynomial, and the degree of R is that of D less 1. We then have

$$\int \frac{N}{D} dx = \int Q\, dx + \int \frac{R}{D} dx.$$

The first integral on the right can be written out directly. The second integral demands our attention. We know from the theory of equations (C. Smith's Algebra, § 436) that every polynomial in x of degree n has n roots, real or imaginary, and can be written

$$A(x - a_1)(x - a_2) \dots (x - a_n).$$

If there is no second root equal to a_1, then a_1 is said to be a single root. If, however, there is another root equal to a_1, say $a_2 = a_1$, then the two factors can be written $(x - a_1)^2$, and we say that a_1 is a double root, or that the polynomial has two equal roots. In like manner, if there are r equal roots equal to a, the corresponding factor is $(x - a)^r$, and we say that a is a multiple root of order r, or the polynomial has r equal roots of value a.

Again, we know that if the coefficients in the polynomial are all real, then imaginary roots must occur in conjugate pairs (C. Smith, Algebra, § 446). Therefore, if there is an imaginary root $a + b\sqrt{-1}$, there must be another $a - b\sqrt{-1}$. Now the product of the factors corresponding to these two roots is

$$\begin{aligned}(x - a - b\sqrt{-1})(x - a + b\sqrt{-1}) &= (x-a)^2 + b^2,\\ &= x^2 - 2ax + a^2 + b^2,\end{aligned}$$

which can be written $\qquad = x^2 + px + q.$

Moreover, if $a + ib$ $(i \equiv \sqrt{-1})$ is a multiple root of order r, so also is $a - ib$, and we have the corresponding factor in the polynomial

$$(x^2 + px + q)^r.$$

Hence any polynomial in x is composed of factors, linear and quadratic, of the types

$$x - a, \quad (x - b)^r, \quad x^2 + px + q, \quad (x^2 + px + q)^r.$$

If
$$\frac{F(x)}{f(x)},$$

be a rational function, in which $F(x)$ is of a degree at least 1 lower than that of $f(x)$, we can always decompose the function into the sum of partial fractions corresponding to the roots of $f(x)$, as follows:

For each single real root a there is a fraction

$$\frac{A}{x - a};$$

for each multiple real root b of order r there are r fractions

$$\frac{B_1}{(x - b)} + \frac{B_2}{(x - b)^2} + \ldots + \frac{B_r}{(x - b)^r};$$

for each pair of conjugate imaginary roots there is a fraction

$$\frac{C + Dx}{x^2 + px + q};$$

for each pair of conjugate multiple imaginary roots of order s there are s fractions of the types

$$\frac{E_1 + xF_1}{x^2 + \alpha x + \beta} + \frac{E_2 + xF_2}{(x^2 + \alpha x + \beta)^2} + \ldots + \frac{E_s + xF_s}{(x^2 + \alpha x + \beta)^s}.$$

In these partial fractions the numbers A, B, C, D, E, F, etc., are constants. Since there are exactly as many of these constants as there are roots of $f(x)$, they are n in number.

If now we equate $F(x)/f(x)$ to the sum of the partial fractions and multiply the equation through by $f(x)$, we shall have $F(x)$ equal to a polynomial in x of degree $n - 1$. When we equate the constant terms and the coefficients of like powers of x on each side of this equation, we have n linear equations in the constants A, B, C, etc., which serve to determine their values.*

The integral of the rational function then depends on

$$\int \frac{dx}{(x-a)^r} \quad \text{and} \quad \int \frac{(E+Fx)\,dx}{(x^2+px+q)^r}.$$

The first of these can be integrated immediately, the second is always of the type

$$\int \frac{(E+xF)\,dx}{[(x-a)^2+b^2]^r} = (E+aF)\int \frac{dz}{(z^2+b^2)^r} + F\int \frac{z\,dz}{(z^2+b^2)^r},$$

wherein $x = a + z$. The last integral on the right is

$$\int \frac{z\,dz}{(z^2+b^2)^r} = \frac{1}{2}\int \frac{d(z^2)}{(z^2+b^2)^r} = \frac{1}{2(r-1)}\,\frac{-1}{(z^2+b^2)^{r-1}}.$$

To integrate the first integral on the right,† put $z = b \tan \theta$.

$$\therefore \quad dz = b \sec^2\theta\, d\theta.$$

Then $$\int \frac{dz}{(z^2+b^2)^r} = \frac{1}{b^{2r-1}}\int \cos^{2r-2}\theta\, d\theta,$$

which can always be integrated by parts, § 139.

Hence the rational function can always be integrated.

EXAMPLES.

1. $\displaystyle\int \frac{x^2+6x-8}{x^3-4x}\,dx.$

We have here single real roots; hence

$$\frac{x^2+6x-8}{x^3-4x} = \frac{x^2+6x-8}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}.$$

Clearing of fractions,

$$x^2+6x-8 = A(x-2)(x+2) + Bx(x+2) + C(x-2)x, \qquad (1)$$
$$= (A+B+C)x^2 + 2(B-C)x - 4A.$$

Equating coefficients,

$$A+B+C = 1, \qquad 2(B-C) = 6, \qquad -4A = -8.$$
$$\therefore \quad A = 2, \qquad B = 1, \qquad C = -2.$$

Hence the integral is

$$\int \frac{x^2+6x-8}{x^3-4x}\,dx = 2\log x + \log(x-2) - 2\log(x+2),$$
$$= \log \frac{x^2(x-2)}{(x+2)^2}.$$

If we assign particular values to x in (1), we can find A, B, C more easily. Thus put $x = 0$, then $-4A = -8$; put $x = 2$, then

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[*] Provided these n equations are independent, which they are.

[†] See also Ex. 88, at the end of the chapter.

$8B = 8$; put $x = -2$, then $8C = -16$, which give the constants at once. The general principle involved in this abbreviated process is: when there are only single roots, put x equal to each root in turn, and the constants are immediately determined.

2. $\displaystyle\int \frac{(x-1)\,dx}{(x-3)(x+2)} = \tfrac{2}{5}\log(x-3) + \tfrac{3}{5}\log(x+2).$

3. $\displaystyle\int \frac{x\,dx}{x^2+2x-3} = \tfrac{3}{4}\log(x+3) + \tfrac{1}{4}\log(x-1).$

4. $\displaystyle\int \frac{x^2+x-1}{x^3+x^2-6x}\,dx = \tfrac{1}{6}\log x + \tfrac{1}{2}\log(x-2) + \tfrac{1}{3}\log(x+3).$

5. $\displaystyle\int \frac{1+x^2}{x-x^3}\,dx = \log\frac{x}{1-x^2}.$

6. $\displaystyle\int \frac{3x-1}{x^2+x-6}\,dx = \log\left[(x+3)^2(x-2)\right].$

7. $\displaystyle\int \frac{x^4\,dx}{(x^2-1)(x+2)} = \frac{x^2}{2} - 2x + \tfrac{1}{6}\log\frac{x-1}{(x+1)^3} + \tfrac{16}{3}\log(x+2).$

8. $\displaystyle\int \frac{x^3+1}{x(x-1)^3}\,dx.$ Here there is one single root, 0, and a triple root, $x = 1$.

Hence

$$\frac{x^3+1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{(x-1)^3} + \frac{C}{(x-1)^2} + \frac{D}{x-1}.$$

Clearing of fractions, we have

$$x^3 + 1 = (A+D)x^3 + (C - 3A - 2D)x^2 + (3A + B - C + D)x - A.$$

$$\begin{aligned} \therefore \quad 1 &= A + D, \\ 0 &= C - 3A - 2D, \\ 0 &= 3A + B - C + D, \\ 1 &= -A. \end{aligned}$$

Whence $\qquad A = -1, \quad B = 2, \quad C = 1, \quad D = 2.$

$$\therefore \quad \frac{x^3+1}{x(x-1)^3} = -\frac{1}{x} + \frac{2}{(x-1)^3} + \frac{1}{(x-1)^2} + \frac{2}{x-1}.$$

$$\begin{aligned} \int \frac{x^3+1}{x(x-1)^3}\,dx &= -\log x - \frac{1}{(x-1)^2} - \frac{1}{x-1} + 2\log(x-1), \\ &= \log\frac{(x-1)^2}{x} - \frac{x}{(x-1)^2}. \end{aligned}$$

9. $\displaystyle\int \frac{(x-8)\,dx}{x^3-4x^2+4x} = \frac{3}{x-2} + \log\frac{(x-2)^2}{x^2}.$

10. $\displaystyle\int \frac{3x^2-2}{(x+2)^3}\,dx = \frac{12x+19}{(x+2)^2} + 3\log(x+2).$

11. $\displaystyle\int \frac{6x^3-8x^2-4x+1}{x^4-2x^3+x^2}\,dx = \log\frac{(x-1)^8}{x^2} + \frac{4x+1}{x(x-1)}.$

12. $\displaystyle\int \frac{x\,dx}{(x+1)(x^2+1)}.$ Here there are a pair of imaginary roots.

$$\therefore \quad \frac{x}{(x+1)(x^2+1)} = \frac{A}{1+x} + \frac{Lx+M}{1+x^2}.$$

Clearing of fractions,

$$x = A(1 + x^2) + (Lx + M)(1 + x),$$
$$= (A + M) + (L + M)x + (A + L)x^2.$$

Equating coefficients,

$$L + A = 0, \quad L + M = 1, \quad A + M = 0.$$
$$\therefore \quad L = \tfrac{1}{2}, \qquad M = \tfrac{1}{2}, \qquad A = -\tfrac{1}{2}.$$
$$\therefore \quad \int \frac{x\,dx}{(x + 1)(x^2 + 1)} = \frac{1}{4} \log \frac{1 + x^2}{(1 + x)^2} + \frac{1}{2} \tan^{-1} x.$$

13. $\int \frac{dx}{1 + x^3}$. We have $1 + x^3 = (1 + x)(1 - x + x^2)$.

$$\therefore \quad \frac{1}{1 + x^3} = \frac{A}{1 + x} + \frac{Lx + M}{1 - x + x^2}.$$

Clear the fractions and put $x = -1$. Then $A = \frac{1}{3}$. Substituting this, we get

$$3(Lx + M) = 2 - x.$$
$$\therefore \int \frac{dx}{1 + x^3} = \frac{1}{3} \int \frac{dx}{1 + x} + \frac{1}{3} \int \frac{(2 - x)dx}{1 - x + x^2},$$
$$= \frac{1}{3} \log (1 + x) - \frac{1}{6} \log (1 - x + x^2) + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x - 1}{\sqrt{3}}.$$

14. $\int \frac{dx}{1 - x^3} = \frac{1}{6} \log \frac{1 + x + x^2}{1 - 2x + x^2} + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x + 1}{\sqrt{3}}.$

15. $\int \frac{x^2\,dx}{(x - 1)^2(x^2 + 1)}$.

$$\frac{x^2}{(x - 1)^2(x^2 + 1)} = \frac{A}{(x - 1)^2} + \frac{B}{x - 1} + \frac{Lx + M}{1 + x^2}.$$
$$x^2 = A(1 + x^2) + B(x - 1)(x^2 + 1) + (Lx + M)(x - 1)^2.$$

Equating coefficients,

$$A - B + M = 0,$$
$$A - B - 2L + M = 1,$$
$$B + L = 0,$$
$$B + L - 2M = 0.$$
$$\therefore \quad M = 0, \quad B = A = \tfrac{1}{2}, \quad L = -\tfrac{1}{2}.$$

Hence the integral is

$$-\frac{1}{2} \frac{1}{x - 1} + \frac{1}{2} \log (x - 1) - \frac{1}{4} \log (x^2 + 1).$$

16. $\int \frac{5x + 12}{x(x^2 + 4)}\,dx = 3 \log \frac{x}{\sqrt{x^2 + 4}} + \frac{5}{2} \tan^{-1} \frac{x}{2}.$

17. $\int \frac{(2x^2 - 3x - 3)dx}{(x - 1)(x^2 - 2x + 5)} = \log \frac{(x^2 - 2x + 5)^{\frac{3}{2}}}{x - 1} + \frac{1}{2} \tan^{-1} \frac{x - 1}{2}.$

18. $\int \frac{x^3 - 1}{x^3 + 3x}\,dx = x + \frac{1}{6} \log \frac{x^2 + 3}{x^2} - \sqrt{3} \tan^{-1} \frac{x}{\sqrt{3}}.$

19. $\int \frac{dx}{(x^2 + 1)(x^2 + x)} = \frac{1}{4} \log \frac{x^4}{(x + 1)^2(x^2 + 1)} - \frac{1}{2} \tan^{-1} x.$

20. $\int \frac{2x^3 + x + 3}{(x^2+1)^2} dx$. Here there is a double pair of imaginary roots. Hence we put

$$\frac{2x^3 + x + 3}{(x^2+1)^2} = \frac{Ax + B}{(x^2+1)^2} + \frac{Cx + D}{x^2 + 1}$$

$$\therefore \quad 2x^3 + x + 3 = Cx^3 + Dx^2 + (A + C)x + B + D.$$

$$\therefore \quad A = -1, \quad B = 3, \quad C = 2, \quad D = 0.$$

$$\therefore \quad \frac{2x^3 + x + 3}{(x^2+1)^2} = \frac{-x + 3}{(x^2+1)^2} + \frac{2x}{x^2+1}.$$

To integrate $\int \frac{dx}{(x^2+1)^2}$, put $x = \tan\theta$, then the integral becomes

$$\int \cos^2\theta \, d\theta = \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta = \tfrac{1}{2}\tan^{-1}x + \frac{x}{2(x^2+1)}.$$

$$\therefore \quad \int \frac{2x^3 + x + 3}{(x^2+1)^2} dx = \frac{3x + 1}{2(x^2+1)} + \tfrac{3}{2}\tan^{-1}x + \log(x^2+1).$$

21. $\int \frac{(3x+2)\,dx}{(x^2 - 3x + 3)^2} = \frac{13x - 24}{3(x^2 - 3x + 3)} + \frac{26}{3\sqrt{3}} \tan^{-1}\frac{2x-3}{\sqrt{3}}.$

22. $\int \frac{x^3 + x - 1}{(x^2+2)^2} dx = \frac{2 - x}{4(x^2+2)} + \log(x^2+2)^{\frac{1}{2}} - \frac{1}{4\sqrt{2}} \tan^{-1}\frac{x}{\sqrt{2}}.$

23. $\int \frac{(4x^2 - 8x)\,dx}{(x-1)^2(x^2+1)^2} = \frac{3x^2 - x}{(x-1)(x^2+1)} + \log\frac{(x-1)^2}{x^2+1} + \tan^{-1}x.$

142. Trigonometric Transformations.—On account of the simple character of the reduction formulæ in §§ 139, 140, it is often advantageous to transform many algebraic integrals to these forms, and conversely many trigonometrical formulæ can be transformed into useful algebraic forms.*

EXAMPLES.

1. Put $x = a\tan\theta$, then

$$\int \frac{x^m\,dx}{(a^2 + x^2)^{\frac{1}{2}n}} = a^{m-n+1} \int \sin^m\theta \cos^{n-m-2}\theta \, d\theta.$$

2. Put $x = a\sin\theta$, then

$$\int \frac{x^m\,dx}{(a^2 - x^2)^{\frac{1}{2}n}} = a^{m-n+1} \int \frac{\sin^m\theta}{\cos^{n-1}\theta} d\theta.$$

3. Put $x = a\sec\theta$, then

$$\int \frac{x^m\,dx}{(x^2 - a^2)^{\frac{1}{2}n}} = a^{m-n+1} \int \frac{\cos^{n-m-2}\theta}{\sin^{n-1}\theta} d\theta.$$

4. Put $x = 2a\sin^2\theta$, then

$$\int \frac{x^m\,dx}{(2ax - x^2)^{\frac{1}{2}n}} = 2^{m-n+2}a^{m-n+1} \int \frac{\sin^{2m-n+1}\theta}{\cos^{n-1}\theta} d\theta.$$

5. Make the same transformations in the above integrals when m or n is negative.

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* The reduction formulæ for the binomial differentials are given in the Appendix, Note 10.

The general integral

$$\int \frac{x^m dx}{(a + cx^2)^n}$$

can always be transformed to the trigonometric integral when the signs of a and c are known, whatever be the signs of m and n.

EXAMPLES.

1. Integrate by trigonometrical transformations

$$\int \sqrt{a^2 - x^2}\, dx, \quad \int \sqrt{x^2 - a^2}\, dx, \quad \int \sqrt{x^2 + a^2}\, dx,$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}}, \quad \int \frac{dx}{\sqrt{x^2 - a^2}}, \quad \int \frac{dx}{\sqrt{x^2 + a^2}}.$$

RATIONALIZATION.

143. Integration of Monomials.—If an algebraic function contains fractional powers of the variable x, it can be made rational by the substitution $x = z^n$, where n is the least common multiple of the denominators of the several fractional powers.

For example, $$\int \frac{\left(1 + x^{\frac{1}{4}}\right) dx}{1 + x^{\frac{1}{2}}}.$$

Put $x = z^4$. The transformed integral is

$$4\int \frac{z^3(1 + z)\, dz}{1 + z^2}.$$

Consequently the integral is

$$\tfrac{4}{3}x^{\frac{3}{4}} - 2x^{\frac{1}{2}} - 4x^{\frac{1}{4}} + 4 \tan^{-1} x^{\frac{1}{4}} - 2 \log \left(1 + x^{\frac{1}{2}}\right).$$

Again, any algebraic function containing integral powers of x along with fractional powers of a linear function $a + bx$ can be rationalized by the transformation $a + bx = z^n$, in the same way as above.

EXAMPLES.

1. $\displaystyle\int \frac{x^3\, dx}{\sqrt{x - 1}} = \frac{2}{35}\,(5x^3 + 6x^2 + 8x + 16)\sqrt{x - 1}.$

2. $\displaystyle\int \frac{x\, dx}{(a + bx)^{\frac{3}{2}}} = \frac{2}{b^2}\,\frac{2a + bx}{\sqrt{a + bx}},$ by $a + bx = z^2$.

Complete the differential, integrate and compare results.

3. $\displaystyle\int \frac{dx}{x + \sqrt{x - 1}} = \log(x + \sqrt{x - 1}) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\sqrt{x - 1} + 1}{\sqrt{3}}.$

4. $\displaystyle\int \frac{(x^2 \pm 1)dx}{x\sqrt{x^4 + ax^2 + 1}} = \int \frac{dz}{\sqrt{z^2 + a \pm 2}}.$ Put $x \mp x^{-1} = z$.

144. Observations on Integration.—As we have remarked before, comparatively few functions have primitives which can be expressed in a finite form of the elementary functions. For example,

$\int \sqrt{y}\,dx$, when y is a polynomial in x of degree higher than the second, is not, in general, an elementary function and cannot be expressed in finite form in terms of the elementary functions. If y is of the third or fourth degree, the integral defines a new class of functions called *elliptic* functions.

Functions that are non-integrable in terms of the elementary functions can frequently be expanded by Taylor's series and the integral evaluated by means of the infinite series.

Any rational algebraic function of x and $\sqrt{ax^2 + bx + c}$ can be rationalized and integrated as follows:

Factor out the coefficient of x^2 and let $y \equiv \sqrt{\pm x^2 + px + q}$.

The rational function $F(x, y)$ is rationalized in x:

I. When the coefficient of x^2 in y is positive, by the substitution

$$\sqrt{x^2 + px + q} = z - x.$$

Then $x = \dfrac{z^2 - q}{p + 2z}$, $z - x = \dfrac{z^2 + pz + q}{p + 2z}$, $dx = \dfrac{2(z^2 + pz + q)}{(p + 2z)^2}dz$.

$$\therefore \quad \int F(x, y)dx = 2\int F\left(\frac{z^2 - q}{p + 2z}, \frac{z^2 + pz + q}{p + 2z}\right)\frac{z^2 + pz + q}{(p + 2z)^2}dz.$$

II. When the coefficient of x^2 is negative and the roots of the quadratic α, β are real, then

$$-x^2 + px + q = (x - \alpha)(\beta - x).$$

The function $F(x, y)$ is rationalized by either of the substitutions

$$\sqrt{-x^2 + px + q} \equiv \sqrt{(x - \alpha)(\beta - x)} = (x - \alpha)z \text{ or } (\beta - x)z.$$

Then $x = \dfrac{\alpha z^2 + \beta}{1 + z^2}$, $dx = \dfrac{2z(\alpha - \beta)}{(1 + z^2)^2}dz$, $(x - \alpha)z = \dfrac{(\beta - \alpha)z}{1 + z^2}$.

$$\therefore \quad \int F(x, y)dx = 2(\alpha - \beta)\int F\left(\frac{\alpha z^2 + \beta}{1 + z^2}, \frac{(\beta - \alpha)z}{1 + z^2}\right)\frac{z\,dz}{(1 + z^2)^2}.$$

When the roots of $-x^2 + px + q$ are imaginary the radical is imaginary.

145. Integration by Infinite Series.—We know that if a function

$$f(x) = a_0 + a_1x + a_2x^2 + \dots$$

in an interval $)-H, +H($, then also its primitive is equal to the primitive of the series for this same interval (§ 72). Hence

$$\int f(x)dx = a_0x + \tfrac{1}{2}a_1x^2 + \tfrac{1}{3}a_2x^3 + \dots$$

EXAMPLES.

1. $\displaystyle\int \frac{dx}{\sqrt{1 - x^5}} = \frac{x}{1} + \frac{1}{2}\frac{x^6}{6} + \frac{1\cdot3}{2\cdot4}\frac{x^{11}}{11} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^{16}}{16} + \dots$

2. $\displaystyle\int \frac{dx}{\sqrt{\sin x}} = 2\sqrt{\sin x}\left(1 + \frac{1}{2}\frac{\sin^2 x}{5} + \frac{1\cdot3}{2\cdot4}\frac{\sin^4 x}{9} + \dots\right)$

Put $\sin x = z$. $\therefore\ dx = dz/\cos x$, and the integral is

$$\int \frac{dz}{z^{\frac{1}{2}}\sqrt{1 - z^2}}.$$

3. $\displaystyle\int (1 + cx^n)^{\frac{p}{q}} x^{m-1}\,dx = x^m\left(\frac{1}{m} + \frac{pc}{q}\,\frac{x^n}{m+n} + \frac{p(p-q)c^2}{2\,!\,q^2}\,\frac{x^{2n}}{m+2n} + \ldots\right)$

For what values of x is this true?

4. Show that

$$\int \frac{dx}{\sqrt{1 + x^4}} = \frac{x}{1} - \frac{1}{2}\,\frac{x^5}{5} + \frac{1\cdot 3}{2\cdot 4}\,\frac{x^9}{9} - \ldots, \qquad x^2 < 1.$$

$$= -\frac{1}{x} + \frac{1}{2}\,\frac{1}{5x^5} - \frac{1\cdot 3}{2\cdot 4}\,\frac{1}{9x^9} + \ldots, \qquad x^2 > 1.$$

5. Show that

$$\int \frac{e^{ax}}{b + x}\,dx = e^{-ab}\left\{\log(b + x) + \frac{a}{1}\,\frac{b + x}{1} + \frac{a^2}{2\,!}\,\frac{(b + x)^2}{2} + \ldots\right\}.$$

Determine the values of x for which this is true.
Put $b + x = z$. $\therefore\ e^{ax} = e^{-ab}e^{az}$, etc.

6. The elliptic integral $\displaystyle\int (1 - k^2 \sin^2 x)^{\frac{1}{2}}dx$, $k^2 < 1$, can always be expanded by the binomial formula, and the general term $\displaystyle\int \sin^{2n} x\,dx$ integrated.

7. $\displaystyle\int \frac{\sin x}{\sqrt{x}}\,dx = 2x^{\frac{3}{2}}\left(\frac{1}{3} - \frac{1}{7}\,\frac{x^2}{3\,!} + \frac{1}{11}\,\frac{x^4}{5\,!} - \ldots\right).$

EXERCISES.

1. $\displaystyle\int \frac{x^4\,dx}{(1 - x^2)^{\frac{1}{2}}} = \tfrac{3}{8}\sin^{-1} x - \tfrac{1}{8}x\sqrt{1 - x^2}\,(3 + 2x^2).$

2. $\displaystyle\int \frac{dx}{x^3\sqrt{1 - x^2}} = \tfrac{1}{2}\log\frac{1 - \sqrt{1 - x^2}}{x} - \frac{\sqrt{1 - x^2}}{2x^2}$

3. $\displaystyle\int \frac{dx}{(a^2 + x^2)^{\frac{5}{2}}} = \frac{x}{a^4(a^2 + x^2)^{\frac{1}{2}}} - \frac{x^3}{3a^4(a^2 + x^2)^{\frac{3}{2}}}.$

4. $\displaystyle\int \frac{x^4\,dx}{(a^2 + x^2)^2} = \frac{-x^3}{2(a^2 + x^2)} + \frac{3}{2}\left(x - a\tan^{-1}\frac{x}{a}\right).$

5. $\displaystyle\int \frac{x^2\,dx}{(2ax - x^2)^{\frac{1}{2}}} = -(2ax - x^2)^{\frac{1}{2}}\left(\tfrac{1}{2}x + \tfrac{3}{2}a\right) + 3a^2\sin^{-1}\sqrt{\frac{x}{2a}}$

6. $\displaystyle\int x^3 e^{ax}\,dx = \frac{e^{ax}}{a}\left(x^3 - \frac{3}{a}x^2 + \frac{6}{a^2}x - \frac{6}{a^3}\right).$

7. $\displaystyle\int x^3(\log x)^2\,dx = \tfrac{1}{4}x^4\left[(\log x)^2 - \tfrac{1}{2}\log x + \tfrac{1}{8}\right].$

8. $\displaystyle\int x^3\cos x\,dx = x^3\sin x + 3x^2\cos x - 6x\sin x - 6\cos x.$

9. $\displaystyle\int x^4\sin x\,dx = -x^4\cos x + 4x^3\sin x + 12x^2\cos x - 24(x\sin x + \cos x).$

10. $\displaystyle\int \frac{\sin^2\theta\,d\theta}{(1 + \cos\theta)^2} = 2\tan\tfrac{1}{2}\theta - \theta.$

11. $\int \cos^3\theta \sin 2\theta \, d\theta = -\frac{2}{5}\cos^5\theta.$

12. $\int \sin^2\theta \cos^3\theta \, d\theta = \frac{1}{3}\sin^3\theta - \frac{1}{5}\sin^5\theta.$

13. $\int \sin^5\theta \cos^5\theta \, d\theta = -\frac{1}{2^6}(\cos 2\theta - \frac{2}{3}\cos^3 2\theta + \frac{1}{5}\cos^5 2\theta).$

14. $\int \cos^4 x \csc x \, dx = \frac{1}{3}\cos^3 x + \cos x + \log\tan\frac{1}{2}x.$

15. $\int \cos^4 x \csc^3 x \, dx = (\cos^3 x - \frac{3}{2}\cos x)\csc^2 x - \frac{3}{2}\log\tan\frac{1}{2}x.$

16. $\int \frac{dx}{(2+3x)(4-x^2)^{\frac{1}{2}}} = \frac{1}{4\sqrt{2}}\log\frac{\sqrt{4+2x}-\sqrt{2-x}}{\sqrt{4+2x}+\sqrt{2-x}}.$

17. $\int \frac{dx}{(1+x^2)^{\frac{7}{2}}} = (\frac{8}{15}x^4 + \frac{4}{3}x^2 + 1)x(1+x^2)^{-\frac{5}{2}}.$

18. $\int \frac{dx}{[(a^2+x^2)^{\frac{1}{2}}+x]^{\frac{1}{4}}}.$ Put $z = (a^2+x^2)^{\frac{1}{2}} + x.$

Show that $\int x^m[(a^2+x^2)^{\frac{1}{2}}+x]^n dx$ can be integrated by the same substitution when m is a positive integer.

19. $\int \frac{[(1+x^2)^{\frac{1}{2}}+x]^n}{(1+x^2)^{\frac{1}{2}}}dx = \frac{1}{n}[(1+x^2)^{\frac{1}{2}}+x]^n.$

20. $\int \frac{x^{\frac{1}{2}}\,dx}{x^{\frac{3}{4}}+1} = \frac{4}{3}x^{\frac{3}{4}} - \frac{4}{3}\log(x^{\frac{3}{4}}+1).$

21. $\int \frac{dx}{x^{\frac{7}{6}}+x^{\frac{4}{3}}} = -\frac{6}{x^{\frac{1}{6}}} + \log\frac{(x^{\frac{1}{6}}+1)^6}{x}.$

22. $\int \frac{x^{\frac{1}{3}}+1}{x^{\frac{7}{6}}+x^{\frac{5}{4}}}dx = -\frac{6}{x^{\frac{1}{6}}} + \frac{12}{x^{\frac{1}{12}}} + \log x^2 - 24\log(x^{\frac{1}{12}}+1).$

23. $\int \frac{dx}{x^{\frac{5}{8}}-x^{\frac{1}{8}}} = \frac{8}{3}x^{\frac{3}{8}} + 2\log\frac{x^{\frac{1}{8}}-1}{x^{\frac{1}{8}}+1} + 4\tan^{-1}x^{\frac{1}{8}}.$

24. $\int \frac{dx}{x\sqrt{x+1}} = \log\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}.$

25. $\int \frac{x^2\,dx}{(4x+1)^{\frac{5}{2}}} = \frac{6x^2+9x+1}{12(4x+1)^{\frac{3}{2}}}.$

26. $\int \frac{dx}{1+\sqrt[3]{1+x}} = \frac{3}{2}(x+1)^{\frac{2}{3}} - 3(x+1)^{\frac{1}{3}} + 3\log(1+\sqrt[3]{1+x}).$

27. $\int \frac{dx}{(x^{\frac{1}{2}}+a)^{\frac{1}{2}}} = \frac{4}{3}(x^{\frac{1}{2}}-2a)(x^{\frac{1}{2}}+a)^{\frac{1}{2}}.$

28. $\int \frac{x^3\,dx}{\sqrt{1-x^2}} = -\frac{1}{3}\sqrt{1-x^2}(x^2+2).$

29. $\int \frac{x^5\,dx}{\sqrt{2x^2+1}} = \frac{1}{30}(3x^4-2x^2+2)\sqrt{2x^2+1}.$

30. $\int x^3(a^2 - x^2)^{\frac{1}{5}}\,dx = \frac{5}{132}(6x^4 - a^2x^2 - 5a^4)(a^2 - x^2)^{\frac{1}{5}}.$

31. $\int \frac{dx}{x\sqrt{x^2 + a^2}} = \frac{1}{2a}\log\frac{\sqrt{x^2 + a^2} - a}{\sqrt{x^2 + a^2} + a}$

32. $\int \frac{x\,dx}{x^2 + 2\sqrt{3 - x^2}} = \log\left(\sqrt{3 - x^2} + 1\right)^{\frac{1}{4}} + \log\left(\sqrt{3 - x^2} - 3\right)^{\frac{3}{4}}.$

33. $\int \frac{dx}{x\sqrt{2 + x - x^2}} = \frac{1}{\sqrt{2}}\log\frac{\sqrt{2 + 2x} - \sqrt{2 - x}}{\sqrt{2 + 2x} + \sqrt{2 - x}}.$

34. $\int \frac{dx}{x\sqrt{x^2 - x + 2}} = \frac{1}{\sqrt{2}}\log\frac{\sqrt{x^2 - x + 2} + x - \sqrt{2}}{\sqrt{x^2 - x + 2} + x + \sqrt{2}}.$

35. $\int \frac{dx}{x\sqrt{x^2 + 2x - 1}} = 2\tan^{-1}\left(x + \sqrt{x^2 + 2x - 1}\right).$

36. $\int \frac{\sqrt{x^2 + 2x}}{x^2}\,dx = -2\sqrt{\frac{2 + x}{x}} + \log\left(\sqrt{x + 2} + \sqrt{x}\right)^2.$

37. $\int \frac{\sqrt{6x - x^2}}{x^2}\,dx = -2\sqrt{\frac{6 - x}{x}} + 2\tan^{-1}\sqrt{\frac{6 - x}{x}}.$

38. $\int \frac{dx}{(x - 1)^2\sqrt{x^2 - 2x + 2}} = -\frac{\sqrt{x^2 - 2x + 2}}{x - 1}.$

39. $\int \frac{x^2 - x}{(x - 2)^3}\,dx = \log(x - 2) - \frac{3x - 5}{(x - 2)^2}.$ Put $x - 2 = z.$

40. $\int \frac{x^3\,dx}{(x + 1)^4} = ?$ Put $x + 1 = z.$

41. $\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = \frac{1}{a}\log\frac{x}{a + \sqrt{a^2 \pm x^2}}.$ Put $xz = a.$

42. $\int \frac{x^3\,dx}{(x^2 + 1)^{\frac{2}{3}}} = \frac{3}{8}(x^2 - 3)(x^2 + 1)^{\frac{1}{3}}.$ Put $x^2 + 1 = z.$

43. $\int \frac{\sin x\,dx}{\sin(x + a)} = (x + a)\cos a - \sin a\log\sin(x + a).$ Put $x + a = z.$

44. $\int \frac{e^{2x}\,dx}{(e^x + 1)^{\frac{1}{4}}} = \frac{4}{21}(3e^x - 4)(e^x + 1)^{\frac{3}{4}}$ Put $e^x + 1 = z.$

45. $\int \frac{dx}{e^{2x} - 2e^x} = \frac{1}{2e^x} - \frac{x}{4} + \frac{1}{4}\log(e^x - 2).$

46. $\int x^2\log x\,dx = \frac{1}{3}x^3\left(\log x - \frac{1}{3}\right).$

47. $\int x^{n-1}\log x\,dx = \frac{1}{n}x^n\left(\log x - \frac{1}{n}\right)$

48. $\int x\sin x\,dx = -x\cos x + \sin x.$

49. $\int x\log(x + 2)\,dx = (x^2 - 4)\log\sqrt{x + 2} - \frac{1}{4}x^2 + x.$

50. $\int x \tan^{-1}x \, dx = \frac{1}{2}(x^2 + 1) \tan^{-1}x - \frac{1}{2}x.$

51. Integrate

$$\int x^{-3}(a^2 - x^3)^{-\frac{1}{3}} \, dx, \quad \int (a^2 - x^2)^{\frac{5}{2}} \, dx, \quad \int x^2 (a^2 + x^2)^{-\frac{1}{2}} \, dx,$$

$$\int x^2 \sqrt{a^2 - x^2} \, dx, \quad \int x^2 \sqrt{a^2 + x^2} \, dx, \quad \int (a^2 - x^2)^{\frac{7}{2}} \, dx.$$

52. $\int \sin^4 x \cos^2 x \, dx = \frac{1}{2} \cos x \left(\frac{1}{3} \sin^5 x - \frac{1}{12} \sin^3 x - \frac{1}{8} \sin x\right) + \frac{1}{16}x.$

53. $$\int \frac{dx}{3 + 5 \cos x} = \frac{1}{4} \log \frac{\tan \frac{1}{2}x + 2}{\tan \frac{1}{2}x - 2}.$$

54. $$\int \frac{dx}{5 - 3 \cos x} = \frac{1}{2} \tan^{-1}(2 \tan \tfrac{1}{2}x)$$

55. $$\int \frac{dx}{(a + cx^2)^{\frac{3}{2}}} = \frac{x}{a(a + cx^2)^{\frac{1}{2}}}. \qquad \text{Put } xz = 1.$$

56. $$\int \frac{dx}{(a + 2bx + cx^2)^{\frac{3}{2}}} = \frac{b + cx}{(ac - b^2)(a + 2bx + cx^2)^{\frac{1}{2}}}.$$

Complete the square and put $cx + b = z$. The integral reduces to 55.

57. $$\int \frac{(p + qx) \, dx}{(a + 2bx + cx^2)^{\frac{3}{2}}} = \frac{bp - aq + (cp - bq)x}{(ac - b^2)(a + 2bx + cx^2)^{\frac{1}{2}}}.$$

For $xz = 1$ transforms

$$\frac{x \, dx}{(a + 2bx + cx^2)^{\frac{3}{2}}} \quad \text{into} \quad \frac{-dz}{(az^2 + 2bz + c)^{\frac{3}{2}}}.$$

$$\therefore \int \frac{x \, dx}{(a + 2bx + cx^2)^{\frac{3}{2}}} = -\frac{a + bx}{(ac - b^2)(a + 2bx + cx^2)^{\frac{1}{2}}}.$$

Combining with Ex. 56, the result follows at once.

58. $$\int \frac{(3 + x) \, dx}{(1 - 2x + 2x^2)^{\frac{3}{2}}} = \frac{7x - 4}{(1 - 2x + 2x^2)^{\frac{1}{2}}}.$$

59. $$\int \frac{dx}{(x - a)(x - b)(2x - a - b)} = \frac{1}{(a - b)^2} \log \frac{(x - a)(x - b)}{(2x - a - b)^2}.$$

60. $$\int \frac{dx}{(x - a)(x - b)(3x - 2a - b)} = \frac{1}{2(a - b)^2} \log \frac{(x - a)^2(x - b)}{(3x - 2a - b)^3}.$$

61. $$\int \frac{x \, dx}{x^3 - 6x^2 + 11x - 6} = \frac{1}{2} \log \frac{(x - 1)(x - 3)^3}{(x - 2)^4}.$$

62. $$\int \frac{x^3 \, dx}{(x - 1)(x - 2)(x - 3)} = x + \frac{1}{2} \log \frac{(x - 1)(x - 3)^{27}}{(x - 2)^{16}}.$$

63. $$\int \frac{dx}{x^3 - 7x + 6} = \frac{1}{20} \log \frac{(x - 2)^4(x + 3)}{(x - 1)^5}$$

64. $$\int \frac{x^2 \, dx}{x^3 - 7x + 6} = \frac{1}{3} \int \frac{(3x^2 - 7 + 7)dx}{x^3 - 7x + 6},$$

$$= \frac{1}{3} \log (x^3 - 7x + 6) + \frac{7}{60} \log \frac{(x - 2)^4(x + 3)}{(x - 1)^5}.$$

65. $\int \frac{dx}{x(1 - x^2)} = \log \frac{x}{\sqrt{1 - x^2}}$

66. $\int \frac{x^4\,dx}{x^3 - 13x + 12} = \frac{x^2}{2} - \frac{1}{10}\log(x-1) + \frac{81}{14}\log(x-3) + \frac{256}{35}\log(x+4).$

67. $\int \frac{dx}{x^2(a - x)} = -\frac{1}{ax} + \frac{1}{a^2}\log\frac{x}{a - x}.$

68. $\int \frac{x\,dx}{(x - 1)^2(x - 2)} = \frac{1}{x - 1} + 2\log\frac{x - 2}{x - 1}.$

69. $\int \frac{dx}{(x - a)^2(x - b)^2} = \frac{2}{(a - b)^3}\log\frac{x - b}{x - a} - \frac{1}{(a - b)^2}\frac{2x - a - b}{(x - a)(x - b)}.$

70. $\int \frac{dx}{x^2(x^2 + a^2)} = -\frac{1}{a^2x} + \frac{1}{a^3}\cot^{-1}\frac{x}{a}.$

71. $\int \frac{dx}{x^4 - 4x + 3} = \frac{-1}{6(x - 1)} + \log\frac{(x^2 + 2x + 3)^{\frac{1}{18}}}{(x - 1)^{\frac{1}{9}}} + \frac{1}{18\sqrt{2}}\tan^{-1}\frac{x + 1}{\sqrt{2}}.$

Notice $\qquad x^4 - 4x + 3 = (x - 1)^2(x^2 + 2x + 3).$

72. $\int \frac{x^3\,dx}{(x - 1)^2(x^2 - 2x + 2)} = \frac{-1}{x - 1} + \log\frac{(x - 1)^3}{x^2 - 2x + 2} + 2\tan^{-1}(x - 1).$

73. $\int \frac{dx}{(x^2 + a^2)^2} = \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + \frac{1}{2a^2}\frac{x}{x^2 + a^2}.$

74. $\int \frac{dx}{x(1 + x^3)^2} = \frac{1}{3}\log\frac{x^3}{1 + x^3} + \frac{1}{3}\frac{1}{1 + x^3}.$ $\qquad$ Put $x^3 = z.$

75. $\int \frac{dx}{x(1 + x + x^2 + x^3)} = \frac{1}{4}\log\frac{x^4}{(1 + x)^2(1 + x^2)} - \frac{1}{2}\tan^{-1}x.$

76. $\int \frac{dx}{x^4 + 5x^2 + 4} = \frac{1}{6}\tan^{-1}\frac{x^3 + 3x}{2}.$

77. $\int \frac{x^2\,dx}{(1 + x^2)(1 + 4x^2)} = \frac{1}{6}\tan^{-1}\frac{2x^3}{1 + 3x^2}.$

78 $\int \frac{x^5\,dx}{(1 - x^2)^2} = \frac{1}{2(1 - x^2)} + \frac{x^2}{2} + \log(1 - x^2).$

79. If $f(x) \equiv (x - a_1)\ldots(x - a_n)$, and $F(x)$ is a polynomial of degree less than n, show that

$$\int \frac{F(x)}{f(x)}\,dx = \sum_{r=1}^{n}\frac{F(a_r)}{f'(a_r)}\log(x - a_r).$$

80. Show that any algebraic function involving integral powers of x and fractional powers of

$$y = \frac{a + bx}{p + qx}$$

can be rationalized by putting $y = z^m$, where m is the least common multiple of the denominators of the fractional powers. Apply to Exs. 81, 82.

81. $\int \sqrt{\frac{x - a}{x - b}}\,dx = \sqrt{(x - a)(x - b)} - \frac{1}{2}(a - b)\log\frac{\sqrt{x - b} + \sqrt{x - a}}{\sqrt{x - b} - \sqrt{x - a}}.$

82. $\int \sqrt{\frac{a - x}{x - b}}\,dx, \quad \int \left(\frac{x - a}{x - b}\right)^{\frac{3}{2}}dx, \quad \int \left(\frac{x - a}{x - b}\right)^{\frac{1}{3}}dx.$

83. If $f(x)$ is a rational function of $\sin x$, $\cos x$, then $f(x)\,dx$ is rationalized by the substitution $\tan \frac{1}{2}x = z$.

Then $$\sin x = \frac{2z}{1 + z^2}, \quad \cos x = \frac{1 - z^2}{1 + z^2}, \quad dx = \frac{2dz}{1 + z^2}.$$

In particular, when $m - 1$, $n - 1$, or $m + n$ is even, say $2r$, we get for these respective cases

$$\begin{aligned} \sin^m x \cos^n x\, dx &= - c^n(1 - c^2)^r\, dc, \\ &= + s^m(1 - s^2)^r\, ds, \\ &= \frac{t^m\, dt}{(1 + t^2)^{r+1}}, \end{aligned}$$

where $$s \equiv \sin x, \quad c \equiv \cos x, \quad t \equiv \tan x.$$

84. To integrate $\int \frac{dx}{Q_1 Q_2^{\frac{1}{2}}}$, where Q_1, Q_2 are any quadratic functions of x. Write out Q_1^{-1} in partial fractions. This reduces the integral to § 136, (B), or to § 138, (D).

85. In general, if $f(x, y)$ is any rational function of x and y, where

$$y^2 = a + 2bx + cx^2 = c(x - \alpha)(x - \beta),$$

then any one of the following substitutions will rationalize $f(x, y)\,dx$:

$$\begin{aligned} y &= a^{\frac{1}{2}} + xz, \\ &= \pm z + xc^{\frac{1}{2}}, \\ &= z(x - \alpha)\sqrt{c}. \end{aligned}$$

86. $$\int \frac{dx}{(a^2 + x^2)^n} = \int \frac{x^{-3}\, dx}{(1 + a^2x^{-2})^n} \cdot x^{-2n+3}.$$ Put $u = x^{-2n+3}$, dv for the other factor.

$$\therefore \int \frac{dx}{(a^2 + x^2)^n} = \frac{1}{2a^2(n - 1)} \left\{ \frac{x}{(a^2 + x^2)^{n-1}} + (2n - 3)\int \frac{dx}{(a^2+x^2)^{n-1}} \right\}.$$

87. Given the signs of the constants a and b, transform the binomial differential

$$x^\alpha(a + bx^\beta)^\gamma\, dx$$

into the trigonometrical differential

$$C \sin^m x \cos^n x\, dx,$$

determining the constants C, m and n for each case.

CHAPTER XIX.

ON DEFINITE INTEGRATION.

146. The Symbol of Substitution.—We use the symbol

$$F(x)\Big]_{x=a}^{x=b},$$

or, in the abbreviated form when the variable is understood,

$$F(x)\Big]_a^b,$$

to mean that the number a is to be substituted for x in the function and the result subtracted from the value of the function when b is substituted for x. Thus

$$F(x)\Big]_a^b \equiv F(b) - F(a).$$

If $F(x)$ is a primitive of $f(x)$, then we have

$$\int_a^x f(t)\,dt = F(t)\Big]_a^x = F(x) - F(a).$$

The definite integral is a function of its limits. If one limit is constant the definite integral is a function of one variable, the other limit.

147. Interchange of Limits.

Since $$\int_a^b f(x)\,dx = F(b) - F(a),$$

$$\int_b^a f(x)\,dx = F(a) - F(b),$$

$$\therefore \quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

That is, *interchange of the limits is equivalent to a change of sign of the definite integral.*

This is also at once obvious from the original definition of an integral. For dx has opposite signs in the two limit-sums

$$\int_a^b f(x)\,dx \quad \text{and} \quad \int_b^a f(x)\,dx,$$

while they are equal in absolute value.

148. New Limits for Change of Variable.—If we transform the integral

$$\int_{x_0}^{X} f(x)\,dx$$

by the substitution of a new variable for x, then we have to find the corresponding new limits.

Let the substitution be $x = \phi(z)$, which solved for z gives $z = \psi(x)$. Then, when $x = x_0$, we have $z_0 = \psi(x_0)$, and when $x = X$, $Z = \psi(X)$. Also,

$$f(x)\,dx = f[\phi(z)]\,\phi'(z)\,dz = F(z)\,dz.$$

$$\therefore \quad \int_{x_0}^{X} f(x)\,dx = \int_{z_0}^{Z} F(z)\,dz.$$

For example, put $x = a \tan z$. Whence $z = \tan^{-1}\frac{x}{a}$. When $x = 0$, then $z = 0$; when $x = a$, then $z = \frac{1}{4}\pi$. Consequently

$$\int_0^a \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}} = \frac{1}{a^2}\int_0^{\frac{1}{4}\pi} \cos z\,dz = \frac{1}{a^2\sqrt{2}},$$

since $\int \cos z\,dz = \sin z$.

149. Decomposition of the Definite Integral Limits.
If

$$\int_{x_0}^{X} f(x)\,dx = F(X) - F(x_0),$$

then

$$\int_{x_0}^{a} f(x)\,dx = F(a) - F(x_0),$$

$$\int_{a}^{X} f(x)\,dx = F(X) - F(a).$$

Whence, on addition,

$$\int_{x_0}^{a} f(x)\,dx + \int_{a}^{X} f(x)\,dx = \int_{x_0}^{X} f(x)\,dx.$$

Therefore a definite integral is equal to the sum of the definite integrals taken over the partial intervals. This is also immediately evident from the definition of the definite integral.

EXAMPLES.

Evaluate the following definite integrals:

1. $\int_1^3 x^2\,dx = \left.\frac{x^3}{3}\right]_1^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}.$
2. $\int_1^e \frac{dx}{x} = \log x\Big]_1^e = \log e - \log 1 = 1.$
3. $\int_0^{\frac{1}{2}\pi} \sin x\,dx = \int_0^{\frac{1}{2}\pi} \cos x\,dx = 1.$

4. $\int_0^b (b^2x - x^3)dx = \frac{1}{4}b^4.$ 5. $\int_1^4 x^{-\frac{3}{2}} dx = 1.$

6. $\int_2^3 \frac{x\,dx}{1 + x^2} = \log\sqrt{2}.$ 7. $\int_0^\infty \frac{8a^3\,dx}{x^2 + 4a^2} = 2\pi a^2.$

8. $\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = \frac{\pi}{2}.$ 9. $\int_0^{2\pi} \text{vers}^2\theta\,d\theta = 3\pi.$

10. $\int_0^{\frac{1}{2}\pi} \sin^2\theta\,d\theta = \frac{1}{4}\pi.$ 11. $\int_0^{\frac{1}{4}\pi} \cos 2\theta\,d\theta = \frac{1}{2}.$

12. $\int_0^{\frac{1}{2}\pi} \cos^3 x \sin x\,dx = \frac{1}{4}.$ 13. $\int_0^5 \left(\frac{3}{5}\sqrt{t} - \frac{3}{25}t^2\right)dt = 2\sqrt{5} - 5.$

14. $\int_0^\infty \frac{dx}{a^2 + x^2} = \frac{\pi}{2a}.$ 15. $\int_0^\infty e^{-ax}dx = \frac{1}{a}.$

16. $\int_0^{\frac{1}{4}\pi} \frac{dx}{\cos^4 x} = \frac{4}{3}.$ 17. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{dx}{\cos x} = \log\frac{1 + \sqrt{2}}{\sqrt{3}}.$

18. $\int_0^1 \frac{dx}{1 + 2x\cos\phi + x^2} = \frac{1}{2}\int_0^\infty \frac{dx}{1 + 2x\cos\phi + x^2} = \frac{\phi}{2\sin\phi}.$

19. $\int_0^\infty e^{-ax}\sin mx\,dx = \frac{m}{a^2 + m^2}.$ 20. $\int_0^\infty e^{-ax}\cos mx\,dx = \frac{a}{a^2 + m^2}.$

21. $\int_{-\infty}^{+\infty} \frac{dx}{a + 2bx + cx^2} = \frac{\pi}{\sqrt{ac - b^2}},$ when $ac > b^2$.

Note 22. Show, by putting $x = 1 - z$, that

$$\int_0^1 x^{p-1}(1 - x)^{q-1}\,dx = \int_0^1 x^{q-1}(1 - x)^{p-1}\,dx.$$

This is called the *First Eulerian Integral.* Integrating by parts,

$$\int x^{n-1}(1 - x)^{m-1}\,dx = \frac{x^n(1 - x)^{m-1}}{n} + \frac{m - 1}{n}\int x^n(1 - x)^{m-2}\,dx.$$

Use this to show that the value of the above integral is

$$\int_0^1 x^{p-1}(1 - x)^{q-1}\,dx = \frac{(q - 1)!}{p(p + 1)\ldots(p + q - 1)},$$

when q is a positive integer, and therefore whenever p or q is a positive integer the integral can be evaluated.

23(1). $\int_0^1 x^3(1 - x)^{\frac{5}{2}}\,dx = \frac{2^5}{3\cdot7\cdot11\cdot13}.$ 23(2). $\int_0^1 x^4(1 - x)^{\frac{1}{4}}\,dx = \frac{2^{13}}{5\cdot7\cdot9\cdot13\cdot17}.$

⊛. 24. The integral $\int_0^\infty e^{-x}x^n\,dx$ is called the *Second Eulerian Integral* or the *Gamma-function*, $\Gamma(n + 1)$.

We have, by parts,

$$\int e^{-x}x^n\,dx = -e^{-x}x^n + n\int e^{-x}x^{n-1}\,dx.$$

Since $e^{-x}x^n = 0$ when $x = 0$ and when $x = \infty$,

$$\int_0^\infty e^{-x}x^n\,dx = n\int_0^\infty e^{-x}x^{n-1}\,dx.$$

$$\therefore\quad \underline{\underline{\Gamma(n + 1) = n\Gamma(n)}}.$$

Also, when n is an integer,

$$\Gamma(n+1) = n!$$

The Eulerian Integrals are fundamental in the theory of definite integrals.

25. $\displaystyle\int_0^1 x^m(\log x)^n\,dx = \frac{(-1)^n n!}{(m+1)^{n+1}} = (-1)^n \int_0^1 x^m\left(\log\frac{1}{x}\right)^n dx.$

Hint. Put $e^{-z} = x$ in Ex. 24.

26. $\displaystyle\int_0^\infty e^{-az}z^n\,dz = \frac{n!}{a^{n+1}}.$ Put $x = az$ in Ex. 24.

27. Show that $\displaystyle\int_0^{\frac{1}{2}\pi} \cos^n x\,dx = \int_0^{\frac{1}{2}\pi} \sin^n x\,dx,$

and that

$$\int_0^{\frac{1}{2}\pi} \sin^{2m} x\,dx = \frac{1\cdot3\cdot5\ldots(2m-1)}{2\cdot4\cdot6\ldots2m}\frac{\pi}{2},$$

$$\int_0^{\frac{1}{2}\pi} \sin^{2m+1} x\,dx = \frac{2\cdot4\cdot6\ldots2m}{3\cdot5\cdot7\ldots(2m+1)},$$

when m is a positive integer.

28. $\displaystyle\int_{\frac{1}{3}}^1 \frac{(x-x^3)^{\frac{1}{3}}}{x^4}\,dx = 6.$ Put $xz = 1$.

29. $\displaystyle\int_3^{29} \frac{(x-2)^{\frac{2}{3}}dx}{(x-2)^{\frac{2}{3}}+3} = 8 + \tfrac{3}{2}\sqrt{3}\,\pi.$ Put $x - 2 = z^3$.

30. $\displaystyle\int_0^{\log 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}\,dx = 4 - \pi.$ Put $e^x - 1 = z^2$.

31. $\displaystyle\int_0^\pi \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}.$ Where $a > b$.

32. $\displaystyle\int_0^{\frac{1}{2}\pi} x\sin x\,dx = 1.$

150. A Theorem of Mean Value.—Since in

$$\int_{x_0}^X f(x)\,dx$$

dx keeps the same sign throughout the summation,

$$m\int_{x_0}^X dx < \int_{x_0}^X f(x)\,dx < M\int_{x_0}^X dx,$$

where m and M are the least and greatest values respectively of the function $f(x)$ in (x_0, X). Therefore the integral lies in value between $m(X - x_0)$ and $M(X - x_0)$.

Since $f(x)$ is continuous in the interval, there must be a value of x, say ξ, in (x_0, X), for which

$$\int_{x_0}^X f(x)\,dx = (X - x_0)f(\xi),$$

$f(\xi)$ being a value of the function between m and M, its least and greatest values.

The value

$$f(\xi) = \frac{1}{X - x_0} \int_{x_0}^{X} f(x)\, dx$$

is called the *mean value* of the function in (x_0, X).

If $F(x)$ is a primitive of $f(x)$, then

$$\begin{aligned} F(X) - F(x_0) &= (X - x_0)\, f(\xi), \\ &= (X - x_0)\, F'(\xi), \end{aligned}$$

since $F'(x) = f(x)$. This is the familiar form of the Law of the Mean as established in the Differential Calculus.

The theorem of mean value for the Integral Calculus can be established directly from the definition of a mean value. For, if

$$\Delta x \equiv (X - x_0)/n,$$

then

$$\begin{aligned} \int_{x_0}^{X} f(x)\, dx &= \underset{n=\infty}{\overset{\Delta x (=) 0}{\pounds}} \sum_{r=1}^{n} \Delta x\, f(x_r), \\ &= (X - x_0) \underset{n=\infty}{\pounds} \frac{f(x_0) + f(x_1) + \ldots + f(x_n)}{n}. \end{aligned}$$

If the limit of the arithmetical mean of the n values of the function at the points of equal division of (x_0, X) be indicated by $f(\xi)$, the result is the same as above indicated.

GEOMETRICAL ILLUSTRATION.

If $y = f(x)$ is represented by the curve AB, then

$$\int_{x_0}^{X} y\, dx = \text{area } (x_0AZBX).$$

This area lies between the rectangles x_0ATX and x_0SBX, constructed with x_0X as base and the least and greatest ordinates to the curve respectively as altitudes. There is evidently a point ξ between x_0 and X at which the ordinate $\xi Z = f(\xi)$ is the altitude of a rectangle x_0RQX, intermediate in area between the greatest and least rectangles, whose area is equal to that bounded by the curve.

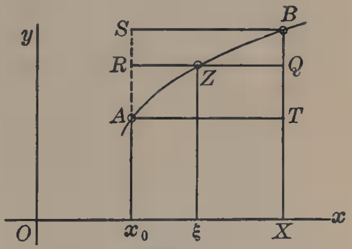

FIG. 68.

EXAMPLES.

1. Find the mean value of the ordinate of a semi-circle, supposing the ordinates taken at equidistant intervals along the diameter.

Let $x^2 + y^2 = a^2$ be the circle. Then

$$\frac{1}{2a} \int_{-a}^{+a} \sqrt{a^2 - x^2}\, dx = \tfrac{1}{4}\pi a,$$

viz., the length of an arc of 45°.

2. In the same case, suppose the ordinates drawn through equidistant points measured along the circumference. Then the arc length is the variable, and the mean ordinate is

$$\frac{1}{\pi}\int_0^{\pi} a \sin\theta\, d\theta = \frac{2}{\pi}a.$$

We shall see later that this is the ordinate of the centroid of the semi-circumference.

3. A number n is divided at random into two parts; find the mean value of their product.

$$\frac{1}{n}\int_0^{n} x(n-x)dx = \frac{1}{6}n^2.$$

4. Find the mean value of $\cos x$ between $-\pi$ and $+\pi$.

5. If $M_{x_1}^{x_2}(y)$ is the mean value of $y = f(x)$ in (x_1, x_2), show that:

(*a*). $M_1^2(2x^2+3x-1) = 8\frac{1}{6}$.

(*b*). $M_0^2(2-3x+5x^2-x^3) = \frac{11}{3}$.

(*c*). $M_1^3(x+1)(x+2) = 12\frac{1}{3}$.

(*d*). $M_0^{\frac{1}{2}\pi}(\sin\theta) = 2/\pi$.

6. Find the mean distance of the points on the semi-circumference of a circle of radius r, from one end of the semi-circumference, with respect to the angle.

$$M_0^{\frac{1}{2}\pi} = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} 2r\cos\theta\, d\theta = \frac{4r}{\pi}.$$

By the mean value of n numbers is meant the nth part of their sum. To estimate the mean value of a continuous variable between assigned values, we take the mean of n values corresponding to equidistant values of some independent variable and find the limit of this average when the number of values is increased indefinitely. The mean value depends on the variable selected. See Exs. 1 and 2 above.

If y is a function of t, then the mean value of y with respect to t for the interval (t_1, t_2) is

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} y\, dt.$$

151. An Extension of the Law of the Mean.—If $\phi(x)$ and $\psi(x)$ are two continuous functions of x, one of which, $\psi(x)$, has the same sign for all values of x in (x_0, X), then we shall have

$$\int_{x_0}^{X}\phi(x)\psi(x)\, dx = \phi(\xi)\int_{x_0}^{X}\psi(x)\, dx,$$

where ξ is some number between x_0 and X.

For if m and M are the least and greatest values of $\phi(x)$ in (x_0, X), then the integral must lie between the numbers

$$m\int_{x_0}^{X}\psi(x)\, dx \quad \text{and} \quad M\int_{x_0}^{X}\psi(x)\, dx,$$

since $\psi(x)\, dx$ does not change sign in (x_0, X). Therefore there must be a number ξ in (x_0, X) for which the integral has the value proposed, since $\phi(x)$ is a continuous function.

152. The Taylor-Lagrange Law of Mean Value.—Integration by parts furnishes a simple and an elegant method of deducing the important formula of Lagrange, and gives the form of the remainder in a much more useful form than that of the Differential Calculus.

Let z be a variable in the fixed interval (a, x). Then

$$f(x) - f(a) = \int_a^x f'(z)\,dz = -\int_a^x f'(z)\,d(x-z).$$

Put $u = f'(z)$, $dv = d(x-z)$, and integrate by parts.

$$\therefore\ f(x) - f(a) = -(x-z)f'(z)\Big]_a^x + \int_a^x (x-z)f''(z)\,dz,$$

$$= (x-a)f'(a) - \int_a^x (x-z)f''(z)\,d(x-z).$$

Put $u = f''(z)$, $dv = (x-z)\,d(x-z)$, and integrate the integral on the right by parts. Then

$$f(x)-f(a)=(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)-\int_a^x\frac{(x-z)^2}{2!}f'''(z)d(x-z).$$

Continue to integrate by parts in the same way, and there results

$$f(x) = \sum_{r=0}^{n}\frac{(x-a)^r}{r!}f^r(a) + \int_a^x \frac{(x-z)^n}{n!}f^{n+1}(z)\,dz. \qquad (1)$$

This is Lagrange's theorem with the terminal term expressed as a definite integral. This form of the terminal term shows that the difference between the function $f(x)$ and the series vanishes when $n = \infty$, provided

$$\mathop{\mathcal{L}}_{n=\infty}\frac{(x-z)^n}{n!}f^{n+1}(z) = 0 \qquad (2)$$

for all values of z in (a, x); and moreover, if this limit is not zero for any finite subinterval of (a, x), however small, the terminal term does not vanish and the series, although convergent, cannot be equal to the function.*

The law of the mean expressed in § 151 enables us to transform the definite integral in (1) directly into the forms of the terminal

---

* The reader should be warned against the language of many writers who confound the *remainder* of Taylor's series with the terminal term of the law of the mean, for they may be quite different. In fact, if Taylor's series S_∞ is convergent and $S_\infty = S_n + R_n$, then we should write

$$f(x) = S_n + R_n + T_n.$$

The terminal term being $R_n + T_n$. In order that $f(x) = S_\infty$ it is necessary that both $\mathcal{L} R_n = 0$, $\mathcal{L} T_n = 0$. $\mathcal{L} R_n = 0$ does not ensure $\mathcal{L}\, T_n = 0$. See Appendix. Note 8.

term given in the differential calculus. For, since $(x - z)^p$ keeps its sign unchanged for all values of z in (a, x), we have

$$\int_a^x \frac{(x-z)^n}{n!} f^{n+1}(z)\,dz = \frac{(x-\xi)^{n-p}}{n!} f^{n+1}(\xi) \int_a^x (x-z)^p\,dz, \qquad (3)$$

where ξ is some number between a and x. This result, (3), takes Lagrange's form when $p = n$, and Cauchy's when $p = 0$. The more general form (3), where p is any integer, is due to Schlömilch and Roche.

153. The Definite Integral Calculated by Series.—If $f(z)$ can be expressed in powers of $(z - a)$ by Taylor's series, for all values of z in (a, x), then also can the primitive of $f(z)$, and the definite integral of the function is equal to that of the series, taken term by term, between a and x. Hence, integrating between a and x,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \ldots,$$

we have

$$\int_a^x f(z)\,dz = (x-a)f(a) + \frac{(x-a)^2}{2!}f'(a) + \frac{(x-a)^3}{3!}f''(a) + \ldots \qquad (1)$$

In particular, put $x = 0$, then we have

$$\int_0^a f(z)\,dz = \frac{a}{1}f(a) - \frac{a^2}{2!}f'(a) + \frac{a^3}{3!}f''(a) - \ldots, \qquad (2)$$

a formula due to *Bernoulli.*

Knowledge of the derivatives at a serve therefore to compute the integral. When $a = 0$ in (1), then

$$\int_0^x f(z)\,dz = xf(0) + \frac{x^2}{2!}f'(0) + \frac{x^3}{3!}f''(0) + \ldots, \qquad (3)$$

which is Maclaurin's form, and is more convenient, in general, for computation than (2).

EXAMPLES.

1. Deduce Bernoulli's formula (2), § 153, by using the formula for parts,

$$\int f(x)\,dx = xf(x) - \int xf'(x)\,dx.$$

2. $\displaystyle\int_{-1}^{+1} e^{x^2}\,dx = 2\left(1 + \frac{1}{3} + \frac{1}{5}\frac{1}{2!} + \frac{1}{7}\frac{1}{3!} + \ldots\right).$

3. $\displaystyle\int_0^x e^{-x^2}\,dx = x - \frac{x^3}{3} + \frac{1}{5}\frac{x^5}{2!} - \frac{1}{7}\frac{x^7}{3!} + \ldots$

4. $\displaystyle\int_0^{\frac{1}{4}\pi} \log(\tan\phi)\,d\phi = -\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots\right).$

154. Observations on Definite Integration.—In order that a function may admit of definite integration in an interval (α, β) it must, in general, be one-valued and continuous throughout the interval. If the function is not one-valued, then generally the branches must be separated so that each may be taken as a one-valued function. If the function becomes infinite for any value of the variable between the limits of integration, then for such particular values of the variable the integral must receive special investigation, a case which we do not consider in this text.

In definite integration when one of the limits is infinite, we consider the integral

$$\int_a^\infty f(x)\,dx$$

as the limit to which converges the integral

$$\int_a^x f(x)\,dx,$$

when $x = \infty$, provided there be such a limit. The same remark holds when one limit is $-\infty$ and the other $+\infty$.

All continuous one-valued functions are integrable in the interval of continuity, as demonstrated in the Appendix, Note 9. But all continuous one-valued functions are not differentiable (see Appendix, Note 1).

The study of definite integrals will be taken up again in Book II.

EXERCISES.

1. $\displaystyle\int_0^a \frac{dx}{\sqrt{a-x}} = 2\sqrt{a}.$ **2.** $\displaystyle\int_0^a \frac{dx}{\sqrt{ax-x^2}} = \pi.$

3. $\displaystyle\int_1^2 \frac{dx}{x\sqrt{x^2-1}} = \tfrac{1}{3}\pi.$ **4.** $\displaystyle\int_0^1 \sin^{-1}x\,dx = \tfrac{1}{2}\pi - 1.$

5. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{dx}{2+\cos x} = \frac{\pi}{3\sqrt{3}}.$ **6.** $\displaystyle\int_0^{\frac{1}{4}\pi} \tan^5 x\,dx = \log 2^{\frac{1}{2}} - \tfrac{1}{4}.$

7. $\displaystyle\int_0^{\pi} \frac{dx}{1+\cos\theta\cos x} = \frac{\pi}{\sin\theta}.$ **8.** $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{dx}{1+\cos\theta\cos x} = \frac{\theta}{\sin\theta}.$

9. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{\pi}{2ab}.$

10. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{\pi(a^2+b^2)}{4a^3b^3}.$

11. $\displaystyle\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} = \pi.$ **12.** $\displaystyle\int_0^{\frac{3}{4}\pi} \frac{\sin x\,dx}{1+\cos^2 x} = \tfrac{1}{4}\pi + \tan^{-1}\frac{1}{\sqrt{2}}.$

13. Show that, when $k^2 < 1$,

$$\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \dots\right].$$

This is an elliptic integral.

14. Show that

$$\mathcal{L}_{n=\infty}\left(\frac{n}{n^2+1} + \frac{n}{n^2+2^2} + \dots + \frac{n}{2n^2}\right) = \frac{\pi}{4}.$$

Put $dx = 1/n$. The limit of the sum is then

$$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}.$$

15. Show that the limit of the sum

$$\frac{1}{\sqrt{n^2-1^2}} + \frac{1}{\sqrt{n^2-2^2}} + \ldots + \frac{1}{\sqrt{n^2-(n-1)^2}},$$

when $n = \infty$, is $\frac{1}{2}\pi$.

16. Show that

$$\int_0^\pi \sin mx \sin nx \, dx \quad \text{and} \quad \int_0^\pi \cos mx \cos nx \, dx$$

are zero when m and n are unequal integers, and are equal to $\frac{1}{2}\pi$ if m and n are equal integers.

17. $\int_0^{\frac{1}{2}\pi} \sin^2 x \cos^3 x \, dx = \frac{2}{15}$.

18. $\int_0^{\frac{1}{4}\pi} \frac{\sin\theta + \cos\theta}{3 + \sin 2\theta} d\theta = \frac{1}{4}\log 3$. Put $\sin\theta - \cos\theta = x$

19. $\int_1^{2+\sqrt{5}} \frac{(x^2+1)dx}{x\sqrt{x^4+7x^2+1}} = \log 3$. Put $x - x^{-1} = z$.

20. $\int_0^\pi \frac{dx}{1 - 2a\cos x + a^2} = \frac{\pi}{1-a^2}$.

21. $\int_0^{-\infty} xe^x \, dx = 1$.

22. $\int_1^3 \frac{x\,dx}{1+x^2} = \log\sqrt{5}$.

23. $\int_0^{\frac{1}{3}\sqrt{3}} \frac{dx}{\sqrt{2-3x^2}} = \frac{\pi}{4\sqrt{3}}$.

24. $\int_0^\infty \frac{dx}{a^2+b^2x^2} = \frac{\pi}{2ab}$.

25. $\int_2^1 \frac{dx}{x(1+x)^2} = \frac{1}{6} + \log\frac{3}{4}$.

26. $\int_1^{\text{[illegible]}} \frac{x\,dx}{x^2-x-2} = \frac{1}{3}\log 2$.

27. $\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$.

28. $\int_{-1}^2 \frac{dx}{\sqrt{2+x-x^2}} = \pi$.

29. $\int_0^a \frac{x^2\,dx}{(x^2+a^2)^2} = \frac{\pi-2}{8a}$.

30. $\int_0^{\frac{1}{4}\pi} \tan x \, dx = \log\sqrt{2}$.

31. $\int_0^{\frac{1}{4}\pi} \sec^2 x \, dx = 1$.

32. $\int_0^\pi \frac{d\theta}{a^2+b^2-2ab\cos\theta} = \frac{\pi}{a^2-b^2}$.

33. $\int_0^a \sqrt{\frac{a-x}{x}}\, dx = \frac{a\pi}{2}$.

34. $\int_0^1 x^2(1-x)^{\frac{3}{2}} dx = 2\int_0^{\frac{1}{2}\pi} \sin^5\theta \cos^4\theta \, d\theta = \frac{16}{315}$.

35. $\int_0^1 x^2(1-x^2)^{\frac{3}{2}} dx = \int_0^{\frac{1}{2}\pi} \sin^2\theta \cos^4\theta \, d\theta = \frac{\pi}{32}$.

36. $\int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{3}{2}} dx = \frac{3\pi}{128}$.

37. Putting $e^x - 1 = y^2$, show that

$$\int_0^{\log 2} \sqrt{e^x - 1}\, dx = 2\int_0^1 \frac{y^2\,dy}{1+y^2} = \frac{4-\pi}{2}.$$

38. If $x + 1 = y$,

$$\int_0^1 \sqrt{2x + x^2}\, dx = \int_1^2 \sqrt{y^2 - 1}\, dy = \sqrt{3} - \tfrac{1}{2}\log(2 + \sqrt{3}).$$

39. Putting $x = a \sin\theta$,

$$\int_0^a x^2 \sqrt{a^2 - x^2}\, dx = a^4 \int_0^{\frac{1}{2}\pi} \sin^2\theta \cos^2\theta\, d\theta = \frac{\pi a^4}{16}.$$

40. If $x = a \tan\theta$,

$$\int_0^a \frac{x^3\, dx}{(a^2 + x^2)^{\frac{3}{2}}} = a \int_0^{\frac{1}{4}\pi} \frac{\sin^3\theta}{\cos^2\theta}\, d\theta = \frac{3\sqrt{2} - 4}{2} a.$$

PART IV.

APPLICATIONS OF INTEGRATION.

CHAPTER XX.

ON THE AREAS OF PLANE CURVES.

155. Areas of Curves. Rectangular Coordinates.—The simplest method of considering the area of a curve is to suppose it referred to rectangular coordinates. The area bounded by the curve, the x-axis, and two ordinates corresponding to the values x_0, x_1 of x, is represented by the definite integral

$$A = \int_{x_0}^{x} y\,dx.$$

This has been shown to be true in Chapter XVI, as an illustration of the definite integral. It has been shown that the definite integral is independent of the manner in which the ordinates are distributed in making the summation.

We demonstrate again that the definite integral gives the area in question. For simplicity we divide the interval (x_0, x_1) into n equal parts, each equal to Δx. Let AB be the curve representing the equation $y = f(x)$, and x_0ABx_1 the boundary of the area required. Let MN be one of the subdivisions of x_0x_1. Draw ordinates to the curve at each of the points of division, and construct the n rectangles such as $MPqN$, and also the n rectangles such as $MpQN$. Since the curve is continuous, we can always take Δx or MN so small that for each corresponding pair of rectangles the curve PQ lies inside the rectangle $PpQq$, and therefore the area $MPQN$ of the curve lies between the areas of the rectangles $MPqN$ and $MpQN$. Hence the whole area x_0ABx_1 for the curve

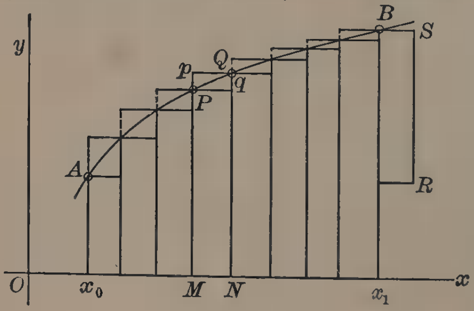

Fig. 69.

lies between the sum of the rectangles represented by $MPqN$ and the sum of those represented by $MpQN$. The difference between the sums of these rectangles is the sum of n rectangles of type $PpQq$. Which sum is equal to a rectangle represented by BR, whose base BS is Δx and altitude RS is $y_1 - y_0$, where $y_1 = x_1B$, $y_0 = x_0A$. (y_1, y_0) being the greatest and least ordinates in the interval. When the number of rectangles, n, is increased indefinitely, the difference between the sums of the rectangles, the one greater, the other less, than the curved area, converges to zero. Therefore the sum of either set of rectangles has for its limit, when $n = \infty$, the area of the curve, or

$$\underset{n=\infty}{\mathcal{L}} \sum_{1}^{n} y \Delta x = \int_{x_0}^{x_1} y\, dx.$$

If $y = f(x)$ is the equation of a curve, the area A included between the curve, the ordinates y_0, y_1 at x_0, x_1, and the x-axis is

$$A = \int_{x_0}^{x_1} f(x)\, dx.$$

EXAMPLES.

1. Area of the *circle*.

Taking $x^2 + y^2 = a^2$ as the equation of the circle,

$$\therefore \quad y = \pm \sqrt{a^2 - x^2}.$$

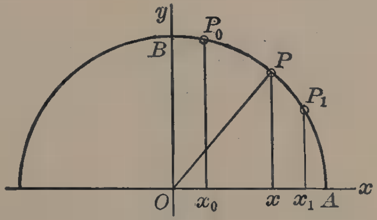

FIG. 70.

Taking the positive value of the radical, we have for the area $x_0P_0P_1x_1$,

$$\int_{x_0}^{x_1} \sqrt{a^2 - x^2}\, dx = \left[\frac{x(a^2 - x^2)^{\frac{1}{2}}}{2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a}\right]_{x_0}^{x_1}.$$

If $x_1 \equiv a$, we get the area of the semi-segment x_0P_0A. If $x_0 = 0$, and $x_1 = a$, we have the area of the quadrant OBA equal to

$$\int_0^a \sqrt{a^2 - x^2}\, dx = \tfrac{1}{4}\pi a^2.$$

If θ is the angle POA, then $y = a \sin\theta$, $x = a\cos\theta$.
$\therefore \quad dx = - a \sin\theta\, d\theta$. The area, A, of the circular quadrant is then given by

$$A = \int_0^a y\, dx = - a^2 \int_{\frac{1}{2}\pi}^{0} \sin^2\theta\, d\theta = a^2 \int^{\frac{1}{2}\pi} \sin^2\theta\, d\theta,$$

$$= \tfrac{1}{2}a^2(\theta - \sin\theta\cos\theta)\Big]_0^{\frac{1}{2}\pi} = \tfrac{1}{4}\pi a^2.$$

The area of the entire circle is therefore πa^2.

2. The area of the *ellipse.*

From the equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$, we get $y = \frac{b}{a}\sqrt{a^2 - x^2}$. Consequently, as in Ex. 1, the area of the elliptic quadrant is

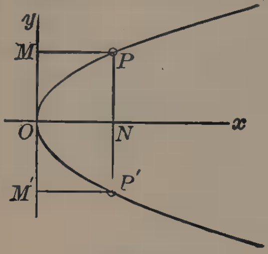

FIG. 71.

$$A = \frac{b}{a}\cdot\int_0^a \sqrt{a^2 - x^2}\,dx,$$

which is b/a times the corresponding area of a circle of radius a. Hence the area of the entire ellipse is πab.

3. Area of the *parabola.*

Taking $y^2 = px$ as the equation of the curve, and the positive value of the radical in $y = \sqrt{px}$, we have the curve OP. The area OPN is then

$$A = \int_0^x \sqrt{px}\;dx = \tfrac{2}{3}\,p^{\frac{1}{2}}x^{\frac{3}{2}}\Big]_0^x,$$
$$= \tfrac{2}{3}xy.$$

But xy is the area of the rectangle $ONPM$. The area of the segment POP' of the parabola cut off by a chord perpendicular to the diameter is two thirds the rectangle $MPP'M'$.

4. Area of the *hyperbola.*

Let $x^2/a^2 - y^2/b^2 = 1$ be the equation to the curve. Then the area of APN is

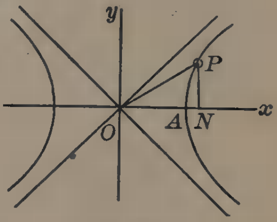

FIG. 72.

$$A = \int_a^x y\;dx,$$
$$= \frac{b}{a}\int_a^x \sqrt{x^2 - a^2}\;dx,$$
$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right)\right]_a^x,$$
$$= \frac{b}{2a}x\sqrt{x^2 - a^2} - \frac{ab}{2}\log\frac{x + \sqrt{x^2 - a^2}}{a},$$
$$= \tfrac{1}{2}xy - \tfrac{1}{2}ab\log\left(\frac{x}{a} + \frac{y}{b}\right).$$

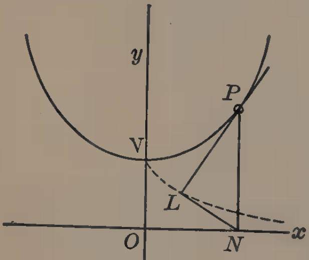

FIG. 73.

5. Area of the *catenary.*

The equation to the curve is

$$y = \tfrac{1}{2}a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

The area $OVPN$ is

$$A = \int_0^x \tfrac{1}{2}a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)dx,$$
$$= \tfrac{1}{2}a^2\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) = a\sqrt{y^2 - a^2}.$$

If NL is perpendicular to the tangent at P, show that the above area is twice that of the triangle PLN. Observe that $\tan LNP = Dy$, $LN = y\cos LNP$, etc.

6. Show that the area of a sector of the *equilateral hyperbola* $x^2 - y^2 = a^2$ included between the x-axis and a diameter through the point x, y of the curve is

$$\tfrac{1}{2}a^2\log\frac{x + y}{a}.$$

7. Find the entire area between the *witch of Agnesi* and its asymptote.

The equation is $(x^2 + 4a^2)y = 8a^3$. *Ans.* $4\pi a^2$.

8. Find the area between the curve $y = \log x$ and the x-axis, bounded by the ordinates at $x = 1$ and x. *Ans.* $x(\log x - 1) + 1$.

9. Find the area bounded by the coordinate axes and the *parabola* $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$. *Ans.* $\frac{1}{6}a^2$.

10. Find the entire area within the curve $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$. *Ans.* $\frac{3}{4}\pi ab$.

11. Find the entire area within the *hypocycloid* $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
Hint. Put $x = a\sin^3\theta$, $y = a\cos^3\theta$. *Ans.* $\frac{3}{8}\pi a^2$.

12. Find the entire area between the *cissoid* $(2a - x)y^2 = x^3$, and its asymptote $x = 2a$. *Ans.* $3\pi a^2$.

13. Find the area included between the *parabola* $x^2 = 4ay$ and the *witch* $y(x^2 + 4a^2) = 8a^3$. *Ans.* $a^2(2\pi - \frac{4}{3})$.
The origin and the point of intersection of the curve give the limits of the integral.

14. Find the area of the loop of the curve

$$cy^2 = (x - a)(x - b)^2.$$

Hint. Let $x - a = z^2$. *Ans.* $\frac{8}{15}\sqrt{\frac{(b-a)^5}{c}}$.

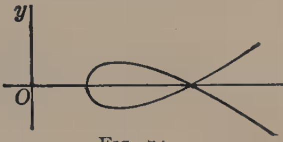

FIG. 74.

15. Find the whole area of the curve $a^2y^2 = x^3(2a - x)$. *Ans.* πa^2.

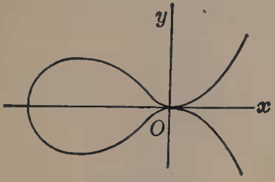

FIG. 75.

16. Find the area of the loop of the curve

$$a^3y^2 = x^4(b + x).$$

The area of the loop is

$$A = \frac{2}{a^{\frac{3}{2}}}\int_{-b}^{0} x^2\sqrt{b + x}\,dx = \frac{32b^{\frac{7}{2}}}{3\cdot 5\cdot 7a^{\frac{3}{2}}}.$$

Put $b + x = z^2$.

17. Show that if $y = f(x)$ is the equation of a curve referred to oblique coordinate axes inclined at an angle ω, then the area bounded by the curve, the x-axis, and two ordinates at x_0, x_1 is

$$A = \sin\omega\int_{x_0}^{x_1} y\,dx.$$

18. The equation to a parabola referred to a tangent and the diameter through the point of contact is $y^2 = kx$.
Show that the area cut off by any chord parallel to the tangent is equal to two thirds the area of the parallelogram whose sides are the chord, tangent, and lines through the ends of the arc parallel to the diameter.

19. The equation to the hyperbola referred to its asymptotes as coordinate axes is $xy = c^2$. If ω is the angle between the asymptotes, show that the area between the curve, x-axis, and two ordinates at x_0, x_1 is

$$c^2\sin\omega\log\left(\frac{x_1}{x_0}\right).$$

20. If $y = ax^n$ is the equation to a curve in rectangular coordinates, show that the area from $x = 0$ to x is

$$\frac{xy}{n + 1}.$$

156. If the area bounded by a curve, the axis of y, and two abscissæ x_0, x_1, corresponding to the ordinates y_0, y_1, is required, then that area is

$$A = \int_{y_0}^{y_1} x\,dy.$$

EXAMPLES.

1. Find the area of the curve $y^2 = px$ between the curve and the y-axis from $y = 0$ to $y = y$.

2. Find the area of the curve $y = e^x$ between the curve, the y-axis, and abscissæ at $y = 1$, $y = a$. Check the result by finding the area between the curve and the x-axis for corresponding limits.

Also find the area bounded by the curve, the y-axis, and the negative part of the x-axis.

157. Observe that in the examples thus far given the portion of the curve whose area was required has been such that the curve was wholly on one side of the axis of coordinates.

It is evident that if the curve crosses the axis between the limits of integration, then, y being positive above the x-axis and negative below it, those portions of the area above Ox are positive, those below are negative. The integral

$$\int_{x_0}^{x_1} y\,dx$$

is then the algebraic sum of these areas, or the difference of the area on one side of Ox from that on the other side.

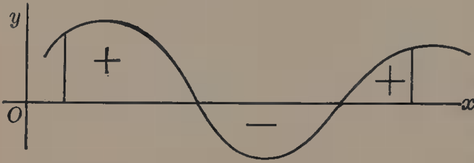

FIG. 76.

EXAMPLE.

Find the area of $y = \sin x$ from $x = 0$ to $x = \frac{3}{2}\pi$.

We have

$$A = \int_0^{\frac{3}{2}\pi} \sin x\,dx = -\cos x\Big]_0^{\frac{3}{2}\pi} = 1.$$

But $\int_0^{\pi} \sin x\,dx = 2$,

$$\int_{\pi}^{\frac{3}{2}\pi} \sin x\,dx = -1.$$

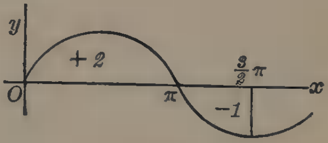

FIG. 77.

$$\therefore\ A_0^{\frac{3}{2}\pi} = A_0^{\pi} + A_{\pi}^{\frac{3}{2}\pi} = 2 - 1 = 1.$$

158. It is evident that the area considered can be regarded as the area generated or swept over by the ordinate moving parallel to a fixed direction, Oy.

If we have to find the area between two curves

$$y_1 = \phi(x), \quad y_2 = \psi(x),$$

and two ordinates at a and b, such as the area $LMNR$ in the figure, that area can be computed by finding the area of each curve separately. But if it is more convenient, the area is

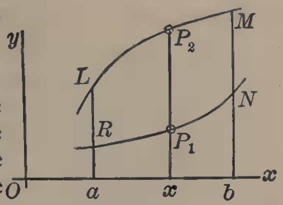

FIG. 78.

$$\int_a^b (y_2 - y_1)\,dx = \int_a^b [\psi(x) - \phi(x)]\,dx.$$

The area in question is generated by the line P_1P_2, equal to the difference of the ordinates $y_2 - y_1$, moving parallel to Oy from the position RL to NM.

EXAMPLE.

Find the area bounded by the curves

$$x(y - e^x) = \sin x \quad \text{and} \quad 2xy = 2\sin x + x^3,$$

the y-axis, and the ordinate at $x = 1$.

$$A = \int_0^1 (e^x - \tfrac{1}{2}x^2)\,dx = e - \tfrac{7}{6} = 1{\cdot}55 +.$$

It would not be so easy to find the areas of each curve separately.

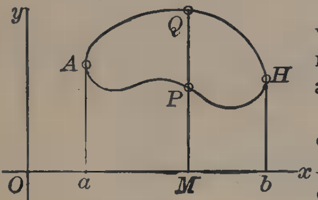

FIG. 79.

159. If it be required to find the whole area of a closed curve, such as that represented in the figure, we may proceed as follows:

Suppose the ordinate MP to meet the curve again in Q, and let $MP = y_1$, $MQ = y_2$. Let a and b be the abscissæ of the extreme tangents aA and bB. Then the area of the curve is

$$A = \int_a^b (y_2 - y_1)\,dx.$$

This result also holds if the curve cuts the axis of x.

EXAMPLE.

Find the whole area of the curve $(y - mx)^2 = a^2 - x^2$.

Here

$$y = mx \pm \sqrt{a^2 - x^2}.$$

$$\therefore \quad y_2 = mx + \sqrt{a^2 - x^2},$$

$$y_1 = mx - \sqrt{a^2 - x^2}.$$

$$\therefore \quad A = \int_{-a}^{+a} (y_2 - y_1)\,dx,$$

$$= 2\int_{-a}^{+a} \sqrt{a^2 - x^2}\,dx = \pi a^2.$$

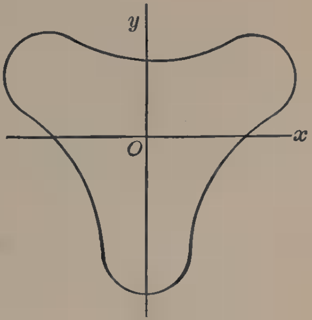

FIG. 80.

160. The area of any portion of the curve

$$f\left(\frac{x}{a}, \frac{y}{b}\right) = c \qquad (1)$$

is ab times the area of the corresponding portion of the curve

$$f(x, y) = c. \tag{2}$$

For (1) is transformed into (2) by putting $x = ax'$, $y = by'$ in (1); and hence $y\,dx$, from (1), becomes $ab\,y'\,dx'$, and we have

$$\int_{x_0}^{x_1} y\,dx = ab\int_{x_0'}^{x_1'} y'\,dx'.$$

EXAMPLES.

1. The entire area of the circle $x^2 + y^2 = 1$ is π. Hence that of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $ab\pi$.

2. Find the whole area of the curve $\left(\dfrac{x}{a}\right)^{\frac{2}{3}} + \left(\dfrac{y}{b}\right)^{\frac{2}{3}} = 1.$

In Ex. 11, § 155, it is shown that the area of

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$

is $\frac{3}{8}\pi$. Hence that of the proposed curve is $\frac{3}{8}\pi ab$.

3. Check the result in Ex. 2 by putting $x = a\sin^3\phi$, $y = b\cos^3\phi$.

Then $$y dx = 3ab\sin^2\phi\cos^4\phi\,d\phi.$$

$$\therefore\quad A = 12ab\int_0^{\frac{1}{2}\pi}\sin^2\phi\cos^4\phi\,d\phi = \tfrac{3}{8}\pi ab.$$

161. Sometimes the quadrature of a curve is to be obtained when the coordinates are given in terms of a third variable, or is facilitated by expressing the coordinates in terms of a third variable. Thus if

$$x = \phi(t),\quad y = \psi(t),$$

the element of area is

$$y\,dx = \psi(t)\phi'(t)dt.$$

EXAMPLES.

1. Find the area of the loop of the *folium of Descartes*,

$$x^3 + y^3 = 3axy.$$

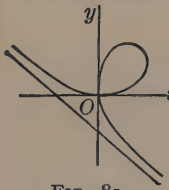

FIG. 81.

Put $y = tx$; then

$$x = \frac{3at}{1 + t^3},\quad y = \frac{3at^2}{1 + t^3}.$$

$$\therefore\quad dx = \frac{1 - 2t^3}{(1 + t^3)^2}\,3a\,dt,\quad \text{and}$$

$$\int y\,dx = 9a^2\int\frac{t^2(1 - 2t^3)dt}{(1 + t^3)^3} = \frac{6a^2}{1 + t^3} - \frac{9a^2}{2(1 + t^3)^2}.$$

The limits for t are 0 and ∞. Hence $A = \frac{3}{2}a^2$.

2. In the *cycloid*,

$$x = a(t - \sin t),\quad y = a(1 - \cos t),$$

$$\therefore\quad \int y\,dx = a^2\int \operatorname{vers}^2 t\,dt = 4a^2\int\sin^4\tfrac{1}{2}t\,dt.$$

Taking t between 0 and π, we get $3\pi a^2$ for the entire area between one arch of the cycloid and its base.

3. Find the area of the ellipse using $x^2/a^2 + y^2/b^2 = 1$, where $x = a \cos \phi$, $y = b \sin \phi$.

4. Find the area of the hyperbola $x^2/a^2 - y^2/b^2 = 1$, from $x = a$ to $x = x$, using $x = a \sec \phi$, $y = b \tan \phi$.

162. Areas in Polar Coordinates.—Let $\rho = f(\theta)$ be the polar equation to a curve. We require the area of a sector, bounded by the curve and two positions of the radius vector, corresponding to $\theta = \alpha$, $\theta = \beta$.

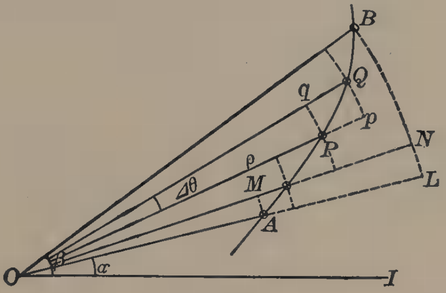

FIG. 82.

Let AB represent $\rho = f(\theta)$, OI the initial line. $\angle IOA = \alpha$, $\angle IOB = \beta$. Then OAB is the sector whose area is required. Divide the angle $AOB = \beta - \alpha$ into n equal parts each equal to $\Delta\theta$, and draw the corresponding radii cutting AB in corresponding points P, Q, etc.; dividing the curve AB into n parts, such as PQ. Through each of the points of division draw circular arcs with center O, such as Qp, qP, etc. From the continuity of $\rho = f(\theta)$, we can always take $\Delta\theta$ so small that the sector OPQ of the curve lies between the corresponding circular sectors OPq and OpQ, and therefore the area of the whole sector OAB lies between the sum of the circular sectors of type OPq and the sum of the circular sectors of type OpQ. But the difference between these sums of circular sectors is equal to the area

$$ALNM = \tfrac{1}{2}(OB^2 - OA^2)\Delta\theta,$$

which has the limit 0 when $\Delta\theta (=) 0$, or when $n = \infty$. Therefore the sum of either the external or internal circular sectors converges to the area of the sector OAB as a limit when $n = \infty$.

Putting $\rho_0 = OA$, $\rho_n = OB$, and $\rho_r (r = 1, 2, \ldots)$, for the radii to the points of division of AB, the area of the curvilinear sector OAB is

$$\underset{n=\infty}{\pounds} \sum_{r=1}^{r=n} \tfrac{1}{2}\rho_r^2 \Delta\theta = \int_\alpha^\beta \tfrac{1}{2}\rho^2 \, d\theta.$$

EXAMPLES.

1. Find the area swept out by the radius vector of the *spiral of Archimedes*, $\rho = a\theta$, in one revolution.

We have
$$A = \tfrac{1}{2}\int_0^{2\pi} \rho^2 \, d\theta = \tfrac{1}{2}\int_0^{2\pi} a^2\theta^2 \, d\theta = \tfrac{4}{3}\pi^3 a^2.$$

2. Find the area described by the radius vector of the *logarithmic spiral* $\rho = e^{a\theta}$, from $\theta = 0$ to $\theta = \frac{1}{2}\pi$. *Ans.* $\dfrac{1}{4a}\left(e^{\pi a} - 1\right)$.

3. Show that the area of the circle $\rho = a \sin\theta$ is $\frac{1}{4}\pi a^2$.

4. Find the area of one loop of $\rho = a \sin 2\theta$. *Ans.* $\frac{1}{8}\pi a^2$.

5. Find the entire area of the *cardioid* $\rho = a(1 - \cos\theta)$. *Ans.* $\frac{3}{2}\pi a^2$.

6. The area of the *parabola* $\rho = a \sec^2 \frac{1}{2}\theta$ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$ is $\frac{4}{3}a^2$.

7. Show that the area of the *lemniscate* $\rho^2 = a^2 \cos 2\theta$, is a^2.

8. In the *hyperbolic spiral* $\rho\theta = a$, show that the area bounded by any two radii vectores is proportional to the difference of their lengths.

9. Find the area of a loop of the curve $\rho^2 = a^2 \cos n\theta$. *Ans.* a^2/n.

10. Find the area of the loop of the *folium of Descartes*,

$$x^3 + y^3 = 3axy.$$

Transform to polar coordinates. Then

$$\rho = \frac{3a \cos\theta \sin\theta}{\sin^3\theta + \cos^3\theta}.$$

Therefore the area is

$$\frac{9a^2}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta \cos^2\theta \, d\theta}{(\sin^3\theta + \cos^3\theta)^2} = \frac{9a^2}{2}\int_0^{\infty} \frac{u^2 \, du}{(1 + u^3)^2} = \tfrac{3}{2}a^2,$$

where $u = \tan\theta$.

11. Show that the whole area between the curve in Ex. 10 and its asymptote is equal to the area of the loop.

12. Find the area between the curves

$$x^2 + y^2 = \left(\frac{\pi}{2}\right)^2 \quad \text{and} \quad \rho^2 + \theta^2 = \left(\frac{\pi}{2}\right)^2.$$

13. The area of $\rho = a \cos 3\theta$, from 0 to $\frac{1}{6}\pi$, is $\frac{1}{24}\pi a^2$.

14. Show that the area of $\rho = a(\sin 2\theta + \cos 2\theta)$, from 0 to 2π, is πa^2.

15. The area of $\rho \cos\theta = a \cos 2\theta$, from 0 to $\frac{1}{4}\pi$, is $\frac{1}{2}(2 - \frac{1}{2}\pi)a^2$.

163. We come now to consider the area generated by a straight-line segment which moves in a plane, under certain general conditions. In rectangular coordinates we have considered the area generated by the moving ordinate to a curve. In polar coordinates the area considered was generated by a moving radius vector. In the former case the generating line moves parallel to a fixed direction, in the latter it passes through a fixed point.

A point Q is taken on the tangent at P to a given curve PP', such that $PQ = t$. To find the area bounded by the given curve, the curve QQ' described by Q, and two positions PQ, $P'Q'$ of the generating line.

Let $PQ = t$, $P'Q' = t + \Delta t$, $PI = \delta t$, $P'I = \delta' t$, and θ be the angle which the tangent at P makes with a fixed direction. Let ΔA represent the area swept over by PQ in moving from PQ to $P'Q'$ through the angle $\Delta\theta$. Draw the chord PP' and the circular arcs QM, $Q'M'$ with I as a center. Then ΔA is equal to the area of the circular sector QIM, plus a fraction of the area of the triangle PIP', plus a fraction of the area $QM\,Q'M'$. Or, in symbols,

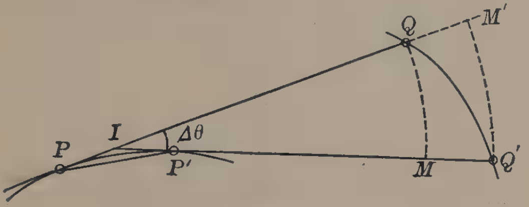

FIG. 83.

$$\Delta A = \tfrac{1}{2}(t - \delta t)^2 \Delta\theta$$
$$+ \frac{f_1}{2}\delta t \cdot \delta' t \sin \Delta\theta + \frac{f_2}{2}[(t + \Delta t + \delta' t)^2 - (t - \delta t)^2]\Delta\theta,$$

where f_1, f_2 are proper fractions. Observing that Δt, δt, and $\delta' t$ converge to 0 when $\Delta\theta(=)0$, divide by $\Delta\theta$ and let $\Delta\theta(=)0$. Then

$$\frac{dA}{d\theta} = \tfrac{1}{2}t^2,$$

or

$$dA = \tfrac{1}{2}t^2\, d\theta.$$

Hence between the limits $\theta = \alpha$, $\theta = \beta$ the area swept over by t is

$$A = \tfrac{1}{2}\int_\alpha^\beta t^2\, d\theta.$$

When the law of change of t, the length of the tangent, is given as a function of θ, the area can be evaluated. If $t = f(\theta)$ be this relation, the curve $t = f(\theta)$, considering t as a radius vector and θ the vectorial angle, is called the *directing* or *director* curve of the generating line.

EXAMPLES.

1. Show that the area swept over by a line of constant length a laid off on the tangent from the point of contact is πa^2, when the point of contact moves entirely around the boundary of a closed plane curve.

2. The *tractrix* is a curve whose *tangent-length* is constant. Find the entire area bounded by the curve. (Fig. 84.)

The area in the first quadrant is generated by the constant length $PT = a$ turning through the angle $\frac{1}{2}\pi$ as the point P moves from J along the curve JPS asymptotic to Ox. Therefore the area in the first quadrant is $\frac{1}{4}\pi a^2$, and the whole area bounded by the four infinite branches is πa^2.

3. Check the above result by Cartesian coordinates and find the equation to the *tractrix*.

We have directly from the figure

$$\frac{dy}{dx} = -\tan PTN = -\frac{y}{\sqrt{a^2 - y^2}}. \quad \therefore\ y\,dx = -\sqrt{a^2 - y^2}\,dy.$$

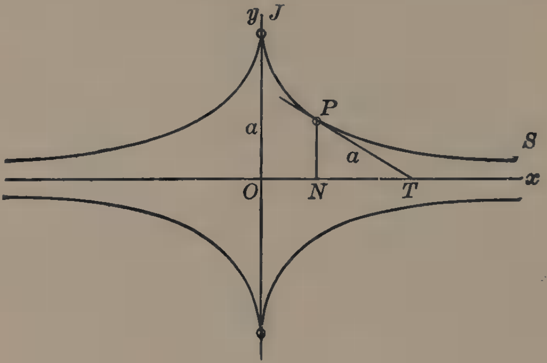

FIG. 84.

Hence the element of area of the tractrix is the same as that of a circle of radius a. It follows directly that the whole area of the tractrix is πa^2. This gives an example of finding the area of a curve without knowing its equation. To find the equation of the tractrix, we have

$$dx = -\frac{\sqrt{a^2 - y^2}}{y}\,dy.$$

Integrating, we get

$$x = -\sqrt{a^2 - y^2} + a \log \frac{a + \sqrt{a^2 - y^2}}{y},$$

since $x = 0$ when $y = a$. This is said to be the first curve whose area was found by integration.

4. Show that the area bounded by a curve, its evolute, and two normals to the curve is

$$\tfrac{1}{2}\int_{\theta_1}^{\theta_2} \rho^2\,d\theta,$$

where ρ is the radius of curvature of the curve, and θ the angle which the normal makes with a fixed direction.

164. Elliott's Theorem.—Two points P_1 and P_2 on a straight line describe closed curves of areas (P_1) and (P_2). The segment P_1P_2 moves in such a manner as to be always parallel and equal to the radius vector of a known curve $\rho = f(\theta)$ called the *director curve*.

It is required to find the area of the closed curve described by a point P on the line P_1P_2 which divides the segment P_1P_2 in constant ratio.

Let (P), (P_1), (P_2), (A) be the areas of the closed curves described by the corresponding points as shown in the figure. Let

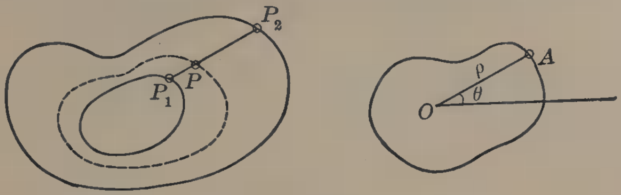

FIG. 85.

P_1P_2 and $P_1'P_2'$, Fig. 86, be two positions of the segment. Produce them to meet in C.

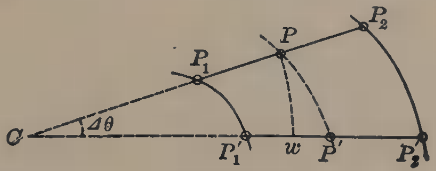

FIG. 86.

Let $\rho = P_1P_2$, $P_1P/PP_2 = m_1/m_2$.

$$\therefore \quad P_1P = \frac{m_1}{m_1 + m_2} P_1P_2 = k_1\rho,$$

$$PP_2 = \frac{m_2}{m_1 + m_2} P_1P_2 = k_2\rho,$$

where $k_1 + k_2 = 1$.

The element of area $P_1P_2P_2'P_1'$ is, § 163, if $CP_1 = r$,

$$d(P_2) - d(P_1) = \tfrac{1}{2}(\rho + r)^2\, d\theta - \tfrac{1}{2}r^2\, d\theta,$$
$$= \rho r\, d\theta + \tfrac{1}{2}\rho^2\, d\theta. \qquad (1)$$

In like manner the element of area $P_1PP'P_1'$ is

$$d(P) - d(P_1) = \tfrac{1}{2}(k_1\rho + r)^2\, d\theta - \tfrac{1}{2}r^2\, d\theta,$$
$$= k_1\rho r\, d\theta + \tfrac{1}{2}k_1^2\rho^2\, d\theta. \qquad (2)$$

Multiply (1) by k_1 and eliminate $k_1\rho r\, d\theta$ between (1) and (2), remembering that $k_1 + k_2 = 1$. Then

$$d(P) = k_2 d(P_1) + k_1 d(P_2) - k_1k_2 d(A).$$

Integrating for a complete circuit of the points P_1 and P_2 about the boundaries of the curves, we have

$$(P) = k_2(P_1) + k_1(P_2) - k_1k_2(A), \qquad (3)$$

where the area of the director curve is given by

$$(A) = \tfrac{1}{2}\int \rho^2\, d\theta,$$

the limits of the integral being determined by the angle through which the line has turned.

In particular, if $P_1P_2 = \rho$ is constant and equal to a, we have *Holditch's theorem*,

$$(P) = k_2(P_1) + k_1(P_2) - \tfrac{1}{2}k_1k_2a^2\int d\theta.$$

If a chord of constant length a moves with its ends on a closed curve of area (C), the area of the closed curve traced by the point P which divides the chord in constant ratio $m : n$ is

$$(P) = (C) - \frac{mn\pi}{(m+n)^2}a^2,$$
$$= (C) - c_1c_2\pi,$$

if P is distant c_1 and c_2 from the ends of the chord.

EXAMPLES.

1. A straight line of constant length moves with its ends on two fixed intersecting straight lines; show that the area of the ellipse described by a point on the line at distances a and b from its ends is πab.

2. A chord of constant length c moves about within a parabola, and tangents are drawn at the ends of the chord; find the total area between the parabola and the locus of the intersection of the tangents. *Ans.* $\frac{1}{8}\pi c^2$.

The area between the parabola and the curve described by the middle point of the chord is the same.

3. It can be shown that the locus of the intersection of the tangents in Ex. 2 to the parabola $y^2 = 4ax$ is

$$(y^2 - 4ax)(y^2 + 4a^2) = a^2c^2.$$

Check the result in Ex. 2 by the direct integration

$$\int z\,dy = \tfrac{1}{8}c^2\pi$$

from $y = -\infty$ to $y = +\infty$. z being half the distance from the intersection of the tangents to the mid-point of the chord.

4. Tangents to a closed oval curve intersect at right angles in a point P; show that the whole area between the locus of P and the given curve is equal to half the area of the curve formed by drawing through a fixed point a radius vector parallel to either tangent and equal to the chord of contact.

5. If ρ_1, θ_1 and ρ_2, θ_2 are the polar coordinates of points P_1 and P_2 on a straight line, then the radius vector ρ of a point on this straight line which divides the segment $P_1P_2 = \lambda$ so that $PP_1 = k_1\lambda$, $PP_2 = k_2\lambda$, is determined by

$$\rho^2 = k_2\rho_1^2 + k_1\rho_2^2 - k_1k_2\lambda^2. \tag{1}$$

This is Stewart's theorem in elementary geometry. If ϕ is the angle which ρ makes with P_1P_2, then

$$\rho_1^2 = \rho^2 + k_1^2\lambda^2 - 2k_1\lambda\rho\cos\phi,$$
$$\rho_2^2 = \rho^2 + k_2^2\lambda^2 + 2k_2\lambda\rho\cos\phi.$$

The elimination of $\cos\phi$ gives (1) at once.

Multiply (1) through by $\frac{1}{2}d\theta$, then

$$\tfrac{1}{2}\rho^2\,d\theta = k_2\,\tfrac{1}{2}\rho_1^2\,d\theta + k_1\,\tfrac{1}{2}\rho_2^2\,d\theta - k_1k_2\,\tfrac{1}{2}\lambda^2\,d\theta, \tag{2}$$

or

$$d(P) = k_2\,d(P_2) + k_1\,d(P_1) - k_1k_2\,d(A),$$

and Elliott's theorem follows immediately on integration.

The geometrical interpretation of (2) is as follows: Let $\lambda = P_1P_2$ be constant. Construct the *instantaneous* center of rotation I of λ as P_1P_2 turns through $\Delta\theta$. Then P_1P_1', PP', P_2P_2' (Fig. 86) subtend the angle $\Delta\theta$ at I. The center I being considered as origin or pole, (2) follows at once. The extension to the case when λ is variable is immediately evident.

6. Theory of the *Polar Planimeter.*

In Fig. 86, let $P_1P_2 = l$ be constant. At P let there be a graduated wheel attached to the bar P_1P_2 in such a manner that the axle of the wheel is rigidly parallel to P_1P_2. This wheel can record only the distance passed over by the bar at right angles to the bar.

Let $P_1P = l_1$, $PP_2 = l_2$. Let $CP = r$.

Then with the symbolism of § 164 we have

$$\begin{aligned} d(P_2) - d(P) &= \tfrac{1}{2}(r + l_2)^2\,d\theta - \tfrac{1}{2}r^2\,d\theta, \\ &= rl_2\,d\theta + \tfrac{1}{2}l_2^2\,d\theta. \\ d(P) - d(P_1) &= \tfrac{1}{2}r^2\,d\theta - \tfrac{1}{2}(r - l_1)^2\,d\theta, \\ &= rl_1\,d\theta - \tfrac{1}{2}l_1^2\,d\theta. \end{aligned}$$

Adding these two equations,

$$d(P_2) - d(P_1) = l\cdot r\,d\theta + \tfrac{1}{2}(l_2^2 - l_1^2)\,d\theta.$$

But $r\,d\theta = dR$ is the wheel record for a shift of the bar.

Integrating, we have for the area bounded by the curves traced by P_2 and P_1 and the initial and terminal position of the bar

$$(P_2) - (P_1) = l(R_2 - R_1) + \tfrac{1}{2}(l_2^2 - l_1^2)(\theta_2 - \theta_1),$$

θ_1, θ_2 being the initial and terminal angles which the bar makes with a fixed direction, and R_1, R_2 the initial and terminal records of the wheel.

Notice that when the wheel is attached to the middle of the bar

$$(P_2) - (P_1) = l(R_2 - R_1).$$

The path of P_1 is a circle in Amsler's instrument.

EXERCISES.

1. Find the area of the limaçon $\rho = a\cos\theta + b$, when $b > a$.

Ans. $(b^2 + \frac{1}{2}a^2)\pi$.

2. Show that the area of a segment of a parabola cut off by any focal chord in terms of l, the chord length, and p, the parameter, is $\frac{1}{6}l^{\frac{3}{2}}p^{\frac{1}{2}}$.

3. Show that the area of the curve $x^2y^2 = (a - x)(x - b)$ is $\pi\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right)^2$.

4. Show that the whole area between the curve $y(a^2 + x^2) = ma^3$ and the x-axis is $m\pi a^2$.

5. Show that the whole area between the curve $y^2(a^2 - x^2) = b^4$ and its asymptotes is $2\pi b^2$.

6. Show that the area between the curve and the axes in the first quadrant for $(x/a)^{\frac{1}{3}} + (y/b)^{\frac{1}{3}} = 1$ is $ab/20$.

7. Show that the area of a loop of the curve $y^4 - 2c^2y^2 + a^2x^2 = 0$ is $2^{\frac{5}{2}}c^3/3a$.

8. The locus of the foot of the perpendicular drawn from the origin to the tangent of a given curve is called the *pedal* of the given curve.

(1). The pedal of the ellipse $(x/a)^2 + (y/b)^2 = 1$ is

$$\rho^2 = a^2\cos^2\theta + b^2\sin^2\theta.$$

Show that its area is $\frac{1}{2}\pi(a^2 + b^2)$.

(2). The pedal of the hyperbola $(x/a)^2 - (y/b)^2 = 1$ is

$$\rho^2 = a^2 \cos^2\theta - b^2 \sin^2\theta.$$

Show that its area is $ab + (a^2 - b^2) \tan^{-1}(a/b)$.

9. If y_1, y_2, y_3 be three ordinates, y_2 being midway between y_1 and y_3, of the curve

$$y = ax^3 + bx^2 + cx + d,$$

show that the area bounded by the curve, the x-axis, and the ordinates y_1 and y_3 is

$$\tfrac{1}{6}(x_3 - x_1)(y_1 + y_3 + 4y_2).$$

If we transfer the origin to x_2, 0, and put $x_1 = -h$, $x_2 = +h$, the equation of the curve can be written

$$y = \alpha x^3 + \beta x^2 + \gamma x + \delta.$$

We have for the area

$$\int_{-h}^{+h} y\,dx = 2h(\tfrac{1}{3}\beta h^2 + \delta),$$

and $\frac{1}{3}h(y_1 + y_3 + 4y_2)$ has this same value. This is called Newton's rule.

10. Show that the area of any parabola

$$y = ax^2 + bx + c,$$

from $x = -h$, to $x = +h$, can be expressed in terms of the coordinates x_1, y_1 and x_2, y_2 of any two points on the curve, whose abscissæ satisfy $x_1 x_2 = -\frac{1}{3}h^2$.

Ans. $A = 2h\dfrac{x_1 y_2 - x_2 y_1}{x_1 - x_2}$.

The mean ordinate in the interval is

$$y_m = \frac{1}{2h}\int_{-h}^{+h} y\,dx = \tfrac{1}{3}ah^2 + c.$$

Let p and q be two undetermined numbers. Then

$$py_1 + qy_2 - y_m = a(px_1^2 + qx_2^2 - \tfrac{1}{3}h^2) + b(px_1 + qx_2) + (p + q - 1)c.$$

The three equations in p, q,

$$px_1^2 + qx_2^2 = \tfrac{1}{3}h^2, \qquad (1)$$

$$px_1 + px_2 = 0, \qquad (2)$$

$$p + q = 1, \qquad (3)$$

give determinate values of p and q, provided

$$\begin{vmatrix} x_1^2, & x_2^2, & \tfrac{1}{3}h^2 \\ x_1, & x_2, & 0 \\ 1, & 1, & 1 \end{vmatrix} = 0,$$

or

$$x_1 x_2 = -\tfrac{1}{3}h^2.$$

Then

$$y_m = py_1 + qy_2,$$

and the values of p and q from (2), (3) give the result.

11. In Elliott's theorem, § 164, (3), show that the mean of the areas of the curves described by all points on the segment P_1P_2 is $\frac{1}{2}[(P_1) + (P_2)] - \frac{1}{6}(A)$.

12. A given arc of a plane curve turns, without changing its form, around a fixed point in its plane; what is the area swept over by the arc?

13. If a curve is expressed in terms of its radius vector r and the perpendicular from the origin on the tangent p, prove that its area is given by

$$\frac{1}{2}\int\frac{pr\,dr}{\sqrt{r^2-p^2}}.$$

14. Lagrange's Interpolation Formula.

We have seen, in the decomposition of rational fractions, that when

$$\psi(x) \equiv (x-a_1)(x-a_2)\,.\,.\,.\,(x-a_n),$$

and $F(x)$ is a polynomial in x of degree less than n,

$$\frac{F(x)}{\psi(x)} = \sum_{r=1}^{n}\frac{1}{x-a_r}\,\frac{F(a_r)}{\psi'(a_r)}.$$

See § 133, and Ex. 79, Chapter XVIII.

If $F(x)$ is any differentiable function of x, then, since

$$F(x) - \sum_{r=1}^{n}\frac{\psi(x)}{x-a_r}\,\frac{F(a_r)}{\psi'(a_r)}$$

vanishes at $x = a_1, \ldots, a_n$, and the second term is a polynomial of degree $n - 1$, we have, § 98, II, lemma,

$$F(x) = \sum_{r=1}^{n}\frac{\psi(x)}{x-a_r}\,\frac{F(a_r)}{\psi'(a_r)} + \frac{\psi(x)}{n!}F^n(\xi), \tag{1}$$

where ξ is some number between the greatest and least of the numbers x, $a_1, \ldots, a_n$.

The formula

$$F(x) = \sum_{r=1}^{n}\frac{\psi(x)}{x-a_r}\,\frac{F(a_r)}{\psi'(a_r)}$$

is called Lagrange's interpolation formula. The member on the left computes the value at x of an unknown function when its values at $a_1, \ldots, a_n$ are known, with an error which is represented by

$$\frac{(x-a_1)\,.\,.\,.\,(x-a_n)}{n\,!}F^n(\xi).$$

15. Gauss' and Jacobi's theorem on areas.

If $F(x)$ is any polynomial of degree $2n - 1$, then the exact area of the curve $y = F(x)$ between $x = p$, $x = q$ can be computed in terms of n properly assigned ordinates.

Let

$$L(x) \equiv \sum_{1}^{n}\frac{\psi(x)}{x-a_r}\,\frac{F(a_r)}{\psi'(a_r)},$$

where, as in Ex. 14, $\psi(x) \equiv (x-a_1)\,.\,.\,.\,(x-a_n)$.

Then $J(x) \equiv F(x) - L(x)$ is a polynomial of degree $2n - 1$, in which $F(x)$ is of degree $2n - 1$, $L(x)$ of degree $n - 1$. Also, $J(x)$ vanishes when $x = a_1, \ldots, a_n$. Hence

$$F(x) - L(x) = A\,\phi(x)\,\psi(x),$$

where A is some constant and $\phi(x)$ some polynomial of degree $\blacksquare - 1$, since $\psi(x)$ is of degree n.

Integrating between p and q,

$$\int_p^q F(x)\,dx - \int_p^q L(x)\,dx = A\int_p^q \phi(x)\,\psi(x)\,dx.$$

Jacobi has shown as follows that we can always assign $a_1, \ldots, a_n$, so that

$$\int_p^q \phi\psi\, dx = 0.$$

For, integrating by parts successively,

$$\int \phi\psi\, dx = \phi\psi_1 - \phi'\psi_2 + \ldots - (-1)^n\phi^{(n-1)}\psi_n,$$

where $\phi^{(r)}$ denotes the result of differentiating ϕ r times, and ψ_r the result of integrating ψ r times, remembering that $\phi^{(n-1)}$ is a constant.

If we take, after Jacobi, for the values $a_1, \ldots, a_n$, the n roots of the equation of the nth degree

$$\left(\frac{d}{dx}\right)^n [(x-p)(x-q)]^n = 0,$$

then the integrals $\psi_1, \ldots, \psi_n$ between p and q are all 0, since each contains $(x-p)(x-q)$ as a factor.

Therefore, for these values of $a_1, \ldots, a_n$, we have

$$\int_p^q F(x)\, dx = \int_p^q \sum_1^n \frac{F(a_r)}{\psi'(a_r)} \frac{\psi(x)}{x - a_r},$$

or the proposition is established.*

If the degree of $F(x)$ is $2n$, then the area can be expressed in terms of $n+1$ ordinates taken at the roots of

$$\left(\frac{d}{dx}\right)^{n+1} [(x-p)(x-q)]^{n+1} = 0.$$

The area of $y = F(x)$ can be expressed in a singly infinite number of ways if one more than the required number of ordinates be used, in a doubly infinite number of ways if two more than the required number be used, and so on.

16. Show that the area of

$$y = a_0 + a_1x + a_2x^2 + a_3x^3,$$

from $-h$ to $+h$, is equal to

$$h(y_1 + y_2),$$

where y_1 and y_2 are the ordinates at $x = \pm h/\sqrt{3}$. Give a rule and compass construction for placing these ordinates.

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* See Boole's Finite Differences, p. 52.

CHAPTER XXI.

ON THE LENGTHS OF CURVES.

RECTANGULAR COORDINATES.

165. Definition of the Length of a Curve.—A mechanical conception of the length of a curve between two points on it can be obtained by regarding the curve as a flexible and inextensible string without thickness, which when straightened out can be applied to a straight line and its length measured. The curvilinear segment is then said to be rectified.

The rigorous analytical definition of a curve and of its length is a more difficult matter.

If y is a function of x such that y, Dy, D^2y, are uniform and continuous functions in an interval $x = \alpha$, $x = \beta$, then the assemblage of points representing

$$y = f(x)$$

in (α, β) is called a curve.

We can demonstrate* that if P and P_1 are any two points on this curve, we can always take P and P_1 so near together that the curve between P and P_1 lies wholly within the triangle whose sides are the tangents at P and P_1 and the chord PP_1. And also, if Q, R be any other two points on the curve between P and P_1, then, however near together are Q and R, the same property is true for Q and R.

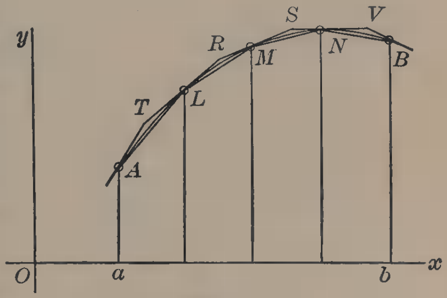

FIG. 87.

If we divide the interval (a, b) into n subintervals and at the points of division erect ordinates to the points $A, L, \ldots, B$, etc., on the curve, then draw the chords through these points, and the tangents to the curve there, we shall have two polygonal broken lines $ALMNB$ inscribed, and $ATRSVB$ circumscribed, to the curve AB.

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* Appendix, Note 11.

Let c_r represent the length of the rth chord, and t_r that of the rth side of the circumscribed line.

Clearly, whatever be the manner in which (a, b) is subdivided or to what extent that subdivision be carried, we shall always have

$$\sum_{1}^{n} c_r < \sum_{1}^{n} t_r \quad \text{and} \quad \mathop{\mathcal{L}}_{n=\infty}^{c_r=0} \sum_{1}^{n} c_r - \mathop{\mathcal{L}}_{n=\infty}^{t_r=0} \sum_{1}^{n} t = 0.$$

If we interpolate more points of division in (a, b), then Σt decreases while Σc increases. Consequently Σt and Σc converge to a common limit. This *limit* we define to be the length of the curve between A and B.

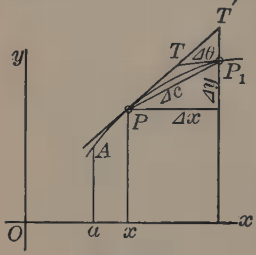


FIG. 88.

166. Let P be a point x, y on a curve, the length of which between A and P is s. Let P_1 be a point on the curve having coordinates $x + \Delta x$, $y + \Delta y$, and let the length of the curve between P and P_1 be Δs, the length of the chord PP_1 be Δc. Draw the tangents at P and P_1. Let the angle which PT makes with Ox be θ, and the angle between TP and TP_1 be $\Delta\theta$. Let $TP = t$, $TP_1 = t_1$, then, by § 165,

$$\Delta c < \Delta s < t + t_1.$$

But, from the triangle PTP_1,

$$\begin{aligned}(\Delta c)^2 &= t^2 + t_1^2 + 2tt_1 \cos \Delta\theta,\\ &= (t + t_1)^2 - 4tt_1 \sin^2 \tfrac{1}{2}\Delta\theta.\end{aligned}$$

$$\therefore \quad \left(\frac{\Delta c}{t + t_1}\right)^2 = 1 - \frac{4tt_1}{(t + t_1)^2} \sin^2 \tfrac{1}{2}\Delta\theta.$$

$4tt_1/(t + t_1)^2$ can never be greater than 1, and when $\Delta\theta(=)0$, $\Delta c(=)0$, $t + t_1(=)0$, also $\sin^2 \frac{1}{2}\Delta\theta(=)0$. Therefore when $\Delta x(=)0$, we have

$$\mathop{\mathcal{L}}_{\Delta x(=)0} \left(\frac{\Delta c}{t + t_1}\right) = 1.$$

Since, by definition, Δs lies between Δc and $t + t_1$, we also have

$$\mathop{\mathcal{L}}_{\Delta x(=)0} \frac{\Delta c}{\Delta s} = 1.$$

Now,

$$(\Delta c)^2 = (\Delta x)^2 + (\Delta y)^2,$$

or

$$\left(\frac{\Delta c}{\Delta x}\right)^2 = \left(\frac{\Delta c}{\Delta s}\right)\left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2.$$

Therefore, in the limit, for $\Delta x (=) 0$,

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2,$$

or

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{1}$$

Hence the length of the arc of the curve from A to P is

$$s = \int_a^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{2}$$

In like manner, using Δy instead of Δx, we obtain

$$s = \int_b^y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy. \tag{3}$$

In differentials

$$ds^2 = dx^2 + dy^2.$$

Since $dy/dx = \tan\theta$, θ being the slope of the curve at x, y, we have

$$dx = \cos\theta\, ds, \quad dy = \sin\theta\, ds.$$

Therefore $\dfrac{dx}{ds}, \dfrac{dy}{ds}$ are the direction cosines of the tangent to the curve at x, y.

EXAMPLES.

1. Rectify the *semi-cubical parabola* $ay^2 = x^3$.

Here $y = \dfrac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}}$. $\therefore \dfrac{dy}{dx} = \dfrac{3}{2}\left(\dfrac{x}{a}\right)^{\frac{1}{2}}, \quad \dfrac{ds}{dx} = \left(1 + \dfrac{9x}{4a}\right)^{\frac{1}{2}}.$

$$\therefore \quad s = \int_0^x \left(1 + \frac{9x}{4a}\right)^{\frac{1}{2}} dx = \frac{8a}{27}\left\{\left(1 + \frac{9x}{4a}\right)^{\frac{3}{2}} - 1\right\},$$

the arc being measured from the vertex. This was the first curve whose length was determined. The result was obtained by *William Neil* in 1660.

2. Rectify the *ordinary parabola* $y^2 = 2ax$.

We have $D_y x = y/a$.

$$\therefore \quad s = \frac{1}{a}\int_0^y \sqrt{a^2 + y^2}\, dy,$$

$$= \frac{1}{2a} y \sqrt{y^2 + a^2} + \frac{a}{2}\log\frac{y + \sqrt{y^2 + a^2}}{a},$$

the arc being measured from the vertex.

3. Rectify the *catenary* $y = \frac{1}{2}a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.

We have $Dy = \frac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)$.

$$\therefore \quad s = \frac{1}{2}\int_0^x \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) dx = \frac{1}{2}a\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right).$$

Show that $s = PL$. Fig. 73. Also, $NL =$ constant. The catenary is therefore the evolute of the tractrix represented by the dotted line in the figure.

4. Rectify the *evolute of the ellipse*

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

Write the equation in the form

$$\left(\frac{x}{\alpha}\right)^{\frac{2}{3}} + \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1,$$

put $x = \alpha \sin^3\phi, \quad y = \beta \cos^3\phi.$

$$\therefore \quad s = \frac{3}{2(\alpha^2 - \beta^2)} \int (\alpha^2 \sin^2\phi + \beta^2 \cos^2 \phi)^{\frac{1}{2}} d(\alpha^2 \sin^2\phi + \beta^2 \cos^2\phi).$$

Measuring the curve from $x = 0$, $y = \beta$, we get

$$S = \frac{(\alpha^2 \sin^3\phi + \beta^2 \cos^3\phi)^{\frac{3}{2}} - \beta^3}{\alpha^2 - \beta^2}$$

5. Find the length of the curve $9ay^2 = x(x - 3a)^2$, from $x = 0$ to $x = 3a$.
Ans. $2a\sqrt{3}$.

6. Find the entire length of the *hypocycloid* $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $6a$.

7. Find the length of the arc of the *circle* $x^2 + y^2 = a^2$, from $x = 0$ to $x = b$, and the whole perimeter.

8. Find the length of the *logarithmic curve* $y = ca^x$.

We have $\quad D_y x = b/y, \quad$ where $\quad b = 1/\log a$.

$$\therefore \quad S = \int \frac{(b^2 + y^2)^{\frac{1}{2}}}{y}\, dy = (b^2 + y^2)^{\frac{1}{2}} + b \log \frac{(b^2 + y^2)^{\frac{1}{2}} - b}{y}.$$

9. Find the length of the *tractrix* (see § 163, Fig. 84)

$$S = -a \int \frac{dy}{y} = -a \log y + \text{const.}$$

Measured from the vertex, $x = 0$, $y = a$,

$$S = a \log (a/y).$$

10. Length of an arc of the *ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Put $\quad x = a \sin \phi, \quad y = b \cos \phi.$ Then

$$S = \int (a^2 \cos^2\phi + b^2 \sin^2\phi)^{\frac{1}{2}}\, d\phi,$$

$$= a \int (1 - e^2 \sin^2\phi)^{\frac{1}{2}}\, d\phi.$$

where e is the eccentricity. This is an elliptic integral and cannot be integrated in finite elementary functions. The length of the elliptic quadrant corresponds to the limits $\phi = 0$, $\phi = \frac{1}{2}\pi$. Since $e^2 \sin^2\phi$ is always less than 1, we can expand the radical in a series of powers of $\sin \phi$, and integrate, obtaining the length of the quadrant (see Ex. 27 § 149)

$$\frac{\pi a}{2} \left\{ 1 - \frac{1}{1}\left(\frac{1}{2}\right)^2 \frac{e^2}{1} - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{e^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{e^6}{5} - \dots \right\}.$$

POLAR COORDINATES.

167. If $\rho = f(\theta)$ is the equation to a curve, and ρ, θ are the coordinates of P; $\rho + \Delta\rho$, $\theta + \Delta\theta$, those of P_1, then, calling Δc the chord PP_1, we have

$$\begin{aligned}(\Delta c)^2 &= (\rho + \Delta\rho)^2 + \rho^2 - 2\rho(\rho + \Delta\rho)\cos\Delta\theta,\\ &= (\Delta\rho)^2 + 2\rho(\rho + \Delta\rho)(1 - \cos\Delta\theta).\end{aligned}$$

Hence

$$\left(\frac{\Delta c}{\Delta\theta}\right)^2 = \left(\frac{\Delta\rho}{\Delta\theta}\right)^2 + 2\rho(\rho + \Delta\rho)\,\frac{1 - \cos\Delta\theta}{(\Delta\theta)^2}.$$

But when $\Delta\theta(=)0$, we have $\Delta\rho(=)0$, $\Delta c(=)0$, and

$$\mathop{\mathcal{L}}_{x(=)0}\frac{1 - \cos x}{x^2} = \frac{1}{2}\mathop{\mathcal{L}}_{x(=)0}\frac{\sin x}{x} = \frac{1}{2}.$$

$$\therefore\quad \left(\frac{dc}{d\theta}\right)^2 = \left(\frac{d\rho}{d\theta}\right)^2 + \rho^2.$$

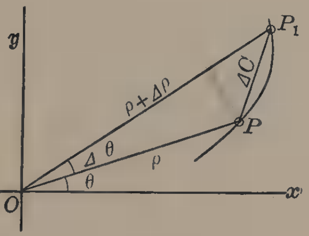


FIG. 89.

Also, $\dfrac{\Delta s}{\Delta\theta}\dfrac{\Delta c}{\Delta s} = \dfrac{\Delta c}{\Delta\theta}$, $\therefore$ $\dfrac{ds}{d\theta} = \dfrac{dc}{d\theta}$, since $\dfrac{ds}{dc} = 1$, § 166.

$$\therefore\quad ds^2 = d\rho^2 + \rho^2\,d\theta^2,$$

or

$$s = \int_{\theta_1}^{\theta_2}\sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\,d\theta = \int_{\rho_1}^{\rho_2}\sqrt{1 + \rho^2\left(\frac{d\theta}{d\rho}\right)^2}\,d\rho.$$

Otherwise we can deduce the formula for the length of an arc in polar coordinates directly from the corresponding formula in rectangular coordinates.

For $\quad x = \rho\cos\theta,\quad y = \rho\sin\theta.$

$$\therefore\quad dx = d\rho\cos\theta - \rho\sin\theta\,d\theta,\quad dy = d\rho\sin\theta + \rho\cos\theta\,d\theta.$$

$$\therefore\quad ds^2 = dx^2 + dy^2 = d\rho^2 + \rho^2\,d\theta^2.$$

EXAMPLES.

1. Find the length of the *cardioid* $\rho = a(1 + \cos\theta)$.

$$D_\theta\rho = -\, a\sin\theta, \text{ and therefore}$$

$$s = a\int[(1+\cos\theta)^2 + \sin^2\theta]^{\frac{1}{2}}\,d\theta = 2a\int\cos\tfrac{1}{2}\theta\,d\theta = 4a\sin\tfrac{1}{2}\theta\Big]_{\theta_1}^{\theta_2}.$$

The entire length is $8a$.

2. Show that the length of the arc of the *spiral of Archimedes*, $\rho = a\theta$, from the pole to the end of the first revolution, is

$$a\left[\pi\sqrt{1+4\pi^2} + \tfrac{1}{2}\log\left(2\pi + \sqrt{1+4\pi^2}\right)\right].$$

3. *Logarithmic spiral* $\rho = a^\theta$. Put $b = 1/\log a$.

Then $$s = \int_{r_1}^{r_2}(1+b^2)^{\frac{1}{2}}\,d\rho = (1+b^2)^{\frac{1}{2}}(r_2 - r_1).$$

4. Show that the length of the arc of $\log\rho = a\theta$, from the origin to (ρ, θ), is

$$\frac{\rho}{a}\sqrt{a^2+1}.$$

5. Find the length of $\rho + \theta^2 = a^2$.

6. Find the length of $\rho = a\sin\theta$, from 0 to $\frac{1}{2}\pi$.

7. Find the length of $\rho = a\sec\theta$, from 0 to $\frac{1}{2}\pi$.

8. Show that the entire length of $\rho = a\sin^3\frac{1}{3}\theta$ is $\frac{3}{2}\pi a$.

9. Show that the entire length of the epicycloid

$$4(\rho^2 - a^2)^3 = 27a^4\rho^2\sin^2\theta,$$

which is traced by a point on a circle of radius $\frac{1}{2}a$ rolling on a fixed circle of radius a, is $12a$.

10. Find the entire length of the curve $\rho = a\sin 2\theta$.

11. Show that the length of the *hyperbolic spiral* $\rho\theta = a$ is

$$\left[\sqrt{a^2+\rho^2} - a\log\frac{a+\sqrt{a^2+\rho^2}}{\rho}\right]_{\rho_2}^{\rho_1}.$$

12. Show that the length of the *parabola* $\rho = a\sec^2\frac{1}{2}\theta$, from $\theta = -\frac{1}{2}\pi$ to $\theta = +\frac{1}{2}\pi$, is $2a(\sec\frac{1}{4}\pi + \log\tan\frac{3}{8}\pi)$.

168. Geometrical Interpretations of the Differential Equations

$$ds^2 = dx^2 + dy^2 \quad\text{and}\quad ds^2 = d\rho^2 + \rho^2 d\theta^2.$$

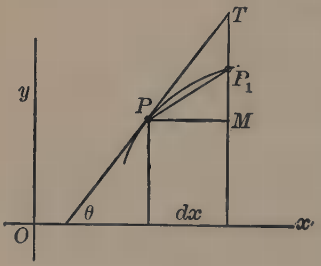


FIG. 90.

I. In Cartesian coordinates, if we take x as the independent variable, then we have $PM = dx$. Also, since

$$Dy = \tan\theta = \tan MPT,$$
$$\therefore\ dy = \tan\theta\,dx = MT.$$
$$\therefore\ ds^2 = PM^2 + MT^2 = PT^2.$$

Hence $ds = PT$, while the corresponding *difference* of the arc is PP_1. Also, we have the relations

$$dx = ds\cos\theta, \quad dy = ds\sin\theta.$$

It is easily shown geometrically that the limit of the quotient of $ds = PT$, by either $\Delta s = PP_1$ or $\Delta c = PP_1$, is 1, when $dx \equiv \Delta x$ converges to 0. See definition of the length of a curve, § 165.

II. We can in the case of polar coordinates exhibit ds, $d\rho$, and $\rho\, d\theta$ as the lengths of certain circular arcs as follows:

Let OA be the initial line and P a point on the curve $f(\rho, \theta) = 0$, PT the tangent at P. Draw OC perpendicular to $\rho = OP$, cutting the normal at P in C. Then $n = PC$ is the normal length, and $S_n = OC$ is the subnormal. Let θ be the independent variable, then $d\vartheta = \varDelta\theta$ is an arbitrarily chosen angle. We have the differentials ds, $d\rho$, $\rho\, d\theta$ proportional to the sides of the triangle POC, or to n, S_n, ρ, respectively. For we have

$$\left(\frac{ds}{d\vartheta}\right)^2 = \left(\frac{d\rho}{d\theta}\right)^2 + \rho^2.$$

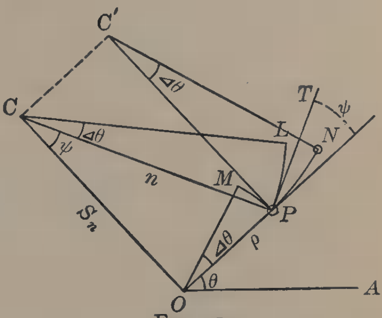


FIG. 91.

But $d\rho = S_n\, d\theta$, by § 92, (5). Also, $S_n^2 + \rho^2 = n^2$. Hence $ds = n\, d\theta$. Draw PC' parallel and equal to OC. Strike the arcs PN, PL, and PM with centers C' radius S_n, C radius n, O radius ρ, having the common central angle $\varDelta\theta = d\theta$. Then

$$ds = PL, \quad d\rho = PN, \quad \rho\, d\theta = PM.$$

It is interesting to notice that the rectilinear triangle PLN is a right-angled triangle similar to PCO; the sides of which, PL, PN, NL, are equal to the chords subtending the arcs PL, PN, and PM respectively.

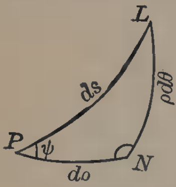


FIG. 92.

Therefore, in the triangular figure PLN whose sides are the circular arcs PL, PN, and an arc LN with radius ρ equal to PM, we have the sides $(PN) = d\rho$, $(PL) = ds$, $(LN) = \rho\, d\theta$.

Also, the angles between the circular arcs are

$$(L) = \tfrac{1}{2}\pi - \psi, \quad (P) = \psi, \quad (N) = \tfrac{1}{2}\pi + d\theta.$$

and
$$ds^2 = d\rho^2 + \rho^2\, d\theta^2,$$
$$d\rho = ds \cos\psi, \quad \rho\, d\theta = ds \sin\psi.$$

In order to prove these statements, it is only necessary to show that the rectilinear segment LN is equal to the chord subtending PM. Let x, y be the chords subtending PL, PN. Then from the rectilinear triangle PNL we have

$$LN^2 = x^2 + y^2 - 2xy \cos LPN.$$

But
$$\angle LPN = \psi + \tfrac{1}{2}\varDelta\theta - \tfrac{1}{2}\varDelta\theta = \psi.$$

Also,
$$x = 2n \sin\tfrac{1}{2}\varDelta\theta, \quad y = 2S_n \sin\tfrac{1}{2}\varDelta\theta.$$

$$\therefore\ LN^2 = 4(S_n^2 + n^2 - 2nS_n \cos\psi)\sin^2\tfrac{1}{2}\varDelta\theta,$$
$$= 4\rho^2 \sin^2\tfrac{1}{2}\varDelta\theta = (\text{chord } MP)^2.$$

The remainder follows easily.

Observe that if we draw PM perpendicular to OP, as in Fig. 93, and put $PM = \delta p$, $MP_1 = \delta\rho$, then we have, for $\varDelta\theta(=)0$,

$$\mathcal{L}\frac{\varDelta s}{\varDelta c} = 1, \quad \mathcal{L}\frac{\delta p}{\rho\,\varDelta\theta} = 1, \quad \mathcal{L}\frac{\delta\rho}{\varDelta\rho} = 1.$$

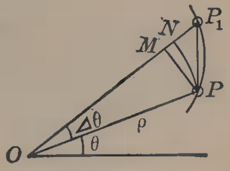


FIG. 93.

Therefore the difference equation

$$\varDelta c^2 = \delta p^2 + \delta\rho^2$$

leads at once to the differential equation

$$ds^2 = d\rho^2 + \rho^2\, d\theta^2.$$

169. Radius of Curvature and Length of Evolute.

If $f(x, y) = 0$ is the equation of a curve, then

$$\frac{dy}{dx} = \tan\theta, \quad \frac{d^2y}{dx^2} = \sec^2\theta\frac{d\theta}{dx}.$$

Hence if R is the radius of curvature at x, y,

$$R = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''} = \sec\theta\frac{dx}{d\theta} = \frac{ds}{d\theta},$$

since $ds = \sec\theta\, dx$. Therefore $ds = R\, d\theta$.

The angle $\Delta\theta = d\theta$ is the angle between the tangents at P and P_1, and is equal to the angle between the normals at P and P_1.

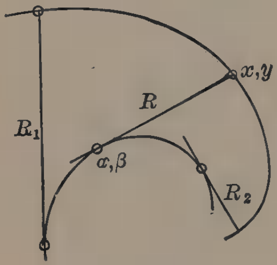


FIG. 94.

170. The length of the arc of the evolute of a given curve is equal to the difference of the corresponding radii of curvature of the involute.

Let x, y be a point on the involute corresponding to the point α, β on the evolute.

Then we have for the radius of curvature

$$R^2 = (\alpha - x)^2 + (\beta - y)^2.$$

Differentiating, we get

$$R\,dR = (\alpha - x)(d\alpha - dx) + (\beta - y)(d\beta - dy),$$
$$= (\alpha - x)d\alpha + (\beta - y)d\beta, \tag{1}$$

since $(\alpha - x)dx + (\beta - y)dy = 0$, this being the equation of R, the normal to the involute. If θ is the angle which the tangent to the involute at x, y makes with Ox, then, since R is tangent to the evolute, R makes with Ox the angle $\phi = \frac{1}{2}\pi + \theta$, and we have

$$\alpha - x = -R\sin\theta = R\cos\phi, \quad \beta - y = R\cos\theta = R\sin\phi.$$

Hence, on substitution in (1),

$$dR = d\alpha\cos\phi + d\beta\sin\phi,$$
$$= d\sigma,$$

if σ is the length of the arc of the evolute.

Integrating between α_1, β_1 and α_2, β_2, we have

$$R_2 - R_1 = \sigma_2 - \sigma_1.$$

This can be shown otherwise, for we have

$$(x - \alpha)d\alpha + (y - \beta)d\beta = 0, \tag{2}$$

the equation to the normal to the evolute at α, β.

The perpendicular R, from x, y on the involute to the line (2), is

$$R = \frac{(x - \alpha)d\alpha + (y - \beta)d\beta}{\sqrt{d\alpha^2 + d\beta^2}},$$

or
$$(x - \alpha)d\alpha + (y - \beta)d\beta = R\,d\sigma.$$

Equating with (1), we get $dR = d\sigma$ as before.

It is to be particularly observed that the theorem as enunciated applies only to an arc of the involute such that between its ends the radius of curvature has neither a maximum nor a minimum value. For when R passes through a maximum or a minimum value dR changes sign. $\int dR$ would be zero when taken between two points at which R has equal values.

In applying the theorem one should be careful to determine the maximum and minimum values of the radius of curvature for the involute, and add the corresponding *absolute* values of the lengths of the evolute, when the radius of curvature has maximum or minimum values between the ends of the arc under consideration.

From a mechanical point of view, since the evolute is the envelope of the normals of the involute, we can regard the involute as a point described by a point in the tangent, as the tangent is unrolled from its contact with the evolute; the arc being considered as a flexible inextensible string wrapped on the curve. The truth of the above theorem from this point of view is made evident.

The theorem of this article rectifies any curve which is the evolute of a known curve whose radius of curvature can be found.

EXAMPLES.

1. Find the length of $27ay^2 = 4(x - 2a)^3$, the evolute of the parabola $y^2 = 4ax$.

We have for the coordinates x', y' of the center of curvature and R, the radius of curvature of the parabola at x, y,

$$x' = 2a + 3x, \quad y' = -\frac{y^3}{4a^2}, \quad R = 2a\left(\frac{a + x}{a}\right)^{\frac{3}{2}}.$$

Measuring the arc of the evolute from the cusp, $x = 2a$, $y = 0$, to x', y', we have

$$S = 2a\left(\frac{x' + a}{3a}\right)^{\frac{3}{2}} - 2a.$$

2. Find the length of

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}},$$

the evolute of the ellipse.

3. Show that the catenary is the evolute of the tractrix, and find the length of an arc of the catenary as such.

171. The Intrinsic Equation of a Curve.—The length s measured from the point of contact O of a curve with a fixed tangent, and the angle ϕ which the tangent at the end P of the arc makes with the fixed tangent, are called the *intrinsic* coordinates of a point on the curve.

The equation $f(s, \phi) = 0$, which expresses the relation between s and ϕ, is called the *intrinsic* equation of the curve.

To find the intrinsic equation of a curve $f(x, y) = 0$ or $f(\rho, \theta) = 0$, we have to find the length of the arc from a fixed point to an arbitrary point on the curve, then the angle ϕ between the tangents there. Eliminating the original coordinates between these three equations, the result is the intrinsic equation.

EXAMPLES.

1. Find the intrinsic equation of the *catenary*.

Take the vertex as the initial point, then

$$y = \tfrac{1}{2}a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

$$Dy = \tfrac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) = \tan\phi.$$

Also, $$s = \tfrac{1}{2}a\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right).$$

Eliminating x, we have $s = a\tan\phi$.

2. Find the intrinsic equation of the involute of the circle.

Let $x^2 + y^2 = a^2$ be the equation to the circle. Unwrap the arc beginning at the point a, 0, and let the radius to the point of contact make the angle ϕ with Ox. Then ϕ is also the angle which the tangent to the involute makes with Ox. The radius of curvature is the unwrapped tangent length, or $R = a\phi$. But

$$ds = R\,d\phi = a\phi\,d\phi. \quad \therefore \quad s = \tfrac{1}{2}a\phi^2.$$

172. General Remark on Rectification.—The problem of finding the length of any curve whose equation is given involves the integral of a function which is in irrational form. This in general does not admit of integration in finite form, and cannot generally be expressed in terms of the elementary functions. There are, generally speaking, but few curves that can be rectified, in terms of elementary functions.

EXERCISES.

1. Show that the length of $8a^3y = x^4 + 6a^2x^2$ measured from the origin is $x(x^2 + 4a^2)^{\frac{3}{2}}/8a^3$.

2. Show that the whole length of $4(x^2 + y^2) - a^2 = 3a^{\frac{4}{3}}y^{\frac{2}{3}}$ is $6a$.

3. Show that $a^my^n = x^{m+n}$ is a curve whose length can be obtained in finite terms when $\dfrac{n}{2m}$ or $\dfrac{n}{2m} + \dfrac{1}{2}$ is an integer.

4. Show that the intrinsic equation to $y^3 = ax^2$ is

$$s = \tfrac{8}{27}a\,(\sec^3\phi - 1).$$

5. Show that $\rho^m = a^m \cos m\theta$ can be rectified when $1/m$ is an integer.

6. If $x = \phi(t)$, $y = \psi(t)$, then

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = [\phi'(t)]^2 + [\psi'(t)]^2.$$

7. In the *cycloid* $x = a(\theta - \sin\theta)$, $y = a \operatorname{vers}\theta$, show that

$$ds = 2a\sin\tfrac{1}{2}\theta\,d\theta.$$

Hence the length of one arch is $8a$.

8. Show that the length of the *cycloid*

$$x = a \cos^{-1} \frac{a-y}{a} + \sqrt{2ay - y^2}$$

from the origin to x, y is $\sqrt{8ay}$.

9. Show that the intrinsic equation of the *cycloid* in Ex. 8 is

$$s = 4a \sin \phi,$$

the tangent at the origin being the initial tangent.

10. In the equiangular spiral $\rho = ae^{\frac{\theta}{n}}$, show that $s = \rho \sec \psi$, where $\tan \psi = n$, measuring s from the pole.

11. Find the length of the reciprocal spiral from $\theta = 2\pi$ to $\theta = 4\pi$, the equation being $\rho\theta = a$.

12. Show that the whole perimeter of the *lemniscate*

$$\rho^2 = a^2 \cos 2\theta$$

is
$$4a\left(1 + \frac{1}{2 \cdot 5} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 9} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 13} + \ldots\right).$$

13. Show that the length of $y^2 = x^3$ between $x = 0$, $x = 1$ is $\frac{1}{27}(13^{\frac{3}{2}} - 8)$.

14. Designating by $L_{x=a}^{x=b}(f)$ the length of the curve $f(x, y) = 0$, from $x = a$ to $x = b$, show that:

(*a*). $L_{x=1}^{x=2}(x^4 - 6xy + 3) = \frac{17}{12}$.

(*b*). $L_{x=0}^{x=a}(x^{\frac{2}{3}} + y^{\frac{2}{3}} - a^{\frac{2}{3}}) = 6a$.

(*c*). $L_{x=1}^{x=3}\left[2y - x\sqrt{x^2 - 1} - \log(x - \sqrt{x^2 - 1})\right] = \int_1^3 x\,dx = 4$.

(*d*). $L_{y=a}^{y=b}\left[x - \sqrt{a^2 - y^2} + a \log \frac{a + \sqrt{a^2 - y^2}}{y}\right] = a \log \frac{b}{a}$.

(*e*). $L_{x=\frac{1}{3}\pi}^{x=\frac{2}{3}\pi}(y - \log \sin x) = \log 3$.

(*f*). $L_{x=0}^{x=1}(y - \sqrt{x - x^2} - \sin^{-1}\sqrt{x}) = \int_0^1 x^{-\frac{1}{2}}\,dx = 2$.

(*g*). $L_{y=1}^{y=2}(y^2 + 4x - 2 \log y) = \frac{1}{4}(2 \log 2 + 3)$.

15. Use the binomial theorem to evaluate

$$L_{x=a}^{x=x}(xy - a^2) = \int_a^x \frac{\sqrt{x^4 + a^4}}{x^2}\,dx.$$

16. Show that

$$L_{x=0}^{x=\pi}(y - \sin x) = \pi(1 + \tfrac{1}{4} - \tfrac{3}{64} + \tfrac{5}{256} - \ldots)$$

17. $L_{\rho=a}^{\rho=2a}(\rho^2 - 2a\rho\theta + a^2) = \frac{a}{4}(3 + \log 4)$.

18. $L_{\theta=0}^{\theta=1}\left(\rho - \frac{e\theta - 1}{e\theta + 1}\right) = \frac{2}{e + 1}$.

19. $L_{\rho=a}^{\rho=3a}\left(\frac{\sqrt{\rho^2 - a^2}}{a} - \cos^{-1}\frac{a}{\rho} - \theta\right) = 4a$.

20. A hawk can fly v feet per second, a hare can run v' feet per second. The hawk, when a feet vertically above the hare, gives chase and catches the hare when the hare has run b feet. Find the length of the curve of pursuit.

Take O, the starting-point of the hawk, as origin, the line OH drawn to the starting-point of the hare H as y-axis, and a parallel Ox to the hare's path as x-axis. When the hawk has flown a distance s to P, the hare will have run a distance σ to P' in the tangent to the curve at P.

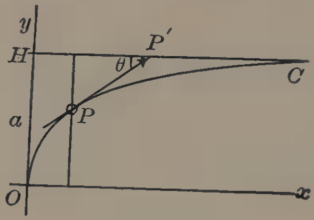


FIG. 95.

Let $PP' = t$. We have $b = v'T = HC$. $S = vT$, the length OPC. T being the time of pursuit.

$$\therefore \quad \frac{b}{S} = \frac{v'}{v} = k = \text{constant.}$$

In like manner,

$$\frac{\sigma}{s} = \frac{v'}{v} = k.$$

$$\therefore \quad d\sigma = k\,ds. \quad \text{Also,}$$

$$t^2 = (\sigma - x)^2 + (a - y)^2.$$

$$\therefore \quad t\,dt = (\sigma - x)(d\sigma - dx) - (a - y)dy.$$

But $\sigma - x = t\cos\theta$, $a - y = t\sin\theta$. Also, $d\sigma\cos\theta$ is $k\,ds\cos\theta$ or $k\,dx$, and

$$dx\cos\theta + dy\sin\theta = ds.$$

$$\therefore \quad dt = k\,dx - ds.$$

Hence
$$S = k\int_0^b dx - \int_a^0 dt = kb + a,$$

$$= \frac{b^2}{S} + a.$$

$$\therefore \quad S = \frac{a}{2} + \sqrt{\frac{a^2}{4} + b^2}.$$

21. If R and r are the radii of the fixed and rolling circles in the epicycloid and hypocycloid

$$x = (R \pm r)\cos\phi \mp r\cos\frac{R \pm r}{\pm r}\phi, \quad y = (R \pm r)\sin\phi \mp r\sin\frac{R \pm r}{\pm r}\phi,$$

show that the lengths of the curves from cusp to cusp are respectively

$$8r(R \pm r)/R.$$

22. In § 164 (Elliott's theorem), if s_1, s_2 s are the corresponding lengths of the arcs described by the points P_1, F_2, P respectively and σ the corresponding length of the director curve, show that

$$ds^2 = k_2\,ds_1^2 + k_1\,ds_2^2 - k_1k_2\,d\sigma^2.$$

CHAPTER XXII.

ON THE VOLUMES AND SURFACES OF REVOLUTES.

173. Definition.—A point is said to revolve about a straight line as an axis when it describes the arc of a circle whose plane is perpendicular to the straight line and whose center is on the straight line.

A plane figure is said to revolve about a straight line in its plane as an axis when each point of the figure revolves about the line as an axis.

The solid geometrical figure generated by the revolution of a plane figure about a straight line in its plane as axis is called a *revolute.* The surface of the figure revolved generates the volume, and the perimeter of the figure revolved generates the surface of the revolute.

Examples of revolutes are familiar from the three round bodies of elementary geometry, the cylinder, cone, and sphere.

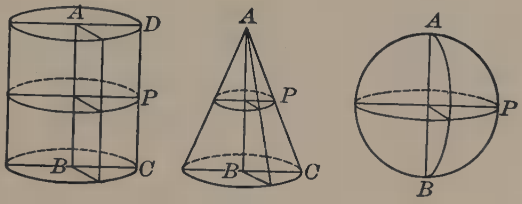


FIG. 96.

AB being the axis of revolution, the cylinder is generated by the revolution of the rectangle $ABCD$, the cone by that of the right-angled triangle ABC, the sphere by that of the semicircle APB. The volumes of the revolutes are generated by the surfaces, and the surfaces of the revolutes by the perimeters of the revolving figures.

We know from elementary geometry that the volume of the cylinder is equal to the area of the circular base multiplied by the altitude or by DC the length of the line generating the curved surface. Also, the curved surface of the cylinder has for its area the product of the circumference of the base into CD, the length of the generating line.

It is evident from the definition of a revolute that any section of a revolute by a plane perpendicular to the axis AB is a circle, such as ODD'. The circular sections cut out of the surface by planes perpendicular to the axis are called *parallels*. In like manner the section of the surface of a revolute by any plane passing through the axis is a line identically the same as the generating line. For if in the figure the surface is generated by the revolution of the line $ACDB$ about the axis AB, then the section $AD'B$ is nothing more than one position of the generating line ACB. Again, the revolute can always be regarded as being generated by a circle moving in such a manner that its center moves along the axis to which its plane is perpendicular, and its radius changes according to a given law.

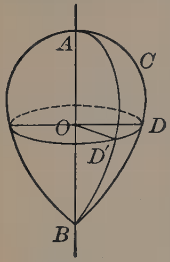


FIG. 97.

174. Volume of a Revolute.—Let $y = f(x)$ be a curve AB. We require the volume of the solid generated by the figure $aABb$ revolving about Ox as axis of revolution.

Divide (a, b) into n subintervals, and pass planes through the points of division cutting the solid into n parts, such as the one generated by the revolution of $xPP'x'$. We can always take $x' - x = \Delta x$ so small that the curve PP' will lie inside the rectangle $PMP'M'$, if $f(x)$ is continuous. Let y be an increasing one-valued function from $x = a$ to $x = b$. The volume, ΔV, of the solid generated by $xPP'x'$, lies between the volumes of the cylinders generated by the rectangles $xPM'x'$ and $xMP'x'$. Hence for each subdivision of the solid we have

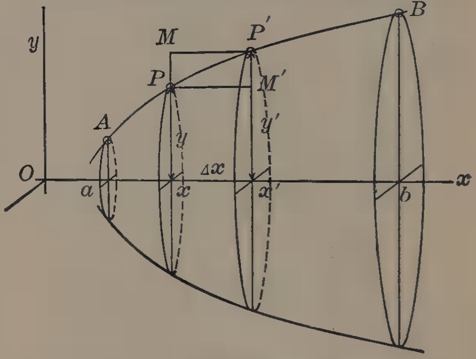


FIG. 98.

$$\pi y^2 \Delta x < \Delta V < \pi y'^2 \Delta x. \tag{1}$$

The whole volume of the revolute, therefore, lies between the sum of the n interior cylinders and that of the exterior cylinders, or

$$\sum_{1}^{n} \pi y^2 \Delta x < V < \sum_{1}^{n} \pi y'^2 \Delta x. \tag{2}$$

But if we interpolate more points of subdivision in (a, b), we increase the sum of the interior volumes and decrease that of the exterior; and since

$$\mathcal{L}_{\Delta x(=)0} \frac{\pi y^2 \Delta x}{\pi y'^2 \Delta x} = 1,$$

these sums have a common limit, which is V.

$$\therefore \quad V_x = \mathcal{L}_{n=\infty}^{\Delta x(=)0} \sum_{1}^{n} \pi y^2 \Delta x,$$

$$= \int_a^b \pi y^2 \, dx. \tag{3}$$

Again, we have directly from the inequality (1)

$$\pi y^2 < \frac{\Delta V}{\Delta x} < \pi y'^2.$$

Hence, for $\Delta x(=)0$, we have

$$\frac{dV}{dx} = \pi y^2,$$

since $\mathcal{L} y' = v$, when $\Delta x(=)0$.

$$\therefore \quad dV = \pi y^2 \, dx, \tag{4}$$

and as before

$$V_x = \pi \int_a^b y^2 \, dx. \tag{5}$$

In like manner we show that if x is a one-valued function of y, say $x = \phi(y)$, then the volume generated by the revolution of the curve about Oy as an axis, included between two planes perpendicular to Oy at $y = p$ and $y = q$, is

$$V_y = \pi \int_p^q x^2 \, dy. \tag{6}$$

EXAMPLES.

1. Find the volume of the *cone* of revolution generated by the revolution of the triangle formed by the lines $x = 0, y = 0, x/a + y/b = 1$, about Ox as axis.

$$V_x = \pi \int_0^a y^2 \, dx = \pi \int_0^a \left(b - \frac{b}{a} x\right)^2 dx,$$

$$= -\frac{\pi}{3} \frac{a}{b} \left(b - \frac{b}{a} x\right)^3 \Bigg]_0^a = \tfrac{1}{3}\pi a b^2.$$

But a is the altitude and b the radius of the base of the cone. Therefore the volume is equal to one third the product of the area of the base into the altitude.

2. Find the volume of the *sphere* generated by the revolution of a semi-circle of $x^2 + y^2 = a^2$ about Oy.

$$V_y = \pi \int_{-a}^{+a} x^2 \, dy = \pi \int_{-a}^{+a} (a^2 - y^2) \, dy = \tfrac{4}{3}\pi a^3.$$

3. The *prolate spheroid* is the revolute generated by the revolution of an ellipse about its long axis, sometimes called the *oblongum*.

Let a be the semi-major axis of $x^2/a^2 + y^2/b^2 = 1$.
Then we have for the volume of the oblongum

$$V_x = \pi \int_{-a}^{+a} \frac{b^2}{a^2}(a^2 - x^2)\,dx = \tfrac{4}{3}\pi ab^2.$$

4. The *oblate spheroid* or *oblatum* is the revolute obtained by revolving the ellipse about its minor axis; show that its volume is $\frac{4}{3}\pi a^2 b$, where b is the semi minor axis.

5. Show that the volume of the revolute obtained by revolving the parabola $y^2 = 4ax$ about Ox, from $x = 0$ to $x = a$, is $2\pi a^3$.
This is the *paraboloid* of revolution.

6. If the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ revolves about Oy, the revolute is called the *hyperboloid of revolution* of *one* sheet. Show that the volume from $y = 0$ to $y = y$ is $\frac{\pi}{3}\frac{a^2}{b^2}(y^3 + 3b^2 y)$.

If the curve revolves about Ox, find the volume from $x = a$ to $x = 2a$. This surface is called the *hyperboloid of revolution* of *two* sheets.

7. Find the entire volume generated by the revolution about Ox of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $\frac{32}{105}\pi a^3$.

8. The surface generated by the revolution of the tractrix about its asymptote is called the *pseudo-sphere*. This important surface has the property of having its curvature constant and negative. Find its volume.

Here $y^2\,dx = -(a^2 - y^2)^{\frac{1}{2}} y\,dy$. Hence the volume from $x = 0$ to $x = x$ is

$$V_x = \pi \int_y^a (a^2 - y^2)^{\frac{1}{2}} y\,dy = \tfrac{1}{3}\pi(a^2 - y^2)^{\frac{3}{2}}.$$

The volume of the entire pseudo-sphere is $\frac{2}{3}\pi a^3$, or one half that of a sphere with radius a.

9. Find the volume generated by the revolution of the catenary

$$y = \tfrac{1}{2}a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) \text{ about } Ox \text{ from } 0 \text{ to } x. \qquad \textit{Ans.}\ \tfrac{1}{2}\pi a(ys + ax).$$

10. The volume generated by revolving the witch $(x^2 + 4a^2)y = 8a^3$ about its asymptote is $4\pi^2 a^3$.

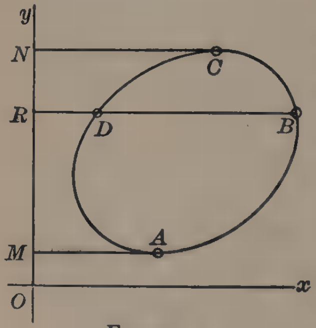


FIG. 99.

175. To find the volume of the revolute generated by a closed curve revolving about an axis in its plane, but external to the curve.

We take the difference between the volumes of the revolutes generated by $MABCN$ and $MADCN$. Hence the volume of the solid ring generated by $ABCD$ revolving about Oy is

$$V_y = \pi \int (x_2^2 - x_1^2)\,dy,$$

where $x_1 = RD$, $x_2 = RB$, and the limits of the integral are $y = OM$, $Y = ON$. A corresponding integral gives the volume about Ox.

EXAMPLES.

✓ **1.** The solid ring generated by the revolution of a *circle* about an axis external to it is called a *torus*. Show that the volume of the torus generated by the circle

$$(x - a)^2 + y^2 = r^2$$

$(a \geqq r)$ about Oy is $2\pi^2r^2a$.

We have
$$x_2 = a + \sqrt{r^2 - y^2},$$
$$x_1 = a - \sqrt{r^2 - y^2}.$$
$$\therefore \quad V_y = \pi\int_{-r}^{+r} 4a\sqrt{r^2 - y^2}\,dy = 2\pi^2r^2a.$$

$\int_0^{\pi} -4a\pi r^2 \sin^2\theta\, d\theta$

$y = r\cos\theta$

$= 8a\pi r^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2}$

$2a\pi^2 r^2.$

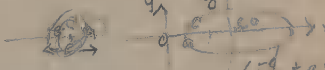
Observe that the volume is equal to the product of the area of the generating circle into the circumference described by its centre.

✓ **2.** Show that the volume of the elliptic torus generated by

$$\frac{(x - c)^2}{a^2} + \frac{y^2}{b^2} = 1$$

$(c > a)$ about Oy is $2\pi^2abc$.

$V = \pi\int_0^b (x_2^2 - x_1^2)\,dy$



$x = c \pm \frac{a}{b}\sqrt{b^2-y^2}$

$x_2 = c + \frac{a}{b}\sqrt{b^2-}$

$x_1 = c - \frac{a}{b}\sqrt{(b^2-}$

176. The Area of the Surface of a Revolute.

—We know, from elementary geometry, that the curved surface of a cone of revolution is equal to half the product of the slant height into the circumference of the base.

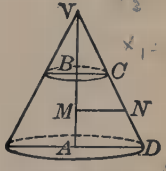


FIG. 100.

The area of the curved surface of the frustum included between the parallel planes AD and BC is therefore

$$\pi(VD\cdot AD - VC\cdot BC).$$

Since $BC/AD = VC/VD$, we deduce for the surface generated by the revolution of CD about BA the area

$$2\pi MN\cdot CD,$$

where MN joins the middle points of AB and CD.

$V = 4\pi\int(b^2-y^2)\frac{ac}{b}\,dy$

$= 8\pi$

$\left[\frac{b^3}{3}\right.$

$8\pi abc$

In the figure of § 174, Fig. 98, subdivide, as before, the interval (a, b) into n parts; erect ordinates to the curve AB at the points of division. Join the points of division on the curve by drawing the chords of the corresponding arcs, thus inscribing in the curve AB a polygonal line AB with n sides. Let PP' be one of the sides of this polygonal line. The curved surface of the frustum of a cone generated by the chord $\varDelta c = PP'$ revolving about Ox has for its area

$$2\pi\frac{y + y'}{2}\varDelta c = 2\pi(y + \tfrac{1}{2}\varDelta y)\varDelta c.$$

2π

We define the surface generated by the revolution of the arc of the curve AB about Ox to be the limit to which converges the surface generated by the revolution about Ox of the inscribed polygonal line, when the number of the sides of the polygonal line increases indefinitely and at the same time each side diminishes indefinitely.

To evaluate this limit, we have for the area of the surface generated by the curve AB

$$S_x = \mathop{\pounds}_{n=\infty}^{\Delta x(=)0} \sum_{1}^{n} 2\pi(y + \tfrac{1}{2}\Delta y)\Delta c,$$

$$= \mathop{\pounds}_{n=\infty}^{\Delta x(=)0} \sum_{1}^{n} 2\pi y\, ds.$$

Since for each pair of corresponding elements of these two sums we have

$$\mathop{\pounds}_{\Delta x(=)0} \frac{2\pi(y + \frac{1}{2}\Delta y)\Delta c}{2\pi y\, ds} = 1.$$

Hence we have, by definition of an integral,

$$S_x = 2\pi \int_{x=a}^{x=b} y\, ds,$$

$$= 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_{y_1}^{y_2} y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy. \qquad (1)$$

In like manner, if AB revolves about Oy, we have for the area of the surface generated

$$S_y = 2\pi \int_{x=a}^{x=b} x\, ds,$$

$$= 2\pi \int_{x_1}^{x_2} x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_{y_1}^{y_2} x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy. \qquad (2)$$

EXAMPLES.

1. Find the surface of the sphere generated by the revolution of the circle $y^2 = a^2 - x^2$ about Ox.

We have $$\frac{dy}{dx} = -\frac{x}{y}, \quad \frac{ds}{dx} = \frac{a}{y}.$$

$$\therefore \quad S_x = 2\pi \int y\, ds = 2\pi(x_2 - x_1)a.$$

Hence the area of the *zone* included between the two parallel planes is equal to the circumference of a great circle into the altitude of the zone. If $x_2 = +a$, $x_1 = -a$, we have the whole surface of the sphere $4\pi a^2$.

2. Show that the curved surface of the cone generated by the revolution of $y = x \tan \alpha$ about Ox, from $x = 0$ to $x = h$, is $\pi h^2 \tan \alpha \sec \alpha$. Verify the formula deduced for the surface of a frustum in § 176.

3. Surface area of the *paraboloid of revolution.*

Let $y^2 = 2mx$ revolve about Ox. Then

$$S_x = \frac{2\pi}{m} \int_0^y (y^2 + m^2)^{\frac{1}{2}} y\, dy = \frac{2\pi}{3m}\left\{ (y^2 + m^2)^{\frac{3}{2}} - m^3 \right\}.$$

4. Let $2a$ be the major axis of $a^2y^2 + b^2x^2 = a^2b^2$, and e its eccentricity. Then we have for the surface of the prolate spheroid

$$S_x = \frac{2\pi be}{a} \int_{-a}^{+a} \sqrt{\frac{a^2}{e^2} - x^2}\, dx = 2\pi ab\left(\sqrt{1 - e^2} + \frac{\sin^{-1} e}{e}\right).$$

5. Show that the surface of the *pseudo-sphere* is

$$S_x = 2\pi a \int_y^a dy = 2\pi a(a - y).$$

Its entire surface is $2\pi a^2$.

6. In the catenary show that

$$S_x = \pi(ys + ax),$$
$$S_y = 2\pi(a^2 + xs - ay),$$

from $x = 0$ to $x = x$.

7. Show that the surface of the *hypocycloid* of revolution generated by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about Ox is $\frac{12}{5}\pi a^2$.

8. A *cycloid* revolves around the tangent at the vertex. Show that the whole surface generated is $\frac{32}{3}\pi a^2$.

9. The *cardioid* $\rho = a(1 + \cos\theta)$ revolves about the initial line. Show that the area of its surface is $\frac{32}{5}\pi a^2$.

177. If a plane closed curve having an axis of symmetry revolves about an axis of revolution parallel to the axis of symmetry and at a distance a from it, then we shall have for the volume and surface of the revolute generated, respectively,

$$V = 2\pi aA, \quad S = 2\pi aL,$$

where A is the area and L the length of the generating curve.

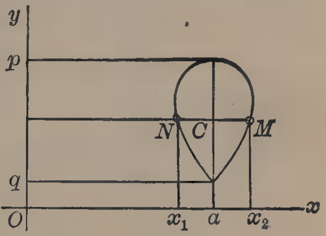


FIG. 101.

Let $x = a$ be the axis of symmetry and Oy the axis of revolution. Then for the volume

$$V_y = \pi \int_q^p (x_2^2 - x_1^2)\,dy.$$

But if $CM = CN = x', \quad x_2 = a + x', \quad x_1 = a - x'$,

$$\therefore \quad V_y = 2\pi a \int_q^p 2x'\,dy = 2\pi aA.$$

For the surface

$$S_y = 2\pi \int_q^p (x_2 + x_1)\,ds,$$

$$= 2\pi a \int_q^p 2ds = 2\pi aL.$$

The results obtained assume that the axis of revolution does not cut the generating curve.

EXAMPLES.

1. The volume and surface of the torus generated by the revolution of a circle of radius a about an axis distant c from the center ($c \geqq a$) are respectively $2\pi^2a^2c$ and $4\pi^2ac$.

2. The volume generated by the revolution of an ellipse, having $2a$, $2b$ as major and minor axes, about a tangent at the end of the major axis is $2\pi^2a^2b$.

EXERCISES.

1. Show that the segment of the parabola $y^2 = 2px$, made by the line $x = a$, when rotated about Ox, generates the volume

$$2\pi p \int_0^a x\,dx = \pi p a^2.$$

2. The figure in Ex. 1 rotated about the y-axis generates the volume

$$2\pi \int_0^{\sqrt{2pa}} \left(a^2 - \frac{y^4}{4p^2}\right) dy = \tfrac{8}{5}\pi a^2 \sqrt{2pa}.$$

3. The volume generated by the closed curve $x^4 - a^2x^2 + a^2y^2 = 0$ about the x-axis is

$$\frac{2\pi}{a^2} \int_0^a (a^2x^2 - x^4)\,dx = \tfrac{4}{15}\pi a^3.$$

4. The curve $x^2 + y^{\frac{2}{3}} = 1$ rotating about the y-axis generates a solid whose volume is $\frac{4}{5}\pi$.

5. The volumes generated by $y = e^x$ about Ox and Oy are respectively

$$\pi \int_{-\infty}^0 e^{2x}\,dx = \tfrac{1}{2}\pi, \quad \pi \int_0^1 (\log y)^2\,dy = 2\pi.$$

6. The curve $y = \sin x$ rotating about Ox and Oy, respectively, for $x = 0$, $x = \pi$, generates the volumes

$$\pi \int \sin^2 x\,dx = \tfrac{1}{2}\pi^2, \quad \pi \int_0^{\frac{1}{2}\pi} (\pi^2 - 2\pi x) \cos x\,dx = 2\pi^2.$$

7. The volume generated by one arch of the cycloid

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta), \quad \text{rotating about } Ox, \text{ is}$$

$$32\pi a^3 \int_0^\pi \sin^6 \tfrac{1}{2}\theta\, d(\tfrac{1}{2}\theta) = 5\pi^2 a^3.$$

8. The same branch rotating about Oy gives the volume

$$4\pi^2 a^3 \int_0^\pi (\pi - \theta + \sin\theta) \sin\theta\, d\theta = 6\pi^3 a^3.$$

9. Show that the whole surface of an oblate spheroid is

$$2\pi a^2 \left(1 + \frac{1 - e^2}{2e} \log \frac{1 + e}{1 - e}\right),$$

e being the eccentricity and a the semi-major axis of the generating ellipse.

10. The curve $y^2(x - 4a) = ax(x - 3a)$, from $x = 0$ to $x = 3a$, revolving around Ox generates the volume $\frac{1}{2}\pi a^3(15 - 16 \log 2)$.

11. The curve $y^2(2a - x) = x^3$ revolves around its asymptote. Show that the volume generated is $2\pi^2 a^3$.

12. The curve $xy^2 = 4a^2(2a - x)$ revolves around its asymptote. Show that the volume generated is $4\pi^2 a^3$.

13. Find the volume and the surface generated by revolving $y^2 = 4ax$ about Oy, from $x = 0$ to $x = a$. *Ans.* $V = \frac{2}{5}\pi a^3$. $S = \frac{1}{4}\pi a^2\left[6\sqrt{2} - \log(3 + 2\sqrt{2})\right]$.

14. Show that the volume generated by revolving the part of the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ between the points of contact with the axes about Ox or Oy is $\frac{1}{15}\pi a^3$.

15. The surface generated by $y = x^3$, from $x = 0$ to $x = 1$, rotating about Ox, is

$$2\pi \int_0^1 \sqrt{1 + 9x^4}\, x^3\, dx = \frac{\pi}{27}\left(\sqrt{1000} - 1\right).$$

16. The surface generated by $x^4 - a^2x^2 + 8a^2y^2 = 0$, about Ox, from $x = 0$ to $x = a$, is

$$\frac{\pi}{4a^2}\int_0^a (3a^2x - 2x^3)dx = \tfrac{1}{4}\pi a^2.$$

17. If a circular arc of radius a and central angle $2\alpha < \pi$ revolves about its chord, the volume and surface of the spindle generated are respectively

$$2\pi a^3(\tfrac{2}{3}\sin\alpha + \tfrac{1}{3}\sin\alpha\cos^2\alpha - \alpha\cos\alpha), \quad 4\pi a^2(\sin\alpha - \alpha\cos\alpha).$$

18. The surface generated by $x^4 + 3 = 6xy$ turning about Oy, from $x = 1$ to $x = 2$, has for area $\pi(\frac{15}{4} + \log 2)$, and $\frac{47}{16}\pi$ when turned about Ox.

19. The surface generated by $y^2 + 4x = 2 \log y$, rotating about Ox, from $y = 1$ to $y = 2$, is $\frac{10}{3}\pi$.

20. The area of the surface of revolution of

$$2y = x\sqrt{x^2 - 1} + \log\left(x - \sqrt{x^2 - 1}\right),$$

about Oy, from $x = 2$ to $x = 5$, is 78π.

21. The surface of the cycloid of revolution is $\frac{64}{3}\pi a^2$, and its volume is $5\pi^2a^3$, the base being the axis of revolution.

22. When the tangent at the vertex is the axis of revolution, in Ex. 21, the surface and volume are $\frac{32}{3}\pi a^2$ and π^2a^3.

23. When, in Ex. 21, the normal at the vertex is the axis of revolution the surface and volume are respectively

$$8\pi a^2(\pi - \tfrac{4}{3}), \quad \pi a^3(\tfrac{3}{2}\pi^2 - \tfrac{8}{3}).$$

24. Show that when the *lemniscate* $\rho^2 = a^2 \cos 2\theta$ is revolved about the polar axis, the surface generated is

$$4\pi a^2 \int_0^{\frac{1}{4}\pi} \sin\theta\, d\theta = 2\pi a^2(2 - \sqrt{2}).$$

25. Show that if the curve $y^2 = ax^2 + bx + c$ be revolved about Ox, the volume generated between x_1, x_2 is

$$V_x = \frac{\pi}{6}(x_2 - x_1)(y_1^2 + y_2^2 + 4y_m^2),$$

where y_m is the ordinate at $\frac{1}{2}(x_1 + x_2)$.

This curve can be made any conic whose axis coincides with Ox, by properly assigning the numbers a, b, c. The result then gives the volume of any conicoid of revolution around one axis of the generating curve.

26. Show that the volume of the *egg* generated by

$$x^2y^2 = (x - a)(b - x),$$

revolving about Ox as an axis, is

$$\pi\left\{(a + b)\log\frac{b}{a} - 2(b - a)\right\}.$$

27. The volume of the *heart-shaped* solid generated by revolving $\rho = a(1 + \cos\theta)$ about the initial line is $\frac{8}{3}\pi a^3$.

28. Find the volume of the *hour-glass* generated by revolving the curve $y^4 - 2c^2y^2 + a^2x^2 = 0$ about Oy.

CHAPTER XXIII.

ON THE VOLUMES OF SOLIDS.

178. We have seen that the volume of a revolute is generated by a circular section moving with its center on a straight line and its plane always perpendicular to that straight line. If H is the distance between any two circular sections A_1 and A_2, and A the area of the circular section at a distance h from A_1, then the volume included between the sections A_1 and A_2 is

$$V = \int_0^H A\,dh. \qquad \S\,174,\ (3).$$

We propose to generalize this and to show that this same formula gives the volume of any solid included between two parallel planes whenever the area A of a section of the solid by a plane parallel to the two given planes can be determined as a continuous function of its distance from one of them.

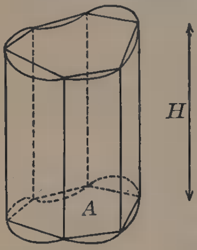


FIG. 102.

In the first place, we observe that if the plane of any plane curve of invariable shape moves in such a manner that the plane of the curve remains parallel to a fixed plane and the curve generates the surface of a cylinder, then the volume of the solid generated is equal to the area of the generating curve multiplied by the altitude of the cylinder generated. For we can always inscribe in the curve a polygon of n sides which will generate a prism as the curve moves in the manner described. If P is the area of the polygon and H its altitude, then PH is the volume of the prism. When $n = \infty$ and each side of the polygon converges to 0, the area of the polygon converges to A, the area of the curve, and the prism and cylinder have the same altitude H. The volume of the cylinder is the limit of the volume of the prism and is therefore AH.

179. Volume of a Solid.—Consider any solid bounded by a surface. Select a point O and draw a straight line Ox in a fixed direction. Cut the solid by two planes perpendicular to Ox at points X_1, X_2 distant X_1 and X_2 from O.

Whenever the area A of the section PM of the solid by any plane PM perpendicular to Ox, distant x from O, is a continuous function of x, then the volume of the solid included between the parallel planes at X_1 and X_2 is

$$V = \int_{X_1}^{X_2} A\,dx.$$

To prove this, divide the interval between X_1 and X_2 into a large number of parts, n. Draw planes through the points of division perpendicular to Ox, thus dividing the solid into n thin slices, of which MPP_1M_1 is a type. Let A be the area of the section PM, and A_1 that of section P_1M_1 at a distance x_1 from O. Let $\varDelta V$ be the volume of the element of the solid included between the sections at x and x_1, and $x_1 - x = \varDelta x$ the perpendicular distance between the sections.

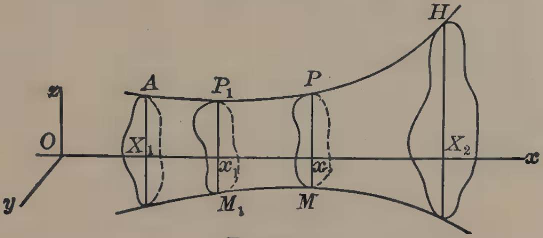


FIG. 103.

We can always take $\varDelta x$ so small that we can move a straight line, always parallel to Ox, around the inside of the ring cut out of the surface by the planes at x and x_1 in such a manner as to always touch this part of the surface and not cut it, and thus cut out of the element of the solid a cylinder whose volume is less than $\varDelta V$. Let the area of the curve traced by this line on the plane PM be A'. Then the volume of this cylinder is

$$\delta V' = A'\varDelta x.$$

In like manner, we can move a straight line parallel to Ox around the ring externally, always touching and not cutting it. Thus cutting out between the planes of the sections at x and x_1 a cylinder of which the element of volume of the solid is a part. Let this straight line trace in the plane PM a curve whose area is A''. The volume of this external cylinder is $A''\varDelta x$.

Hence we have

$$A'\varDelta x < \varDelta V < A''\varDelta x,$$

or

$$A' < \frac{\varDelta V}{\varDelta x} < A''.$$

Also, necessarily, from the manner of construction of the lines bounding the areas A' and A'',

$$A' < A < A''.$$

If now the surface of the solid is such that the boundary of the section P_1M_1 at x_1 converges to the boundary of the section PM at x, when $x_1 (=) x$, then also $A' (=) A$, $A'' (=) A$, and we have

$$\frac{dV}{dx} = A.$$

Therefore

$$V = \int_{X_1}^{X} dV = \int_{X_1}^{X} \frac{dV}{dx}\,dx = \int_{X_1}^{X} A\,dx.$$

When A is determined as a function of x, say $A = \phi(x)$, then the evaluation of V is a matter of integration, and we have

$$V = \int_{X_1}^{X} \phi(x)\,dx.$$

EXERCISES.

1. If the parallel plane sections of any solid have equal areas, then

$$V = \int_{X_1}^{X_2} A\,dx = (X_2 - X_1)A.$$

Therefore, if a plane figure moves in any manner without changing its area or the direction of its plane, the volume generated is equal to that of a cylinder or prism whose base is equal in area to that of the generating figure and whose altitude is equal to the distance between the initial and terminal positions of the generating plane.

2. The general definition of a cone is as follows:

A straight line which passes through a fixed point and moves according to any law generates a surface called a *cone*. In general, the cone is defined by a straight-line generator passing through a fixed point, the *vertex*, and always intersecting a given curve, called the *directrix*.

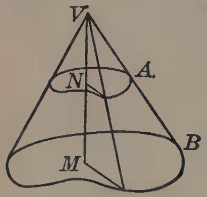


FIG. 104.

A cone is generated by a straight line passing through a fixed point V, and always intersecting a closed plane curve of area B. Find its volume.

Draw a perpendicular $VM = H$ to the plane of the curve. Draw a plane parallel to B cutting the surface in a curve of area A, at a distance $VN = h$ from V. Then we shall have

$$\frac{A}{B} = \frac{h^2}{H^2}.$$

For, inscribe any polygon in the curve B and join the corners to V. The edges of the pyramid thus formed intersect the parallel plane containing A in the corners of a similar polygon inscribed in section A. If P and p are the areas of these polygons, we have

$$\frac{p}{P} = \frac{h^2}{H^2},$$

from elementary geometry. But A and B are the respective limits of p and P. The volume of the cone is then

$$V = \int_0^H A\,dh = \int_0^H B\frac{h^2}{H^2}\,dh = \tfrac{1}{3}BH.$$

3. A *conoid* is the surface generated by a straight line moving in such a manner as to always intersect a fixed straight line and remain parallel to a fixed plane. If the generating line is always perpendicular to the fixed straight-line director and traces a curve in a plane parallel to the directing straight line, the conoid is said to be a *right conoid*, and the curve is called its base.

Find the volume of a right conoid having a closed plane curve of area B for its base.

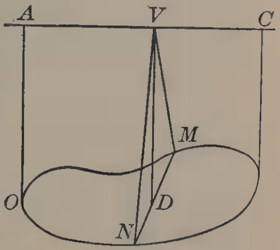


FIG. 105.

Let AVC be the straight-line director at a distance $VD = H$ from the plane of the base.

Any plane VNM perpendicular to VC cuts out of the surface a triangle of constant altitude H, and base $MN = y$. This triangle moving parallel to itself generates the volume required. Hence

$$V = \int_0^{AC} T\,dx = \int_0^{AC} \tfrac{1}{2}Hy\,dx,$$

where $T =$ area MVN, $x = OD = AV$.

But $\int y\,dx = B$, the area of the base. Therefore

$$V = \tfrac{1}{2}HB.$$

The volume of the conoid is therefore half that of a cylinder on the base B having the same altitude H.

This is at once geometrically evident by constructing the rectangle on MN as base with altitude H.

4. On the ordinate of any plane curve, of area B, as base a vertical triangle is drawn with constant altitude H. Show that whatever be the curve traced by the vertex V in the plane parallel to the base, as the ordinate generates the area of the base, the triangle generates a volume $\frac{1}{2}HB$.

5. A rectangle moves parallel to a fixed plane. One side varies directly as the distance, the other as the square of the distance of the rectangle from the fixed plane.

If the rectangle has the area A when at distance H, show that the volume generated is $\frac{1}{4}AH$.

6. The axes of two equal cylinders of revolution intersect at right angles. The solid common to them both is called a *groin*. Find its volume.

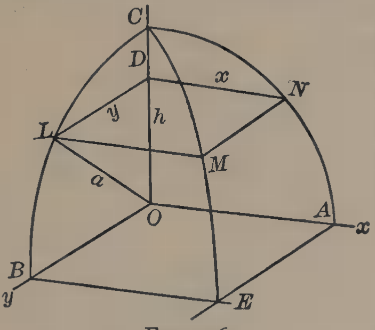


FIG. 106.

Let Ox and Oy be the axes of the two cylinders at right angles. The quarter-circles OAC and OBC are one fourth of their bases. The plane xOy cuts the surfaces of the cylinders in the straight lines AE and BE. The surfaces intersect in CME. A plane $DLMN$ parallel to xOy cuts the cylinders and the vertical planes xOC, yOC in a square, which moving parallel to xOy generates one eighth of the groin. Let x be a side of this square, whose distance from O is h. Then $x^2 = a^2 - h^2$. Hence

$$\tfrac{1}{8}V = \int_0^a (a^2 - h^2)dh = \tfrac{2}{3}a^3.$$

The volume of the groin is $\frac{16}{3}a^3$, where a is the radius of the cylinders.

Knowing that any figure drawn on a cylinder rolls out into a plane figure, show that the entire surface of the groin is $16a^2$.

7. Ox, Oy, Oz are three straight lines mutually at right angles to each other. A cylinder cuts the plane xOy in an ellipse of semi-axes $OA = a$, $OB = b$; and the plane xOz in an ellipse with semi-axes $OA = a$, $OC = c$. The generating lines of the cylinder are parallel to BC. Show that the volume of the cylinder bounded by the three planes xOy, yOz, zOx is $\frac{1}{6}abc$.

8. A right cylinder stands on a horizontal plane with circular base. Show that the volume cut off by a plane through a diameter of the base and making an angle α with the plane of the base is $\frac{2}{3}a^3 \tan \alpha$.

9. On the double ordinates of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, and in planes perpendicular to that of the ellipse, isosceles triangles of vertical angle 2α are constructed. Show that the volume of the solid generated by the triangle is $\frac{4}{3}ab^2 \cot \alpha$.

10. Two wedge-shaped solids are cut from a right circular cylinder of radius a and altitude h, by passing two planes through a diameter of one base and touching the other base. Show that the remaining volume is $(\pi - \frac{4}{3})a^2h$.

11. Two cylinders of equal altitude h have a circle of radius a for their common base; their other bases are tangent to each other. Show that the volume common to the cylinders is $\frac{4}{3}a^2h$.

12. A cylinder passes through two great circles of a sphere which are at right angles. The volume common to the solids is $(1+\frac{1}{2}\pi)/\pi$ times that of the sphere.

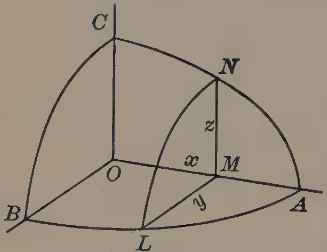


FIG. 107.

13. Two ellipses have a common axis and their planes are at right angles. Find the volume of the solid generated by a third ellipse which moves with its center on the common axis, its plane perpendicular to that axis, and its vertices on the other two curves.

Let AOC and AOB represent quadrants of the given ellipses.

$$OA = a, \quad OB = b, \quad OC = c.$$

Then LMN represents a quadrant of the moving ellipse, having z and y as semi-axes. Let $x = OM$ be the distance of the plane LMN from O. The area of the moving ellipse is πyz.

Also, $c^2x^2 + a^2z^2 = a^2c^2$ and $b^2x^2 + a^2y^2 = a^2b^2$.

Hence we have for the volume

$$V = \int_{-a}^{+a} (\pi yz)\, dx = \tfrac{4}{3}\pi abc.$$

The surface is called the *ellipsoid* with three unequal axes.

14. Two parabolæ have a common axis and vertex. Their planes are at right angles. Find the volume generated by an ellipse which moves with its center on the common axis, its plane perpendicular to that axis, and its vertices on the parabolæ.

Let OM and ON be the two parabolæ whose equations referred to AOL, BOL as axes are $x^2 = 2a^2z$ and $y^2 = 2b^2z$.

MLN is the position of a quadrant of the generating ellipse at a distance $z = OL$ from O. The area of the ellipse is πxy. The volume generated from $z = 0$ to $z = c$ is

$$V = \int_0^c (\pi xy)dz = \pi abc^2.$$

The surface generated is called the *elliptic paraboloid.*

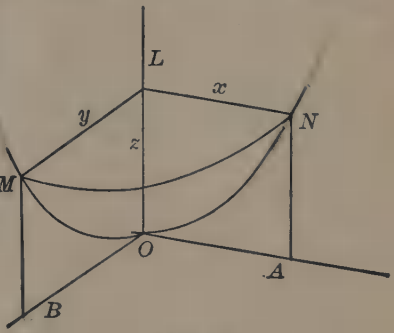


FIG. 108.

15. Volume of the *hyperbolatoid*.

Given two parallel planes at a distance apart H. The solid cut out between the planes by a straight line intersecting them and moving in such a manner as to return to its initial position is called the *hyperbolatoid*.

If in one of the planes a fixed point P be taken, then a straight line through P, moving always parallel to the line generating the curved surface of the hyperbolatoid, cuts out a cone between the planes, called the director cone of the hyperbolatoid. Show that the volume of the hyperbolatoid between the parallel planes is equal to

$$V = H\left(\frac{B_1 + B_2}{2} - \frac{C}{6}\right),$$

where B_1, B_2 are the areas of the sections of the solid by the parallel planes distant apart H, and C is the area of the base of the director cone.

Hint. Any plane parallel to the given planes cuts the generating line in segments that are in constant ratio. Therefore the area B of any such section is

$$B = k_1B_2 + k_2B_1 - k_1k_2C$$

(projecting on a plane parallel to the bases), by Elliott's theorem, § 164, (3).

k_1 and k_2 can be expressed in terms of h, the distance of the section B from either base B_1 or B_2. Then

$$V = \int_0^H B\,dh,$$

where B is a quadratic function of h, and the result follows directly.

Since B is a quadratic function of h, the results of Exercises 10, 11, 16, Chapter XX, apply also to the hyperbolatoid, when ordinates are read sectional areas.

An important general case is: If the generating straight line moves in such a manner as to remain always parallel to a fixed plane, then $C = 0$ and

$$V = \tfrac{1}{2}H(B_1 + B_2).$$

16. Find the section of minimum area in a given hyperbolatoid, and show that sections equidistant from the least section have equal areas.

17. On the double ordinate of $x^2 + y^2 = a^2$, as a central diagonal, is constructed a regular polygon of n (even) sides, whose plane is perpendicular to that of the circle. Show that the volume generated by the polygon is

$$\tfrac{4}{3}\pi a^3 \frac{\sin\dfrac{2\pi}{n}}{\dfrac{2\pi}{n}},$$

and therefore the volume of the sphere is $\frac{4}{3}\pi a^3$.

18. Show that the hyperbolic paraboloid passing through any skew quadrilateral divides the tetrahedron having for vertices the corners of the quadrilateral into two parts of equal volume.

19. On a sphere of radius R draw two circles whose planes are parallel and distant $R/\sqrt{3}$ from the center of the sphere. Draw tangent planes to the sphere at the ends of the diameter perpendicular to the planes of the circles.

Show that any ruled surface passing through the circles cuts out a solid between the tangent planes whose volume is equal to that of the sphere.

BOOK II.

FUNCTIONS OF MORE THAN ONE VARIABLE.

PART V.

PRINCIPLES AND THEORY OF DIFFERENTIATION.

CHAPTER XXIV.

THE FUNCTION OF TWO VARIABLES.

180. Definition.—When there is a variable z related to two other variables x and y in such a manner that corresponding to each pair of values of x, y there is a determinate value of z, then z is said to be a function of the variables x and y.

We represent functions of two variables x, y by the symbols $f(x, y)$, $\phi(x, y)$, etc., in the same sense that we employed the corresponding symbols $f(x)$, $\phi(x)$, etc., to represent functions of one variable x.

When it is so well understood that we are considering a function $f(x, y)$ of the two variables x and y that it is unnecessary to place the variables in evidence, we frequently omit the variables and the parenthesis and represent the function by the abbreviated symbol f. In like manner we frequently consider the single letter z as representing a function of the variables x and y, and write

$$z \equiv f(x, y).$$

181. Geometrical Representation.—Let z be a function of two variables x and y. Let the value c of z correspond to the values a of x and b of y. Through a point O in space draw three straight lines Ox, Oy, Oz mutually at right angles, in such a manner that Oz is vertical as in the figure. We then have a system of three planes xOy, yOz, zOx mutually at right angles, of which xOy is horizontal. These planes divide space into eight octants. The plane xOy we take as the plane of the variables x and y, in which we represent any pair of values of the variables x and y by a point having these values as coordinates referred respectively to Ox, Oy as axes, as in plane analytical geometry.

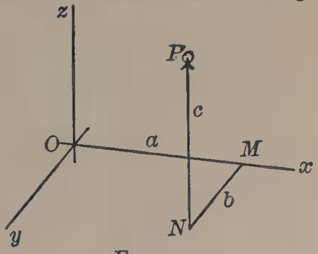


FIG. 109.

We take, as in the figure, Ox drawn to the right as positive, drawn to the left as negative; Oy drawn in front of the xOz plane as positive, drawn behind that plane as negative; Oz drawn upward above the horizontal plane as positive, drawn downward as negative.

To represent the value $z = c$ of the function corresponding to the values $x = a$, $y = b$ of the variables: Construct the point N in the plane, xOy, of the variables, having for its coordinates $OM = a$, $MN = b$. The value c of the function z can then always be represented by a point P, which is constructed by drawing a perpendicular NP to the plane of the variables at N, such that $NP = c$ is drawn upwards or downwards according as c is positive or negative.

The representation is nothing more than the Cartesian system of coordinates in analytical geometry. The numbers a, b, c, or in general x, y, z, are the coordinates of the point P with respect to the orthogonal coordinate planes xOy, yOz, zOx.

We can then always represent any determinate function $f(x, y)$ of two variables by a point in space whose distance from a plane is the value of the function.

182. Function of Independent Variables.—Let $z \equiv f(x, y)$ be a function of the two variables x and y. When there is no connection whatever between x and y, then z is said to be a function of the two *independent* variables x and y.

This means that, within the limits for which z is a function of x and y, whatever be the *arbitrarily* assigned values of x and y there corresponds a value of z.

Geometrical Illustration.

Consider the function of two independent variables

$$+\sqrt{a^2 - x^2 - y^2}.$$

This function has no real existence for values of x and y such that $x^2 + y^2 > a^2$. Also, for $x^2 + y^2 = a^2$ the function is 0, while for any arbitrarily assigned values of x and y whatever, such that $x^2 + y^2 < a^2$. the function has a unique determinate positive value. Geometrically speaking, the function exists for any point on or inside the circumference of the circle $x^2 + y^2 = a^2$ in the plane xOy, and the point representing the function for any such assigned pair of values of x, y, is a point on the surface of the hemisphere

$$x^2 + y^2 + z^2 = a^2$$

which lies above xOy. The circle $x^2 + y^2 = a^2$ is called the boundary of the region of the variables for which the function

$$z = +\sqrt{a^2 - x^2 - y^2}$$

is defined, or exists in real numbers.

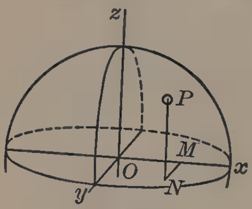


Fig. 110.

In general, a function $z = f(x, y)$ of two independent variables is represented by the ordinate to a surface of which $z = f(x, y)$ is the equation in Cartesian orthogonal coordinates. The study of a function of *two* independent variables corresponds, therefore, to the study of surfaces in geometry, in the same sense that the study of a function of *one* variable corresponds to the study of plane curves as exhibited in Book I.

183. Function of Dependent Variables.—Let $z \equiv f(x, y)$ be a function of two *independent* variables x and y. Since x and y are independent of each other, we can assign to them any values we choose in the region for which z is a defined function of x and y.

I. In particular, we can hold y fixed and let x alone vary. In which case z is a function of the single variable x. For example, let $y = b$ be constant, then

$$z = f(x, b) \tag{1}$$

is a function of the single variable x. If $z = f(x, y)$ be represented by a surface, then equation (1), which is nothing more than the two simultaneous equations

$$\left.\begin{aligned} z &= f(x, y), \\ y &= b, \end{aligned}\right\} \tag{2}$$

is represented by a curve AB in a plane $x'O'z'$, parallel to and at a distance b from the coordinate plane xOz. Or, is the curve of intersection of the surface $z = f(x, y)$ and the vertical plane $y = b$, as exhibited by the simultaneous equations (2). The equation $z = f(x, b)$ of this curve is referred to axes $O'x'$, $O'z'$ of x and z respectively, in its plane $x'Oz'$.

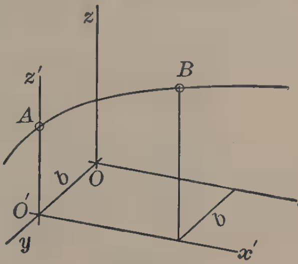


FIG. 111.

II. In like manner, if we make x remain constant, say $x = a$, and let y vary, then $z = f(x, y)$ becomes

$$z = f(a, y), \tag{3}$$

a function of y only, and is represented by a curve AB in a plane $y'O'z'$, Fig. 112, parallel to and at a distance a from the coordinate plane yOz. Or, it is the curve of intersection of the surface $z = f(x, y)$ and the plane $x = a$, whose equations are

$$\left.\begin{aligned} z &= f(x, y), \\ x &= a. \end{aligned}\right\} \tag{4}$$

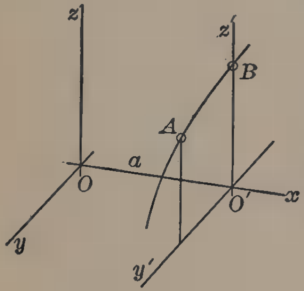


FIG. 112.

III. Again, since x and y are independent, we can assign any relation we choose between them. For example, instead of making, as in I, II, x and y take the values of coordinates of points on the line $x = a$ or $y = b$ in xOy, we can make them take the values of coordinates of points on the straight line

$$\frac{x - a}{\cos \alpha} = \frac{y - b}{\sin \alpha} = r, \tag{5}$$

which is a straight line through the point a, b in xOy and making an angle α with the axis Ox.

Substituting

$$x = a + r \cos \alpha, \quad y = b + r \sin \alpha$$

in $z = f(x, y)$ for x and y respectively, and observing that r is the distance of x, y from a, b measured on the line (5), we have

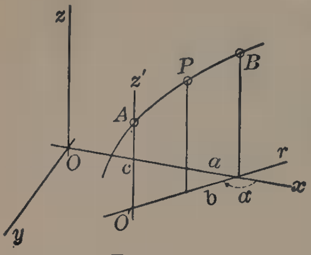


FIG. 113.

$$z = f(a + r\cos\alpha,\ b + r\sin\alpha). \qquad (6)$$

If α is constant, (6) is a function of the single variable r, and is the equation of a curve APB cut out of the surface $z = f(x, y)$ by a vertical plane through (5), and the curve has for its equations

$$\left.\begin{aligned} z &= f(x, y), \\ \frac{x-a}{\cos\alpha} &= \frac{y-b}{\sin\alpha}. \end{aligned}\right\} \qquad (7)$$

The curve (6) is referred in its own plane, $rO'z'$, to $O'r$, $O'z'$ as coordinate axis. The coordinates of any point P on the curve being r, z.

IV. In general, x and y being independent, we can assume any relation between them we choose.

For example, we may require the point x, y in xOy to lie on the curve

$$\phi(x, y) = 0.$$

Then, as in III, $z = f(x, y)$ is a function of the *dependent* variables x and y which are connected by the functional relation $\phi(x, y) = 0$. The geometrical meaning of this is: The point P representing the function z must lie in the vertical through the point P' representing x, y on the curve $\phi(x, y) = 0$. Or, the function z of the *dependent* variables x, y is represented by the ordinate to a curve in space drawn on the vertical cylinder which has the curve $A'P'$ for its base. The curve $A'P'$, whose equation in yOx is $\phi(x, y) = 0$, is the horizontal projection of the curve in space AP representing the function.

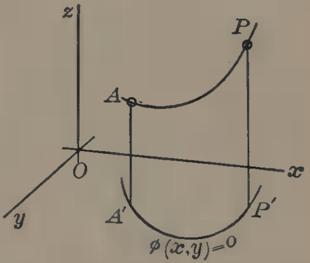


FIG. 114.

Geometrically speaking, the function $z = f(x, y)$ of two *dependent* variables x and y, connected by the relation $\phi(x, y) = 0$, is represented by the space curve which is the intersection of the surface $z = f(x, y)$ and the vertical cylinder $\phi(x, y) = 0$, whose equations are

$$\left.\begin{aligned} z &= f(x, y), \\ 0 &= \phi(x, y). \end{aligned}\right\} \qquad (8)$$

If we solve $\phi(x, y) = 0$ for y and get $y = \psi(x)$, then substituting for y in $f(x, y)$, we express z as a function of x only, thus :

$$z = f[x, \psi(x)]. \tag{9}$$

This equation (9) is the equation of the projection of the space curve AP (8) on the plane zOx.

In like manner we can express z as a function of y only, and get the equation of the orthogonal projection of (8) on the plane yOz.

184. The Implicit Function.—We saw in Book I how the functional dependence of one variable on another was expressed by the implicit functional relation, or equation in two variables,

$$f(x, y) = 0,$$

and that this implied or defined either variable as a function of the other. We also saw that this functional relation could be represented by a plane curve having x and y as coordinates of its points. The implicit function of two variables is a particular case of a function of two independent variables. For, in such a function,

$$z = f(x, y),$$

of the two independent variables x and y, if we make z constant, say $z = c$, we have the implicit function in two variables

$$f(x, y) = c. \tag{1}$$

Geometrically, this is nothing more than the equation to the curve LMN, Fig. 115, cut out of the surface $z = f(x, y)$ by the horizontal plane $z = c$, at a distance c from xOy.

Its equations are

$$\left.\begin{array}{l} z = f(x, y), \\ z = c. \end{array}\right\} \tag{2}$$

The lines cut on a surface by a series of horizontal planes are called the *contour* lines of the surface. In particular, if $z = 0$, then $f(x, y) = 0$ is the equation in the xOy plane of the horizontal trace of the surface $z = f(x, y)$, or the curve ABC cut in the horizontal plane by the surface.

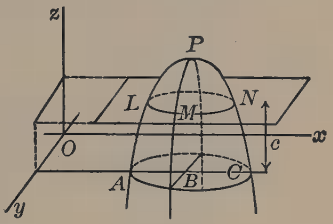


FIG. 115.

In the same way that the implicit equation in two variables defines either variable as a function of the other, the implicit function

$$f(x, y, z) = 0$$

is an equation defining either of the three variables as a function of the other two as independent variables, and can be represented by a surface in space having x, y, z as the coordinates of its points.

185. Observations on Functions of Several Variables.—The general method of investigating a function of two independent variables is to make one of the variables constant and then study the function as a function of one variable. Geometrically, this amounts to studying the surface represented by investigating the curve cut from the surface by a vertical plane parallel to one of the coordinate planes.

Or, more generally, to impose a linear relation between the variables x and y, and thus reduce the function to a function of one variable, as in § 183, III, which can be investigated by the methods of Book I. Geometrically, this amounts to cutting the surface by any vertical plane and studying the curve of section.

As we have seen in § 184, and as we shall see further presently, the study of functions of two variables is facilitated by reducing them to functions of one variable, and reciprocally we shall find that the study of functions of two or more variables throws much light on the study of functions of one variable.

186. Continuity of a Function of Two Independent Variables. Definition.—The function $z = f(x, y)$ is said to be continuous at any pair of values x, y of the variables when corresponding to x, y we have $f(x, y)$ determinate and

$$\pounds f(x_1, y_1) = f(x, y),$$

for $x_1 (=) x$, $y_1 (=) y$, *independent* of the manner in which x_1 and y_1 are made to converge to their respective limits x and y.

The definition also asserts that

$$\pounds\, [f(x_1, y_1) - f(x, y)] = 0,$$

for $x_1 (=) x$, $y_1 (=) y$.

In words: The function $z = f(x, y)$ is continuous at x, y whenever the number $z_1 \equiv f(x_1, y_1)$ converges to z as a limit, when the variables x_1, y_1 converge simultaneously to the respective limits x, y in an arbitrary manner.

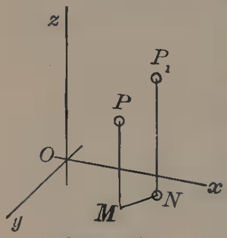


FIG. 116.

Geometrically interpreted, the point P_1, representing x_1, y_1, z_1, must converge to P, representing x, y, z, as a limit, at the same time that the point N, representing x_1, y_1, converges to M, representing x, y; whatever be the path which N is made to trace in xOy as it converges to its limit M.

A function $f(x, y)$ is said to be continuous in a certain region A in the plane xOy when it is continuous at every point x, y in the region A.

An important corollary to the definition of continuity of $f(x, y)$ at x, y is this: Whatever be the value of $f(x, y)$ different from 0, we can always take x_1, y_1 so near their respective limits x, y that we shall have $f(x_1, y_1)$ of the same sign as $f(x, y)$.

187. The Functional Neighborhood.—A consequence of the definition of continuity of $z = f(x, y)$ is as follows:

If $f(x, y)$ is continuous in a certain region containing a, b, we can always assign an absolute number ϵ so small that corresponding to ϵ there are two assigned absolute numbers h and k, such that for all values of x and y for which

$$|x - a| < h, \quad |y - b| < k,$$

we have

$$|f(x, y) - f(a, b)| < \epsilon.$$

The proof of this is the same as that given for a function of one variable. For, let y and a be fixed numbers, and let x vary. Then whatever number $\frac{1}{2}\epsilon$ be assigned, we can always assign a corresponding number $h \neq 0$, such that for $|x - a| < h$ we have

$$|f(x, y) - f(a, y)| < \tfrac{1}{2}\epsilon,$$

since $f(x, y)$ is a continuous function of one variable x, and its limit is $f(a, y)$.

In like manner for $|y - b| < k$ we have

$$|f(a, y) - f(a, b)| < \tfrac{1}{2}\epsilon,$$

and on addition

$$|f(x, y) - f(a, b)| < \epsilon$$

for all values of x, y, such that

$$|x - a| < h, \quad |y - b| < k.$$

Geometrically speaking, whatever be the value $c = f(a, b)$, we can always assign an arbitrarily small number ϵ, corresponding to which there is a rectangle $KLMN$ in the plane xOy, the coordinates of whose corners are K, $(a - h, b + k)$; L, $(a + h, b + k)$; M, $(a + h, b - k)$; N, $(a - h, b - k)$, such that, whatever be the point x, y in the rectangle $KLMN$, the corresponding point x, y, z on the surface $z = f(x, y)$ lies between the parallel planes $z = c - \epsilon$, $(STUV)$ and $z = c + \epsilon$, $(WXYZ)$. The point P representing a, b, c.

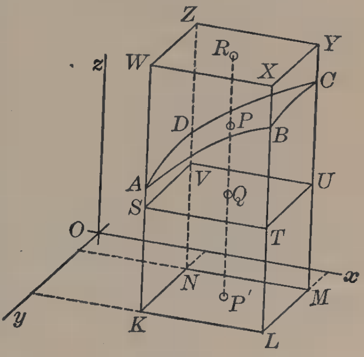


FIG. 117.

Such a region $KLMN$ is called the *neighborhood* of the point a, b. The point is called its center. In like manner the corresponding parallelopiped $STUV$-$WXYZ$ is called the *neighborhood* of the point P in space.

The above results may be stated thus: When the variables x, y are in the neighborhood of a, b, then must the continuous function $f(x, y)$ be in the neighborhood of $f(a, b)$.

An important consequence is this: If $f(x, y)$ is continuous in the neighborhood of $f(a, b) \neq 0$, then we can always assign a neighborhood of a, b such that for all values of x, y in this neighborhood the value $f(x, y)$ of the function has the same sign as $f(a, b)$.

EXERCISES.

1. Trace the surface representing the function

$$f(x, y) \equiv y - mx + b.$$

Put $z = y - mx + b$. When $z = 0$, the surface cuts xOy in the straight line $y = mx - b$. If $x = a$, we have for the section of the surface by the plane $x = a$ the straight line

$$z = y - ma + b.$$

Whatever be a, this line is sloped 45° to the plane xOy. As $x = a$ varies, this line moves parallel to itself, intersecting the fixed line $y = mx - b$ in xOy, and therefore generates a plane.

In like manner it can be shown that the implicit function of the first degree in x, y, z,

$$f(x, y, z) \equiv Ax + By + Cz + D = 0,$$

is always represented by a plane.

2. Show that the function

$$\sqrt{a^2 - x^2 - y^2}$$

can be represented by a sphere, by showing that it can be generated by a circle whose diameters are the parallel chords of a fixed circle, and whose planes are perpendicular to that of the fixed circle.

3. Trace the surfaces representing the implicit functions

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \pm \frac{z^2}{c^2} - 1 = 0, \quad \frac{x^2}{a} \pm \frac{y^2}{b} - 2z = 0$$

by their plane sections.

4. Trace by sections the surface representing

$$(x^2 - az)^2(a^2 - x^2) - x^4y^2 = 0.$$

5. Find the maximum value of the function

$$f(x, y) \equiv 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

when the variables are subject to the condition $x + y = 1$.

Let $z = f(x, y)$. Then z is immediately reduced to a function of one variable by substituting $1 - x$ for y.

$$\therefore \quad z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

$$\therefore \quad \frac{dz}{dx} = -\frac{2x}{a^2} + 2\,\frac{(1 - x)}{b^2} = 0$$

gives $x = a^2/(a^2 + b^2)$, $y = b^2/(a^2 + b^2)$, $z = 1 - 1/(a^2 + b^2)$, which is a maximum value of z since $D_x^2 z$ is negative.

Consider the geometrical aspect of this problem. We have

$$z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}, \tag{1}$$

the equation of the elliptic paraboloid whose vertex is $0, 0, 1$, and which cuts xOy in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

We wish the highest point on the curve cut out of the surface by the plane $x + y = 1$. Take $O'r$, the horizontal trace of this plane, as the positive axis of r, and $O'z'$, its vertical trace on yOz, as axis of z in the plane $rO'z'$. Then for the equation to the curve in the plane $x + y = 1$, or

$$\frac{x - 0}{\sqrt{2}} = \frac{y - 1}{-\sqrt{2}} = r,$$

we substitute $x = r\sqrt{2}$, $y = 1 - r\sqrt{2}$ in (1). Hence the equation to the curve of section in its own plane is

$$z = \left(1 - \frac{1}{b^2}\right) + \frac{2\sqrt{2}}{b^2}r - 2\left(\frac{a^2 + b^2}{a^2b^2}\right)r^2.$$

$D_rz = 0$ gives $r = a^2/(a^2 + b^2)\sqrt{2}$, and $D_z^2r = -$. Hence the values of x, y, z as before.

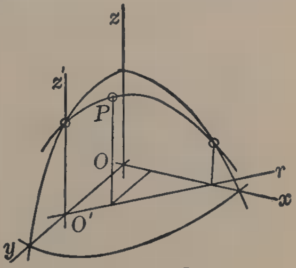


FIG. 118.

The first method, in which we substitute for y in terms of x, is only possible when we can solve the condition to which the variables are subject, with respect to one of them. The second method, in which we express x and y in terms of a third variable, is always possible, although perhaps cumbersome.

The class of problems such as the one proposed and solved here should be carefully considered, for we propose to develop more powerful methods for attacking them. But it should not be forgotten that those methods themselves are developed in the same way as is the solution of this particular problem. The student should accustom himself to seeing curves referred to coordinate systems in other planes than the coordinate planes, for in this way a visual intuition of the meaning of the change of variables, and a concrete conception of the corresponding analytical changes which the functions undergo, is acquired.

CHAPTER XXV.

PARTIAL DIFFERENTIATION OF A FUNCTION OF TWO VARIABLES.

188. On the Differentiation of a Function of Two Variables.—A function of two independent variables has no determinate derivative. It is only when the variables are *dependent* on each other that we can speak of the *derivative* of a function of two variables. The derivative of a function of two variables is indeterminate unless the variable is mentioned with respect to which the differentiation is performed and the law of connectivity of the variables given.

189. The Partial Derivatives of a Function of Two Independent Variables.—Among all the derivatives a function of two variables can have, the simplest and most important are the *partial* derivatives.

Let $z = f(x, y)$ be a function of the two independent variables x and y. The simplest relation we can impose between x and y is to make one of them remain constant while the other varies. We then reduce the function z to a function of one variable, to which we can apply all the methods of Book I for functions of one variable.

For example, let y be constant and x variable. Then $z = f(x, y)$ is a function of x only, and it can be differentiated with respect to x by the ordinary method, and we have

$$D_x z = \mathop{\mathcal{L}}\limits_{x_1 (=) x} \frac{f(x_1, y) - f(x, y)}{x_1 - x},$$

$$= f'_x(x, y).$$

This is called the *partial* derivative of the function z or f with respect to x. To obtain the partial derivative of $f(x, y)$ with respect to x, make y constant and differentiate with respect to x.

Correspondingly, the *partial differential* of $f(x, y)$ with respect to x is the product of the partial derivative with respect to x, $D_x f$, and the differential of x or $x_1 - x \equiv \Delta x$. If we represent the partial differential of f with respect to x by $d_x f$, then we have

$$d_x f = f'_x(x, y)\, dx,$$

and the corresponding *partial differential quotient* is

$$\frac{d_x f}{dx} = f'_x(x, y) \equiv D_x z.$$

It is customary to employ the peculiar symbolism designed by

Jacobi for representing the *partial* differential quotient or derivative of $f(x, y)$ with respect to x. Thus the above will hereafter be written (the symbol ∂ is called the *round d*)

$$\frac{\partial f}{\partial x} \equiv \frac{d_x f}{dx}.$$

The symbol ∂ being used instead of d to indicate the partial differential as distinguished from what will presently be defined as the total differential, which will be represented as formerly by d.

In the same way, if we make x *constant*, then $f(x, y)$ becomes a function of one variable y, and has a determinate derivative with respect to y. This derivative we call the partial derivative of $f(x, y)$ with respect to y, which is written and defined to be

$$\frac{\partial f(x, y)}{\partial y} = \underset{y_1(=)y}{\overset{x=\text{const.}}{\mathcal{L}}} \frac{f(x, y_1) - f(x, y)}{y_1 - y}.$$

190. Geometrical Illustration of Partial Derivatives.—If $z = f(x, y)$ is represented by the ordinate to a surface, then at any point $P(x, y, z)$ on the surface draw two planes PMQ and PMR parallel respectively to the coordinate planes xOz and yOz. These planes cut out of the surface the two curves, PK and PJ respectively, passing through P.

$z = f(x, y)$ (y constant)

is the equation of the curve PK in the plane PMQ.

$z = f(x, y)$ (x constant),

is the equation of the curve PJ in the plane PMR.

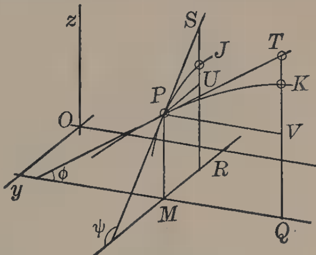


FIG. 119.

Draw the tangents PT and PS to the curves PK and PJ in their respective planes, and let them make angles ϕ and ψ with their horizontal axes, as in plane geometry. Then we have

$$\frac{\partial z}{\partial x} = \tan \phi, \quad \frac{\partial z}{\partial y} = \tan \psi.$$

Therefore the partial derivatives of $f(x, y)$ with respect to x and y are represented by the slopes of the tangent lines to the surface $z = f(x, y)$, at the point x, y, z, to the horizontal plane xOy. These tangents being drawn respectively parallel to the vertical coordinate planes xOz, yOz.

Also, draw PV parallel to MQ, and PU parallel to MR. Then we have

$$VT = (x_1 - x) \tan \phi, \quad US = (y' - y) \tan \psi,$$

if Q is x_1, y, and R is x, y'. Or

$$VT = \frac{\partial f}{\partial x}dx, \quad US = \frac{\partial f}{\partial y}dy,$$

represent the corresponding partial differentials of f with respect to x and y at $P(x, y, z)$.

Thus the partial derivatives and differentials of $f(x, y)$ are interpreted directly through the corresponding interpretations as given for a function of one variable.

191. Successive Partial Derivatives.—If $z = f(x, y)$ is a function of two independent variables x and y, then, in general, its partial derivative with respect to x,

$$\frac{\partial f}{\partial x} = f'_x(x, y),$$

is also a function of x and y as independent variables. This derivative can also be differentiated *partially* with respect to either x or y, as was $f(x, y)$. Thus, differentiating again with respect to x, y being constant, we have the second partial derivative of f with respect to x. In symbols

$$\frac{\partial^2 f(x, y)}{\partial x^2} = f''_{xx}(x, y).$$

In like manner $f'_x(x, y)$ can be differentiated partially with respect to y, x being constant. Thus we have for the second partial differential quotient of f with respect first to x and then to y

$$\frac{\partial^2 f(x, y)}{\partial y\, \partial x} = f''_{xy}(x, y).$$

Similarly, differentiating $f'_y(x, y)$ partially with respect to y, we have

$$\frac{\partial^2 f(x, y)}{\partial y^2} = \frac{\partial f'_y(x, y)}{\partial y} = f''_{yy}(x, y),$$

and with respect to x we have

$$\frac{\partial^2 f(x, y)}{\partial x\, \partial y} = \frac{\partial f'_y(x, y)}{\partial x} = f''_{yx}(x, y).$$

Thus we see that the function $z = f(x, y)$ has two *first* partial derivatives,

$$\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y},$$

and four *second* partial derivatives,

$$\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^2 z}{\partial y\, \partial x}, \quad \frac{\partial^2 z}{\partial x\, \partial y}.$$

Each of these give rise to two partial derivatives of the third order, and generally the function has 2^n partial derivatives of the nth order, of the forms

$$\frac{\partial^n z}{\partial x^p\,\partial y^q},\quad \frac{\partial^n z}{\partial y^p\,\partial x^q},$$

where p and q are any positive integers satisfying $p + q = n$. These nth derivatives, however, are not all different, for we shall demonstrate presently that ∂x^p and ∂y^q in the denominators are interchangeable when the partial derivatives are continuous functions, and that

$$\frac{\partial^n z}{\partial x^p\,\partial y^q} = \frac{\partial^n z}{\partial y^q\,\partial x^p},$$

or the *order* of effecting the partial differentiations is indifferent The number of partial derivatives of $f(x, y)$ of order n is then $n + 1$

EXAMPLES.

1. If $z = x^2 + axy + \cos x \sin y$,

$$\therefore\quad \frac{\partial z}{\partial x} = 2x + ay - \sin x \sin y,$$

$$\frac{\partial z}{\partial y} = ax + \cos x \cos y.$$

2. If $f(x, y) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1$,

$$\therefore\quad \frac{\partial f}{\partial x} = \frac{2x}{a^2},\quad \frac{\partial f}{\partial y} = \frac{2y}{b^2}.$$

3. In Ex. 1, $\dfrac{\partial^2 z}{\partial y\,\partial x} = a - \sin x \cos y = \dfrac{\partial^2 z}{\partial x\,\partial y}$,

$$\frac{\partial^2 z}{\partial x^2} = 2 - \cos x \sin y,\quad \frac{\partial^2 z}{\partial y^2} = -\cos x \sin y.$$

4. In Ex. 2, show that $\dfrac{\partial^2 f}{\partial y\,\partial x} = \dfrac{\partial^2 f}{\partial x\,\partial y}$.

192. Theorem.—The partial derivatives are independent of the order in which the operations are effected with respect to x and y.

In symbols, if $z = f(x, y)$, we have

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}.$$

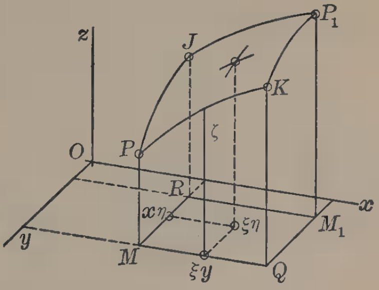


FIG. 120.

Consider the rectangle of the four points

M, (x, y); M_1, (x_1, y_1);
Q, (x_1, y); R, (x, y_1).

The theorem of mean value applied to a function of one variable x gives

$$\frac{\Delta f}{\Delta x} = \frac{f(x_1, y) - f(x, y)}{x_1 - x},$$

$$= f_\xi'(\xi, y), \qquad (1)$$

where ξ is some number between x_1 and x. (See Book I, § 62.)

Form the difference quotient of (1) with respect to y,

$$\frac{\Delta}{\Delta y}\frac{\Delta f}{\Delta x}=\frac{f(x_1, y_1)-f(x_1, y)-f(x, y_1)+f(x, y)}{(y_1-y)(x_1-x)},$$

$$=\frac{f_\xi'(\xi, y_1)-f_\xi'(\xi, y)}{y_1-y}=f_{\xi,\eta}''(\xi, \eta), \tag{2}$$

where η is some number between y_1 and y. The value (2) is therefore equal to the second partial derivative of f, taken first with respect to x, then with respect to y, at a pair of values ξ, η of x, y. Geometrically, at a point ξ, η in the rectangle MQM_1R.

In like manner, taking the difference-quotient of f, first with respect to y, we have

$$\frac{\Delta f}{\Delta y}=\frac{f(x, y_1)-f(x, y)}{y_1-y}=f_{\eta'}'(x, \eta'), \tag{3}$$

where η' is some number between y_1 and y.

Now taking the difference-quotient of (3) with respect to x, we have

$$\frac{\Delta}{\Delta x}\frac{\Delta f}{\Delta y}=\frac{f(x_1, y_1)-f(x, y_1)-f(x_1, y)+f(x, y)}{(x_1-x)(y_1-y)},$$

$$=\frac{f_\eta'(x_1, \eta')-f_\eta'(x, \eta')}{x_1-x}=f_{\eta',\xi'}''(\xi', \eta'), \tag{4}$$

where ξ' lies between x_1 and x, η' between y_1 and y. The value of (4) is then equal to the second partial derivative of f, taken first with respect to y and then with respect to x at some point ξ', η', also inside the rectangle MQM_1R.

But (2) and (4) are identically equal. Hence we have

$$f_{\xi,\eta}''(\xi, \eta)=f_{\eta',\xi'}''(\xi', \eta'),$$

or

$$\frac{\partial^2 f(\xi, \eta)}{\partial\eta\,\partial\xi}=\frac{\partial^2 f(\xi', \eta')}{\partial\xi'\,\partial\eta'}. \tag{5}$$

This relation is true whatever be the values x_1, y_1.

If now the functions

$$\frac{\partial^2 f}{\partial y\,\partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x\,\partial y}$$

are continuous functions of x and y in the neighborhood of x, y, then since ξ', η' and ξ, η converge to the respective limits x, y when $x_1(=)x$, $y_1(=)y$, the two members of (5) converge to a common limit at the same time, and therefore

$$\frac{\partial^2 f}{\partial y\,\partial x}=\frac{\partial^2 f}{\partial x\,\partial y}. \tag{6}$$

Incidentally, equations (2) and (4) show that the difference-quotients

$$\frac{\Delta}{\Delta y}\frac{\Delta f}{\Delta x} \equiv \frac{\Delta^2 f}{\Delta y\, \Delta x}, \quad \frac{\Delta}{\Delta x}\frac{\Delta f}{\Delta y} \equiv \frac{\Delta^2 f}{\Delta x\, \Delta y}$$

converge to a common limit whatever be the manner in which $\Delta x (=) 0$, $\Delta y (=) 0$, and that common limit is

$$\frac{\partial^2 f}{\partial y\, \partial x} \quad \text{or} \quad \frac{\partial^2 f}{\partial x\, \partial y}.$$

Observe that in the symbols

$$\frac{\partial^2}{\partial y\, \partial x} f \equiv f''_{xy}$$

the operations are performed in the order of the proximity of the variable to the function.

In like manner, making use of the result in (6), we have

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x\, \partial y}\right) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial}{\partial y}\frac{\partial^2 f}{\partial x^2}.$$

$$\therefore \quad \frac{\partial^3 f}{\partial x^2\, \partial y} = \frac{\partial^3 f}{\partial y\, \partial x^2},$$

and similarly for other cases. Hence, in general,

$$\frac{\partial^{p+q} f}{\partial x^p\, \partial y^q} = \frac{\partial^{p+q} f}{\partial y^q\, \partial x^p},$$

in whatever order the differentiations be made.

EXERCISES.

1. If $z = \tan^{-1}\frac{x}{y}$, show that

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x} = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

2. If $z = \frac{x^2 y}{a^2 - x^2}$, find $D^2_{xy}\, z$, $D^2_{yx}\, z$.

3. Verify in the following functions the equation

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x}.$$

$x \sin y + y \sin x$,
$x \log y$,
x^y,

$\log \tan (y/x)$,
$(ay - bx)/(by - ax)$,
$y \log (1 + xy)$.

4. If $z = \tan^{-1}\frac{xy}{\sqrt{1 + x^2 + y^2}}$, show that

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{1}{(1 + x^2 + y^2)^{\frac{3}{2}}}; \quad \frac{\partial^4 z}{\partial x^2\, \partial y^2} = \frac{15xy}{(1 + x^2 + y^2)^{\frac{7}{2}}}.$$

5. If $u = x^3y^2 - 2xy^4 + 3x^2y^3$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 5u.$$

6. If $t = \log(e^\alpha + e^\beta)$, then $\dfrac{\partial t}{\partial \alpha} + \dfrac{\partial t}{\partial \beta} = 1$.

7. If $c(e^a + e^b) = e^{ab}$, then $\dfrac{\partial c}{\partial a} + \dfrac{\partial c}{\partial b} = (a + b - 1)c$.

8. If $z = e^x \sin y + e^y \sin x$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{2x} + e^{2y} + 2e^{x+y}\sin(x + y).$$

9. From $z = x^y y^x$, show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = (x + y + \log z)z.$$

10. Show that if γ is the angle between the plane xOy and the tangent plane to the surface $z = f(x, y)$ at x, y, then

$$\tan^2\gamma = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

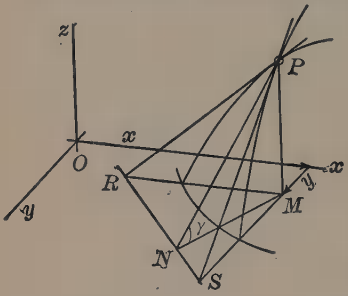


FIG. 121.

Let P, (x, y, z) be on the surface, and PRS the tangent plane.

Draw $MN = p$ perpendicular to RS. Then $\gamma = PNM$.

$$\frac{\partial f}{\partial x} = \frac{PM}{RM}, \quad \frac{\partial f}{\partial y} = \frac{PM}{SM}.$$

Since $RS \cdot NM = RM \cdot MS$,

and $RS^2 = RM^2 + MS^2$,

$\therefore \quad (MN)^{-2} = (RM)^{-2} + (MS)^{-2}$,

and therefore

$$\tan^2\gamma = \frac{PM^2}{MN^2} = \left(\frac{PM}{RM}\right)^2 + \left(\frac{PM}{MS}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

Also,

$$\sec^2\gamma = 1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

11. In Fig. 121, let N be *any* point in the trace of the tangent plane with xOy. Let NM make an angle θ with Ox, and the tangent line NP to the surface make an angle ϕ with the horizontal plane xOy. Then the triangle RSM is the sum of the triangles RMN, NMS, or

$$RM \cdot SM = SM \cdot NM \cos\theta + RM \cdot NM \sin\theta,$$

or

$$\frac{PM}{NM} = \frac{PM}{RM}\cos\theta + \frac{PM}{SM}\sin\theta.$$

Therefore the slope of a tangent line to the surface at x, y, z, whose vertical plane makes the angle θ with zOx, is

$$\tan\phi = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta. \tag{1}$$

12. Find the tangent line to a surface at P which has the steepest slope.

From Ex. 11 we have

$$\frac{d}{d\theta}\tan\phi = -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta = 0.$$

The values of $\sin\theta$, $\cos\theta$ from this equation put in (1), Ex. 11, give for the tangent line of steepest slope

$$\tan^2\phi = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

Observe that this is the slope of the tangent plane in Ex. 10.

13. If $\phi(x, y) = 0$ is the equation of any plane curve, show that

$$\frac{dy}{dx} = -\frac{\dfrac{\partial\phi(x, y)}{\partial x}}{\dfrac{\partial\phi(x, y)}{\partial y}}.$$

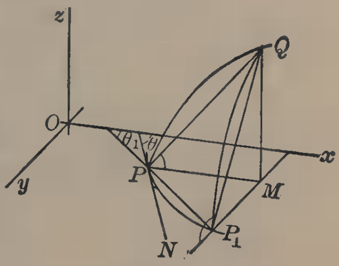


FIG. 122.

Let $z = \phi(x, y)$ be the equation of a surface cutting the horizontal plane in the curve $\phi(x, y) = 0$.

Let P, (x, y) and P_1, (x_1, y_1) be two points on the curve $\phi(x, y) = 0$. Draw the vertical planes through P and P_1 parallel respectively to xOz and yOz, cutting the surface in curves PQ, P_1Q. Then Q is a point x_1, y, z on the surface. The derivative of y with respect to x in $\phi(x, y) = 0$ is the limit of the difference-quotient

$$\frac{y_1 - y}{x_1 - x} = \frac{MP_1}{PM} = \frac{MQ\cot MP_1Q}{MQ\cot MPQ} = \frac{\tan MPQ}{\tan MP_1Q} = -\frac{\tan MPQ}{\tan NP_1Q}.$$

Also, $\quad \tan MPQ = \dfrac{\partial\phi(\xi, y)}{\partial\xi}, \quad \tan NP_1Q = \dfrac{\partial\phi(x_1, \eta)}{\partial\eta},$

ξ being between x and x_1, η between y and y_1 (by the theorem of the mean).

Therefore, when $x_1(=)x$, $y_1(=)y$,

$$\frac{dy}{dx} = \mathop{\mathcal{L}}\frac{y_1 - y}{x_1 - x} = -\frac{\dfrac{\partial\phi}{\partial x}}{\dfrac{\partial\phi}{\partial y}}.$$

This usually saves much labor in computing the derivatives of implicit functions in x and y.

The important results of Exs. 10, 11, 12, and 13 are deduced here geometrically to serve as illustrations of the usefulness of partial differentiation. They will be given rigorous analytical treatment later.

14. Employ the methods of Book I, and also that of Ex. 13, to find $D_x y$ in the following curves:

$$x^2/a^2 - y^2/b^2 - 1 = 0, \qquad x\sin y - y\sin x = 0,$$

$$ax^3y + by^2x - 4xy = 0, \qquad e^x\sin y - \log y\cos x = 0.$$

15. Show that the slope of the tangent at x, y on the conic

$$ax^2 + by^2 + 2hxy + 2ux + 2vy + d = 0$$

is

$$\frac{dy}{dx} = -\frac{ax + hy + u}{hx + by + v}.$$

CHAPTER XXVI.

TOTAL DIFFERENTIATION.

193. In the *partial* differentiation of $f(x, y)$ we made x or y remain constant during the operation, and differentiated the function of the one remaining variable by the ordinary methods of Book I.

We now come to consider the differentiation of $f(x, y)$ when both x and y vary during the operation of evaluating the derivative. Such derivatives are called *total* derivatives.

In order to make clear the nature of the total derivative of a function

$$z = f(x, y),$$

consider the simple case when there is a linear relation between x and y,

$$\frac{x - x'}{l} = \frac{y - y'}{m} = r,$$

where $l = \cos\theta$, $m = \sin\theta$, and the differentiation is performed with respect to r. Let x', y'; l, m; be constant. Then r varies with x and y, and

$$r^2 = (x - x')^2 + (y - y')^2.$$

Also, x and y are linear functions of r, and

$$x = x' + lr, \quad y = y' + mr.$$

Substituting these values of x and y in $f(x, y)$, we reduce that function to a function of the one variable r, and it becomes

$$f(x' + lr, y' + mr). \tag{1}$$

The derivatives of this function with respect to r can now be formed by the methods of Book I. Thus we get by the ordinary process of differentiation

$$\frac{df}{dr}, \quad \frac{d^2f}{dr^2}, \quad \frac{d^3f}{dr^3}, \text{ etc.,}$$

for the successive derivatives of f with respect to r. These are called the *total* derivatives of f with respect to r. Both variables x and y vary with r.

We can give a geometrical interpretation to this total derivative as follows: The equation

$$\frac{x - x'}{l} = \frac{y - y'}{m} = r \tag{2}$$

is the equation of a straight line through x', y' in the horizontal plane xOy, making an angle θ with Ox. r being the distance between the points x', y' and x, y on the line. Let O' be the point x', y'. Draw $O'z'$ vertical. The vertical plane $rO'z'$ through the line (2) cuts the surface representing $z = f(x, y)$ in a curve PP_1, whose equation in its plane, referred to $O'r$ and $O'z'$ as axes of coordinates r and z, is

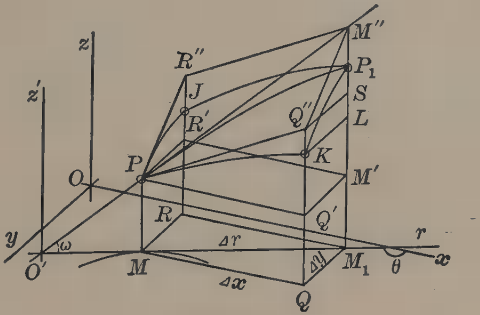


FIG. 123.

$$z = f(x' + lr, y' + mr). \tag{3}$$

Let P_1 be a point on this curve whose coordinates in space are x_1, y_1, z_1 and in $rO'z'$ are r_1, z_1. Let $r_1 - r = \Delta r$. Then, by definition, the derivative of z with respect to r at x, y is the limit of the difference-quotient, when $r_1 (=) r$,

$$\frac{z_1 - z}{r_1 - r} = \frac{f(x_1, y_1) - f(x, y)}{r_1 - r}.$$

Hence we have

$$\frac{dz}{dr} = \tan \omega,$$

where ω is the angle which the tangent PM'' to the curve PP_1 at P, and therefore to the surface, makes with $O'r$, or the horizontal plane xOy.

Observe that as x_1, y_1 converge to x, y, the point M_1 converges to M along the line M_1M.

By assigning different values to θ we can get the slope of any tangent line to the surface, at P, with the horizontal plane.

In particular, when the line (2) is parallel to Ox or Oy, or, what is the same thing, when $\theta = \pi$ or $\frac{1}{2}\pi$, the total derivative becomes a partial derivative, as considered in the preceding chapter.

194. The Total Derivative in Terms of Partial Derivatives.— It is in general tedious to obtain the total derivative, after the manner indicated in § 193, by reducing the function directly to a function of one variable, and generally it is impracticable. We now develop a method of determining the total derivative in terms of the partial derivatives. Let $z = f(x, y)$, where x and y are connected by any relation $\phi(x, y) = 0$. To find the derivative of z with respect to t, where t is any differentiable function of x and y.

Let z take the value z_1, and t become t_1, when x, y become x_1, y_1.

Let y be constant and x_1 be a variable. Then the law of the mean is applicable to the function $f(x_1, y)$ of the one variable x_1, and we have

$$f(x_1, y) - f(x, y) = (x_1 - x)\frac{\partial}{\partial \xi} f(\xi, y), \tag{1}$$

where ξ is some number between x_1 and x.

In like manner, let x be constant and y_1 vary, then, by the law of the mean,

$$f(x_1, y_1) - f(x_1, y) = (y_1 - y)\frac{\partial}{\partial \eta} f(x_1, \eta), \tag{2}$$

where η is some number between y_1 and y.

Adding (1) and (2), we have

$$f(x_1, y_1) - f(x, y) = (x_1 - x)\frac{\partial}{\partial \xi} f(\xi, y) + (y_1 - y)\frac{\partial}{\partial \eta} f(x_1, \eta). \tag{3}$$

Therefore the difference-quotient with respect to t is

$$\frac{z_1 - z}{t_1 - t} = f'_\xi(\xi, y)\frac{\Delta x}{\Delta t} + f'_\eta(x_1 \eta)\frac{\Delta y}{\Delta t}. \tag{4}$$

Δx, Δy, Δz, Δt converge to 0 together, and at the same time $x_1 (=) x$, $\xi (=) x$, $y_1 (=) y$, $\eta (=) y$. Also,

$$\frac{\partial f(\xi, y)}{\partial \xi} \quad \text{and} \quad \frac{\partial f(x_1, \eta)}{\partial \eta}$$

have the respective limits

$$\frac{\partial f(x, y)}{\partial x} \quad \text{and} \quad \frac{\partial f(x, y)}{\partial y}$$

if these latter functions are continuous in the neighborhood of x, y. Passing to the limit in (4), we have for the total derivative of $f(x, y)$ with respect to t, at x, y,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{5}$$

The geometrical interpretation of (1) is this: In Fig. 123 we have M, (x, y); M_1, (x_1, y_1); Q, (x_1, y); R, (x, y_1).

Also,

$$f(x_1, y) - f(x, y) = Q'K = PQ' \tan Q'PK.$$

But, since on the curve PK there must be a point X, (ξ, y, z) at which the tangent is parallel to the chord,

$$\tan Q'PK = \frac{\partial}{\partial \xi} f(\xi, y).$$

In like manner for equation (2),

$$f(x_1, y_1) - f(x_1, y) = LP_1 = -LK \tan LKP_1.$$

But, since there is a point Y, (x_1, η, z) on the curve KP_1 at which the tangent is parallel to the chord, we have

$$-\tan LKP_1 = \frac{\partial}{\partial \eta} f(x_1, \eta).$$

195. The Linear Derivative.—An important particular total derivative is the case considered in § 193. Suppose there is a linear relation between x and y, such as

$$\frac{x-a}{l} = \frac{y-b}{m} = r.$$

Then $x = a + lr$, $y = b + mr$. To find the total derivative of $f(x, y)$ with respect to the variable r, we have

$$\frac{dx}{dr} = l, \quad \frac{dy}{dr} = m.$$

$l = \cos\theta$, $m = \sin\theta$, being constant. Therefore

$$\frac{df}{dr} = l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y}. \tag{1}$$

This is a much simpler way of evaluating this derivative than that proposed in § 193.

As before (see Ex. 11, § 192, § 193),

$$\tan\omega = \frac{df}{dr} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \tag{2}$$

is the slope to the horizontal plane of a tangent line to the surface, in a vertical plane making an angle θ with xOz.

Again, suppose, as in § 194, that x and y are related by $\phi(x, y) = 0$, and we wish the derivative of f with respect to s, the length of the curve $\phi(x, y) = 0$, measured from a fixed point to x, y. Then, putting $t \equiv s$ in (5) § 194,

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}. \tag{3}$$

But $\dfrac{dx}{ds} = \cos\theta$, $\dfrac{dy}{ds} = \sin\theta$, where θ is the angle which the tangent to $\phi(x, y) = 0$ at x, y makes with Ox. Hence we have the same value of the derivative as in (1),

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta, \tag{4}$$

which is also the slope to the horizontal plane of the tangent line to the surface.

196. The Total Differential of $f(x, y)$.—By definition, the differential of a function is the product of the derivative into the differential of the variable. Hence, mutiplying (5), § 194, through by dt, we have for the total differential of f at x, y

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{1}$$

Observe that

$$\frac{\partial f}{\partial x}dx = \partial_x f, \quad \frac{\partial f}{\partial y}dy = \partial_y f$$

are the partial differentials of f. Hence

$$df = \partial_x f + \partial_y f; \tag{2}$$

or, the total differential of f at x, y is equal to the sum of the partial differentials there.

The value of the differential at a fixed point depends on the values of dx and dy, which are quite arbitrary.

The geometrical interpretation of the differential is as follows: In Fig. 123, let $dx = MQ$ and $dy = MR$. Draw PR', $Q'M'$, $Q''S$ parallel to MR. Then

$$\partial_x f = Q'Q'' = M'S \quad \text{and} \quad \partial_y f = R'R'' = SM''.$$

$$\therefore \quad df = M'S + SM'' = M'M'';$$

or, the differential of the function is represented by the distance from a point in the tangent plane to the surface at P from a horizontal plane through P.

197. The Total Derivatives with respect to x and y.—If, in the total derivative

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt},$$

we take $t \equiv x$, then the total derivative of f with respect to x is

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}. \tag{1}$$

If we take $t \equiv y$, then

$$\frac{df}{dy} = \frac{\partial f}{\partial x}\frac{dx}{dy} + \frac{\partial f}{\partial y}. \tag{2}$$

Equations (1) and (2) represent the *total* derivatives of f with regard to x and y respectively. These derivatives are quite distinct and different from the partial derivatives, as is shown by the formulæ, and as is exhibited in their geometrical interpretations as follows:

The total derivative of $z = f(x, y)$ with respect to x is the limit of the difference-quotient

$$\frac{z_1 - z}{x_1 - x},$$

x and y varying as the coordinates of a point on some curve MH in the horizontal plane.

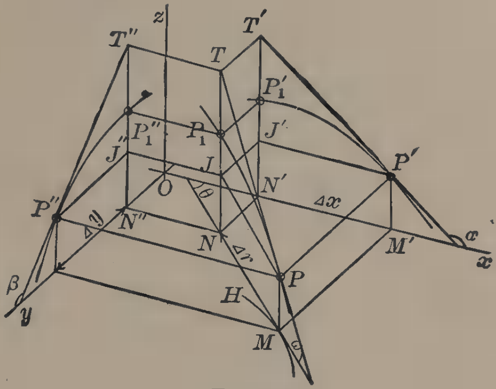


FIG. 124.

If, P_1 is x_1, y_1, z_1, then, in Fig. 124,

$$z_1 - z = JP_1 = J'P_1', \quad x_1 - x = N'M' = J'P'.$$

Therefore $\dfrac{dz}{dx} = \tan\alpha$ is the total derivative of z with respect to x. That is, the total derivative of f with respect to x is represented by the slope to Ox of the projection $P'T'$ of the tangent PT to the surface on the vertical plane xOz. The tangent PT being in a vertical plane through P which makes with xOz the angle θ determined by $\dfrac{dy}{dx} = \tan\theta$, as determined from $\phi(x, y) = 0$. That is, $\dfrac{dy}{dx}$ is the slope to Ox of the horizontal projection MN of the tangent PT.

In like manner the total derivative of f with respect to y is equal to $\tan\beta$, this being the slope to Oy of the projection of the same tangent PT on the perpendicular plane yOz.

Equations (1) and (2) are immediately determined from the total differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

by dividing through first by dx and then by dy.

In Fig. 124 we have

$$df = JT = J'T' = J''T'',$$

and equations (1) and (2) can be verified by the differential quotients taken from the figure directly.

198. Differentiation of the Implicit Function $f(x, y) = 0$.—An important and valuable corollary to the total differentiation of the function $z = f(x, y)$ is that which results in giving the derivative of y with respect to x in the implicit function $f(x, y) = 0$.

Since $z = 0$ in $z = f(x, y)$ gives $f(x, y) = 0$, and in $f(x, y) = 0$ are admissible only those values of x and y which make z *constantly* zero, the derivative of z with respect to any variable must be 0.

Therefore, from (5), §194, or (1), § 196, § 197,

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}.$$

This has been geometrically interpreted in Ex. 13, Chap. XXV.

In general, the plane $z = c$, c being any constant, cuts the surface $z = f(x, y)$ in a contour line, or curve in a horizontal plane, at distance c from the horizontal plane xOy. The equation of this curve in its plane is $f(x, y) = c$. In the same way as above,

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{df}{dt} = \frac{dc}{dt} = 0.$$

$$\therefore \quad \frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}},$$

which corresponds to the slope of the tangent to the contour (at the point x, y, c) to the vertical plane xOz.

EXERCISES.

1. If $x^3 + y^3 - 3axy = c$, find $D_x y$.

Here $$\frac{\partial f}{\partial x} = 3(x^2 - ay), \quad \frac{\partial f}{\partial y} = 3(y^2 - ax)$$

$$\therefore \quad \frac{dy}{dx} = \frac{x^2 - ay}{ax - y^2}.$$

2. Find $D_x y$ in $x^m/a^m + y^m/b^m = 1$.

$$\frac{\partial f}{\partial x} = \frac{mx^{m-1}}{a^m}, \quad \frac{\partial f}{\partial y} = \frac{my^{m-1}}{b^m}. \quad \therefore \quad \left(\frac{dy}{dx}\right) = -\left(\frac{b}{a}\right)^m \left(\frac{x}{y}\right)^{m-1}.$$

3. If $x \log y - y \log x = 0$, then $\dfrac{dy}{dx} = \dfrac{y}{x}\dfrac{\log y^x - y}{\log xy - x}$.

4. Let $x = \rho \cos \theta$. Find the total differential of x.

$$\frac{\partial x}{\partial \rho} = \cos \theta, \quad \frac{\partial x}{\partial \theta} = -\rho \sin \theta,$$

$$\therefore \quad dx = \cos \theta \, d\rho - \rho \sin \theta \, d\theta.$$

5. Find the slope to the horizontal plane of the curve

$$\left.\begin{aligned} z &= \frac{x^2}{a^2} + \frac{y^2}{b^2}, \\ 1 &= x + y. \end{aligned}\right\} \qquad \textit{Ans.}\ \left(\frac{y}{b^2} - \frac{x}{a^2}\right)\sqrt{2}.$$

6. Find the slope to xOy (the steepness) of the curve cut from the hyperbolic paraboloid $z = x^2/a^2 - y^2/b^2$ by the parabolic cylinder $y^2 = 4px$.

We have

$$\tan\omega = \frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds},$$

s being the length of the parabola $y^2 = 4px$. Here

$$\frac{\partial z}{\partial x} = \frac{2x}{a^2}, \quad \frac{\partial z}{\partial y} = -\frac{2y}{b^2}.$$

$$\frac{dx}{ds} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-\frac{1}{2}} = \frac{\sqrt{x}}{\sqrt{x+p}}, \quad \frac{dy}{ds} = \frac{\sqrt{p}}{\sqrt{x+p}}.$$

$$\therefore \quad \tan\omega = \frac{2}{\sqrt{p+x}}\left(\frac{x\sqrt{x}}{a^2} - \frac{y\sqrt{p}}{b^2}\right),$$

which is the declivity of the curve in space at x, y, z.

Find the points at which the tangent to this curve is horizontal.

7. If $u = \tan^{-1}(y/x)$, $du = (x\,dy - y\,dx)/(x^2 + y^2)$.

8. If $z = x^y$, $dz = yx^{y-1}\,dy + x^y \log x\,dy$.

9. Find the locus of all the tangent lines to a surface $z = f(x, y)$ at a point (a, b, c), P.

Through P draw a vertical plane, Fig. 123, rMP, whose equation is

$$\frac{x-a}{l} = \frac{y-b}{m} = r. \tag{1}$$

Then the equation to the tangent line, PM'', to the surface at P, in the plane rMP in terms of its slope at a, b, c, is

$$\frac{z-c}{r} = \frac{df}{dr}.$$

z and r being the coordinates of any point on the tangent line. But at a, b, c

$$\begin{aligned} \frac{df}{dr} = \tan\omega &= \frac{\partial f(a, b)}{\partial a}\frac{da}{dr} + \frac{\partial f(a, b)}{\partial b}\frac{db}{dr}, \\ &= l\frac{\partial f}{\partial a} + m\frac{\partial f}{\partial b}. \end{aligned}$$

Therefore the equation to the tangent line to the surface at a, b, c, whose horizontal projection makes $\angle\,\theta$ with Ox (where $l = \cos\theta$, $m = \sin\theta$), is

$$z - c = rl\frac{\partial f}{\partial a} + rm\frac{\partial f}{\partial b}. \tag{2}$$

Eliminating rl and rm between (1) and (2), we have

$$z - c = (x-a)\frac{\partial f}{\partial a} + (y-b)\frac{\partial f}{\partial b}, \tag{3}$$

an equation of the first degree in x, y, z, which is the locus in space of the tangent lines at a, b, c on the surface. This locus is a plane, Exercise 1, Chap. XXIV, touching the surface at a, b, c, and is defined to be the *tangent* plane to the surface at a, b, c.

10. Show that the equation to the tangent plane to the surface $z = ax^2 + by^2$ at any point x', y', z' on the surface is

$$z + z' = 2(axx' + byy').$$

11. Use the equation to the tangent plane

$$(x - a)\frac{\partial f}{\partial a} + (y - b)\frac{\partial f}{\partial b} - (z - c) = 0$$

to verify Ex. 12, § 192.

The direction cosines of the plane are proportional to $\frac{\partial f}{\partial a}, \frac{\partial f}{\partial b}, -1$. Hence if l, m, n are these cosines,

$$\frac{l}{\frac{\partial f}{\partial a}} = \frac{m}{\frac{\partial f}{\partial b}} = \frac{n}{-1} = \frac{1}{\sqrt{1 + \left(\frac{\partial f}{\partial a}\right)^2 + \left(\frac{\partial f}{\partial b}\right)^2}}.$$

Also, $\sec^2\gamma = 1/n^2$, giving the same result as Ex. 12.

12. Show that when

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \tag{1}$$

the tangent plane to the surface is horizontal at values of x, y satisfying $z = f(x, y)$ and (1).

13. Show that the curve on the surface $z = f(x, y)$ at all points of which the tangent plane to the surface makes the angle 45° with xOy is the curve cut on the surface by the cylinder

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = 1.$$

14. Apply Ex. 13 to show that the cylinder $x^2 + y^2 = \frac{1}{2}a^2$ cuts the sphere $x^2 + y^2 + z^2 = a^2$ in a line at every point of which the tangent plane to the sphere is sloped 45° to the horizontal plane. Draw a figure and verify geometrically.

15. The equation $x^2 + y^2 = a^2$ represents a vertical cylinder of revolution whose axis is Oz and radius is a. Find the equations of the path of a point which starts at $x = a$, $y = 0$, $z = 0$ and ascends the cylinder on a line of constant grade k. This curve is the helix, a spiral on the cylinder, having for its equations

$$x = a \cos\frac{z}{ka}, \quad y = a \sin\frac{z}{ka}.$$

CHAPTER XXVII.

SUCCESSIVE TOTAL DIFFERENTIATION.

199. Second Total Derivative and Differential of $z = f(x, y)$. It has been shown in § 194, (5), that

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}, \tag{1}$$

where x and y are any differentiable functions of t.

If we differentiate again with respect to t, then

$$\begin{aligned}\frac{d^2f}{dt^2} &= \frac{d}{dt}\left(\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}\right) + \frac{d}{dt}\left(\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}\right),\\ &= \frac{dx}{dt}\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2}.\end{aligned} \tag{2}$$

Since $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are functions of x and y to which (1) is applicable, in the same way we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{dy}{dt},\\ &= \frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y\,\partial x}\frac{dy}{dt}.\end{aligned} \tag{3}$$

Also,

$$\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}. \tag{4}$$

Substituting in (2) and remembering that

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x},$$

we have finally for the second total derivative of $f(x, y)$ with respect to t

$$\frac{d^2f}{dt^2} = \frac{\partial^2 f}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + 2\frac{\partial^2 f}{\partial x\,\partial y}\frac{dx}{dt}\frac{dy}{dt} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dt}\right)^2 + \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2}. \tag{5}$$

Multiplying through by dt^2, we have the second total differential of $f(x, y)$,

$$d^2f = \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x\,\partial y}dx\,dy + \frac{\partial^2 f}{\partial y^2}dy^2 + \frac{\partial f}{\partial x}d^2x + \frac{\partial f}{\partial y}d^2y. \quad (6)$$

In (5), t is taken as the independent variable, and while dt is perfectly arbitrary in (1) in actual value, we agree, as in Book I, that dt shall be taken as having a constant value in the successive differentiations.

Thus if we take x as the *independent variable* instead of t, then dx is taken constant, in which case $\frac{d^2x}{dx^2} = \frac{d}{dx}\left(\frac{dx}{dx}\right) = 0$, and we have for the second total derivative of f with respect to x

$$\frac{d^2f}{dx^2} = \frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x\,\partial y}\frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + \frac{\partial f}{\partial y}\frac{d^2y}{dx^2}. \quad (7)$$

In like manner taking y as the *independent variable*, changing t to y in (5), we have dy constant, and the total derivative of f with respect to y is

$$\frac{d^2f}{dy^2} = \frac{\partial^2 f}{\partial x^2}\left(\frac{dx}{dy}\right)^2 + 2\frac{\partial^2 f}{\partial x\,\partial y}\frac{dx}{dy} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial x}\frac{d^2x}{dy^2}. \quad (8)$$

200. $\frac{d^2y}{dx^2}$, when $f(x, y) = 0$.

The formulæ of the preceding article furnish means of expressing the second derivative of y with respect to x in an explicit function $f(x, y) = 0$, in terms of the partial derivatives of $f(x, y)$. This generally saves much labor in computing this derivative when f is a complicated function.

For brevity, represent the partial derivatives of f with respect to x and y by

$$f'_x,\ f'_y,\ f''_{xx},\ f''_{xy},\ f''_{yy},\ \text{etc.},$$

and the first and second derivatives of y with respect to x by y', y''.

Putting $f = z = 0$ in (7), § 199, we have

$$0 = f''_{xx} + 2f''_{xy}y' + f''_{yy}y'^2 + f'_y y''.$$

But $y' = -f'_x/f'_y$. Substituting this and solving for y'',

$$\frac{d^2y}{dx^2} = \frac{2f''_{xy}f'_xf'_y - f''_{xx}(f'_y)^2 - f''_{yy}(f'_x)^2}{(f'_y)^3}.$$

In like manner we get, by interchanging x and y, the second derivative D_y^2x. Otherwise deduced from (8), § 199.

201. Higher Total Derivatives.—We shall not have occasion to use the higher total derivatives of $z = f(x, y)$ above the second. They, however, are deduced in the same way as has been the second, by repeated applications of the formula for forming the first derivative. For the third total derivative of f with respect to t see Exercise 35 at the end of this chapter.

The higher total derivatives of $f(x, y)$ with respect to an arbitrary function t of x and y become very complicated and are seldom employed in elementary analysis. There is, however, an important particular case in which the higher derivatives of $f(x, y)$ require to be worked out completely—that is, when x and y are connected by a linear relation. This case we now consider and call it linear differentiation.

202. Successive Linear Total Derivatives.—To find the nth derivative of $f(x, y)$ with respect to r, when x and y are linearly related by

$$\frac{x-a}{l} = \frac{y-b}{m} = r,$$

a, b, l, m being constants.

The first derivative is, as found before,

$$\frac{df}{dr} = l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y}. \tag{1}$$

Differentiating again with respect to r, we have

$$\frac{d^2f}{dr^2} = l^2\frac{\partial^2 f}{\partial x^2} + 2lm\frac{\partial^2 f}{\partial x\,\partial x} + m^2\frac{\partial^2 f}{\partial y^2}. \tag{2}$$

Otherwise this follows immediately from (5), § 199, wherein

$$t \equiv r, \quad \frac{dx}{dr} = l, \quad \frac{dy}{dr} = m, \quad \frac{d^2x}{dr^2} = \frac{d^2y}{dr^2} = 0.$$

Differentiating (2) again with respect to r, and rearranging the terms, we have

$$\frac{d^3f}{dr^3} = l^3\frac{\partial^3 f}{\partial x^3} + 3l^2m\frac{\partial^3 f}{\partial x^2\,\partial y} + 3lm^2\frac{\partial^3 f}{\partial x\,\partial y^2} + m^3\frac{\partial^3 f}{\partial y^3}. \tag{3}$$

We observe that (1), (2), (3) are formed according to a definite law. The powers of l, m, and their coefficients follow the law of the binomial formula.

If we consider the symbols $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ as operators on f, and write conventionally

$$\frac{\partial^{p+q}}{\partial x^p\,\partial y^q} \equiv \left(\frac{\partial}{\partial x}\right)^p\left(\frac{\partial}{\partial y}\right)^q,$$

then we can write

$$\frac{\partial f}{\partial r} = \left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right)f, \tag{4}$$

$$\frac{\partial^2 f}{\partial r^2} = \left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right)^2 f, \tag{5}$$

$$\frac{\partial^3 f}{\partial r^3} = \left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right)^3 f, \tag{6}$$

in which the parentheses are to be expanded by the binomial formula and the indices of the powers of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ taken to mean the number of times these operations are performed.

We can demonstrate that this law is general and that we shall have

$$\frac{d^n f}{dx^n} = \left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right)^n f, \tag{7}$$

as follows.

First, observe that

$$\frac{d}{dr}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial r}, \quad \frac{d}{dr}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial r}. \tag{8}$$

For

$$\begin{aligned}\frac{d}{dr}\frac{\partial f}{\partial x} &= \frac{\partial^2 f}{\partial x^2}\frac{dx}{dr} + \frac{\partial^2 f}{\partial y\,\partial x}\frac{dy}{dr},\\ &= l\frac{\partial^2 f}{\partial x^2} + m\frac{\partial^2 f}{\partial y\,\partial x}.\end{aligned}$$

Also,

$$\begin{aligned}\frac{\partial}{\partial x}\frac{df}{dr} &= \frac{\partial}{\partial x}\left(l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y}\right),\\ &= l\frac{\partial^2 f}{\partial x^2} + m\frac{\partial^2 f}{\partial x\,\partial y},\end{aligned}$$

which proves the first equality in (8), and the second is proved in the same way.

Now assume (7) to be true. Differentiating again with respect to r, we have

$$\begin{aligned}\frac{d^{n+1}f}{dr^{n+1}} &= \frac{d}{dr}\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right)^n f,\\ &= \left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right)^n \frac{df}{dr},\\ &= \left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right)^n \left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right) f,\\ &= \left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right)^{n+1} f.\end{aligned}$$

The *memoria technica* (7) being true for $n = 3$, it is true for 4, and so on generally.

EXERCISES.

1. Given $x^2 + y^2 = a^2$, find Dy.

2. If $x^3 + xy^2 - ay^2 = 0$, $y' = (3x^2 + y^2)/2y(a - x)$.

3. If $(t^2 + u^2)^2 = 2a^2(t^2 - u^2)$, $\dfrac{du}{dt} = \dfrac{t(a^2 - t^2 - u^2)}{u(a^2 + t^2 + u^2)}$.

4. If $z^n = \dfrac{x + z}{x - z}$, $\dfrac{dz}{dx} = \dfrac{2z^2}{2zx - n(x^2 - z^2)}$.

5. If $u^2v^2 + \sqrt[3]{u} - \sqrt{v} = 0$, $\dfrac{dv}{du} = \dfrac{12uv^2 + 2u^{-\frac{2}{3}}}{3v^{-\frac{1}{2}} - 12u^2v}$.

6. If $f(x, y) = 0$, is the equation to any curve, show that

$$(X - x)\frac{\partial f(x, y)}{\partial x} + (Y - y)\frac{\partial f(x, y)}{\partial y} = 0,$$

$$(X - x)\frac{\partial f(x, y)}{\partial y} = (Y - y)\frac{\partial f(x, y)}{\partial x},$$

are the equations to the tangent and normal at x, y. The running coordinates being X, Y.

7. Show by Ex. 6 that the equations of the tangent and normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ are

$$\frac{Xx}{a^2} + \frac{Yy}{b^2} = 1 \quad \text{and} \quad a^2\frac{X}{x} - b^2\frac{Y}{y} = a^2 - b^2.$$

8. Show that the second derivative of y with respect to x, in $f(x, y) = 0$, can be expressed in the form

$$\frac{d^2y}{dx^2} = -\frac{\left(\dfrac{\partial f}{\partial x}\dfrac{\partial}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial}{\partial x}\right)^2 f}{\left(\dfrac{\partial f}{\partial y}\right)^3}.$$

9. Show that the ordinate of the curve $f(x, y) = 0$ is a maximum or a minimum when $f'_x = 0$, according as f''_{xx} and f'_y are like or unlike signed.

For a maximum value of y we must have

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x}\bigg/\frac{\partial f}{\partial y} = 0,$$

or $f'_x = 0$, $f'_y \neq 0$. When this is the case, by § 200,

$$\frac{d^2y}{dx^2} = -\frac{\partial^2 f}{\partial x^2}\bigg/\frac{\partial f}{\partial y},$$

which gives a maximum when f''_{xx} and f'_y are like signed and a minimum when unlike signed.

10. Show that the maximum and minimum ordinates of the conic

$$f \equiv ax^2 + by^2 + 2hxy + 2gx + 2fy + d = 0$$

are found by aid of

$$f'_x = ax + hy + g = 0.$$

If $f'_y = by + hx + f$, is positive, the ordinate is a maximum; if negative, a minimum.

11. Find the maximum ordinate in the folium of Descartes,

$$y^3 - 3axy + x^3 = 0 \equiv f(x, y).$$

$$\tfrac{1}{3}f'_x = -ay + x^2, \quad \tfrac{1}{3}f'_y = y^2 - ax.$$

Eliminating y between $f = 0$, $f'_x = 0$, we have

$$x^6 - 2a^3x^3 = 0.$$

$\therefore$ $x = 0$, $x = a\sqrt[3]{2}$. These values give $y = 0$, $y = a\sqrt[3]{4}$. For $x = 0$, $y = 0$, we have $f'_y = 0$, but for $x = a\sqrt[3]{2}$, $y = a\sqrt[3]{4}$, we have a maximum y if $a = +$, since

$$\frac{d^2y}{dx^2} = -\frac{\partial^2 f}{\partial x^2}\bigg/\frac{\partial f}{\partial y} = -\frac{2}{a}.$$

12. If two curves $\phi(x, y) = 0$, $\psi(x, y) = 0$ intersect at a point x, y, and if ω be their angle of intersection, prove that

$$\tan\omega = \frac{\phi'_x\psi'_y - \phi'_y\psi'_x}{\phi'_x\psi'_y + \phi'_y\psi'_x}.$$

13. Show that two curves $\phi = 0$, $\psi = 0$ cut at right angles if at their point of intersection

$$\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial x} = 0.$$

14. Apply this to show that the ellipses

$$x^2/a^2 + y^2/b^2 = 1, \quad x^2/\alpha^2 + y^2/\beta^2 = 1$$

will cut at right angles if $a^2 - b^2 = \alpha^2 - \beta^2$.

15. Show that the length of the perpendicular p from the origin on the tangent to the curve $\phi(x, y) = 0$ at x, y is

$$p = \frac{x\phi'_x + y\,\phi'_y}{\sqrt{(\phi'_x)^2 + (\phi'_y)^2}}.$$

16. Show that the radius of curvature of $f(x, y) = 0$ at x, y is

$$R = \frac{[(f'_x)^2 + (f'_y)^2]^{\frac{3}{2}}}{f''_{xx}(f'_y)^2 - 2f''_{xy}f'_xf'_y + f''_{yy}(f'_x)^2}.$$

17. If $f(x, y) = 0$, show that

$$\frac{\partial f}{\partial y}\frac{d^3y}{dx^3} + 3\left(\frac{\partial^2 f}{\partial x\,\partial y} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + \left(\frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y}\right)^3 f = 0.$$

18. If $y^2 = 2xy + a^2$, show that

$$\frac{dy}{dx} = \frac{y}{y - x}, \quad \frac{d^2y}{dx^2} = \frac{a^2}{(y - x)^3}, \quad \frac{d^3y}{dx^3} = -\frac{3a^2x}{(y - x)^5}, \quad \frac{d^2x}{dy^2} = -\frac{a^2}{y^3}.$$

Also, that $x = \pm a$ are maximum and minimum values of x.

19. Investigate $y = \sin(x + y)$ for maximum and minimum y.

$$\frac{dy}{dx} = \frac{\cos(x + y)}{1 - \cos(x + y)}, \quad \frac{d^2y}{dx^2} = \frac{-y}{[1 - \cos(x + y)]^3}.$$

20. If $z = x^3y^2 - 2xy^4 + 3x^2y^3$, show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 5z,$$

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 20z.$$

21. If $z = \phi(y + ax) + \psi(y - ax)$, $\therefore$ $\dfrac{\partial^2 z}{\partial x^2} = a^2\dfrac{\partial^2 z}{\partial y^2}$.

22. If $z = \dfrac{a - x}{x - y}$, $\dfrac{\partial z}{\partial x} = \dfrac{y - a}{(x - y)^2}$.

23. If $y - nz = f(x - mz)$, then $m\dfrac{\partial z}{\partial x} + n\dfrac{\partial z}{\partial y} = 1.$

24. If $u = y^x$, prove $u''_{xy} = y^{x-1}(1 + \log y^x) = u''_{yx}.$

25. If $z = \sqrt{x^2 + y^2}$, prove $\left(x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}\right)^2 z = 0.$

If $z = \sqrt{x^3 + y^3}$, prove $\left(x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}\right)^2 z = \frac{3}{4}z.$

26. The curve $x^3 + y^3 - 3x = 0$ has a maximum ordinate at the point $1, \sqrt[3]{2}$, and a minimum ordinate at $-1, -\sqrt[3]{2}.$

27. The curve $\rho(\sin^3\theta + \cos^3\theta) = a \sin 2\theta$ has a maximum radius vector at the point $a\sqrt{2}, \frac{1}{4}\pi.$

28. The curve $2x^2y + y^2 + 4x - 3 = 0$ has no minimum ordinate, it has a maximum ordinate at the point $-\frac{1}{2}, 2.$

29. The curve $x^4 + y^4 - 4xy^3 - 2 = 0$ has neither maximum nor minimum ordinate.

30. Show that $(0, 2)$ gives y a maximum, and $\pm\frac{1}{4}\sqrt{3}, -\frac{1}{4}$ a minimum, while $\frac{3}{4}\sqrt{3}, \frac{3}{4}$ makes x a maximum, and $-\frac{3}{4}\sqrt{3}, \frac{3}{4}$ gives x a minimum in the cardioid

$$(x^2 + y^2)^2 - 2y(x^2 + y^2) - x^2 = 0.$$

31. In $x^4 + 2ax^2y - ay^3 = 0$, y is a minimum at $x = \pm a.$

32. In $3a^2y^2 + xy^3 + 4ax^3 = 0$, y is a maximum for $x = 3a/2.$

33. Investigate the conic $ax^2 + 2hxy + by^2 = 1$, for maximum and minimum coordinates.

34. If R is the radius of curvature of $f(x, y) = 0$, and θ the angle which the tangent makes with a fixed line, show from $ds = R\,d\theta$ and $\theta = \tan^{-1} dy/dx$, that

$$R = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''} = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{d^2y\,dx - dy\,d^2x}.$$

The first when x is the independent variable, the second when the independent variable is not specified and dx, dy are variables.

35. The third total derivative of $f(x, y)$ with respect to any variable t is

$$\frac{d^3f}{dt^3} = \left(\frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y}\right)^3 f + \frac{\partial f}{\partial x}\frac{d^3x}{dt^3} + \frac{\partial f}{\partial y}\frac{d^3y}{dt^3}$$
$$+ 3\left[\frac{\partial^2 f}{\partial x^2}\frac{d^2x}{dt^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial x\,\partial y}\left(\frac{d^2x}{dt^2}\frac{dy}{dt} + \frac{d^2y}{dt^2}\frac{dx}{dt}\right) + \frac{\partial^2 f}{\partial y^2}\frac{d^2y}{dt^2}\frac{dy}{dt}\right].$$

CHAPTER XXVIII.

DIFFERENTIATION OF A FUNCTION OF THREE VARIABLES.

203. We are particularly interested here in the differentiation of a function

$$w = f(x, y, z)$$

of three independent variables, for the reason that when $w = 0$ we have

$$f(x, y, z) = 0,$$

the implicit function of three variables, which can be represented by a surface in space, and also because the treatment of the function of three variables assists in the discussion of the implicit function of three variables.

We do not attempt to represent geometrically a function w of three independent variables.

However, corresponding to any triplet $x = a$, $y = b$, $z = c$, there is a point in space which represents the three variables x, y, z for those particular values.

When, corresponding to any triplet x, y, z, the function $f(x, y, z)$ has a determinate value or values it is defined as a function of x, y, z.

The function f is a continuous function of x, y, z at x, y, z when for all values of x_1, y_1, z_1 in the neighborhood of x, y, z we have the number $f(x_1, y_1, z_1)$ in the neighborhood of $f(x, y, z)$.

204. Differentiation of $w = f(x, y, z)$.—Let x, y, z and x_1, y_1, z_1 be represented by two points P, P_1 in space. Complete the parallelopiped $PRQP_1$ with diagonal PP_1, by drawing parallels to the axes through P and P_1. Then in the figure we have the coordinates of R, (x_1, y, z), and of Q, (x_1, y_1, z). Let $PP_1 = \Delta r$, and let l, m, n be the direction cosines of the angles which PP_1 makes with the axes Ox, Oy, Oz, respectively.

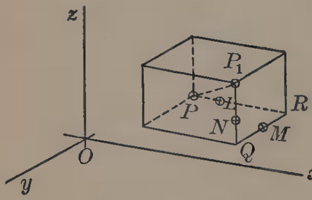


FIG. 125.

Then we have

$$x_1 - x = l\Delta r,$$
$$y_1 - y = m\Delta r,$$
$$z_1 - z = n\Delta r.$$

Applying the theorem of mean value for one variable, letting z, y, x in succession alone vary, we have

$$\begin{aligned} f(x_1y_1z_1) - f(x_1y_1z) &= (z_1 - z)f'_\zeta(x_1y_1\zeta),\\ f(x_1y_1z) - f(x_1yz) &= (y_1 - y)f'_\eta(x_1\eta z),\\ f(x_1yz) - f(xyz) &= (x_1 - x)f'_\xi(\xi yz),\end{aligned}$$

where ξ, y, z; x_1, η, z; x_1, y_1, ζ, are points such as L, M, N, respectively, on the segments PR, RQ, QP_1. By addition, we have

$$w_1 - w = f'_\xi(\xi yz)\Delta x + f'_\eta(x_1\eta z)\Delta y + f'_\zeta(x_1y_1\zeta)\Delta z.$$

Now let t be any differentiable function of x, y, z, such that $t = t_1$, when x, y, z become x_1, y_1, z_1. Then for the difference-quotient of w with respect to t,

$$\frac{w_1 - w}{t_1 - t} = f'_\xi(\xi yz)\frac{\Delta x}{\Delta t} + f'_\eta(x_1\eta z)\frac{\Delta y}{\Delta t} + f'_\zeta(x_1y_1\zeta)\frac{\Delta z}{\Delta t}.$$

If now the partial derivatives $\dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, \dfrac{\partial w}{\partial z}$ are continuous functions throughout the neighborhood of x, y, z, we have, on passing to limits in the above equation, the total derivative of f with respect to t,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}. \tag{1}$$

The process is obviously general for a function of any number of variables, and if F is a function of n independent variables $v_1, \ldots, v_n$, then the derivative of F with respect to t, a function of these variables, is

$$\frac{dF}{dt} = \sum_1^n \frac{\partial F}{\partial v_r}\frac{dv_r}{dt}.$$

Second Total Derivative of $w = f(x, y, z)$.—We can differentiate (1) with respect to t and obtain in the same way

$$\frac{d^2f}{dt^2} = \left(\frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z}\right)^2 f + \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2} + \frac{\partial f}{\partial z}\frac{d^2z}{dt^2}. \tag{2}$$

205. Successive Linear Differentiation.—Of chief importance are the successive linear total derivatives of $f(x, y, z)$ with respect to r when

$$\frac{x - a}{l} = \frac{y - b}{m} = \frac{z - c}{n} = r,$$

where a, b, c, l, m, n are constants. Then

$$x = a + lr, \quad y = b + mr, \quad z = c + nr,$$

and

$$\frac{dx}{dr} = l, \quad \frac{dy}{dr} = m, \quad \frac{dz}{dr} = n$$

are constants, their higher derivatives are 0.

Equation (1), § 204, becomes

$$\frac{df}{dr} = l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} + n\frac{\partial f}{\partial z}. \tag{1}$$

We can differentiate (1) again with respect to r and get

$$\frac{d^2f}{dr^2} = \left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y} + n\frac{\partial}{\partial z}\right)^2 f, \tag{2}$$

or obtain the result directly from the equation (2) in § 204.

We can show, as for two variables, that the nth linear total derivative can be expressed by

$$\frac{d^nf}{dr^n} = \left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y} + n\frac{\partial}{\partial z}\right)^n f, \tag{3}$$

where the parenthesis is to be expanded by the multinomial theorem and the exponents of the operative symbols indicate the number of times the operation is to be performed on f.

EXERCISES.

1. If $u = (y - z)(z - x)(x - y)$, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

2. If $xe^y + \log z - yz = 0$, $\frac{\partial z}{\partial y} = \frac{z(z - xe^y)}{1 - yz}.$

3. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, $u'_x + u'_y + u'_z = 3(x + y + z)^{-1}.$

4. If $w = \log(\tan x + \tan y + \tan z)$, $w'_x \sin 2x + w'_y \sin 2y + w'_z \sin 2z = 2.$

5. If $w = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$, show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0.$$

6. If $w = e^{xyz}$, $\frac{\partial^3 w}{\partial x\, \partial y\, \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}.$

7. If $w = x^3z^4 + e^xy^2z^3 + x^2y^2z^2$, $w^{\text{IV}}_{xxyz} = 6e^xyz^2 + 8yz.$

8. Show that $\frac{\partial z}{\partial x} = \infty$ at the point (3, 4, 2) on the surface

$$x^2 + 3z^2 + xy - 2yz - 3x - 4z = 0.$$

9. Show that $\frac{\partial^2 z}{\partial x^2} = 0$ at the point $(-2, -1, 0)$ on the surface

$$4x^2 + z^2 - 5xz + 4yz + y - 2z - 15 = 0.$$

10. $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{50}{343}$ at the point $(1, 2, -1)$ of the surface

$$x^2 - y^2 + 2z^2 + 2xy - 4xz + x - y + z - 5 = 0.$$

11. Show that the second total derivatives of $w = f(x, y, z)$ with respect to x, y, z are respectively

$$\frac{d^2w}{dx^2} = \left(\frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y} + \frac{dz}{dx}\frac{\partial}{\partial z}\right)^2 f + \frac{\partial f}{\partial y}\frac{d^2y}{dx^2} + \frac{\partial f}{\partial z}\frac{d^2z}{dx^2},$$

$$\frac{d^2w}{dy^2} = \left(\frac{dx}{dy}\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{dz}{dy}\frac{\partial}{\partial z}\right)^2 f + \frac{\partial f}{\partial x}\frac{d^2x}{dy^2} + \frac{\partial f}{\partial z}\frac{d^2z}{dy^2},$$

$$\frac{d^2w}{dz^2} = \left(\frac{dx}{dz}\frac{\partial}{\partial x} + \frac{dy}{dz}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 f + \frac{\partial f}{\partial x}\frac{d^2x}{dz^2} + \frac{\partial f}{\partial y}\frac{d^2y}{dz^2}.$$

CHAPTER XXIX.

EXTENSION OF THE LAW OF THE MEAN TO FUNCTIONS OF TWO AND THREE VARIABLES.

206. Functions of Two Variables.—Let $z = f(x, y)$ be a function of two independent variables.

When $x = a, y = b$, let z become $c = f(a, b)$. Also, let

$$\frac{x - a}{l} = \frac{y - b}{m} = r. \tag{1}$$

Then
$$z = f(x, y) = f(a + lr, b + mr) \tag{2}$$

is a function of the one variable r, if a, b, l, m are constants. This function becomes $c = f(a, b)$ when $r = 0$. If this function of r and its first $n + 1$ derivatives with respect to r are continuous for all values of r from $r = 0$ to $r = r$, then, by the Law of the Mean for functions of one variable,

$$z = c + r\left(\frac{dz}{dr}\right)_0 + \cdot \cdot \cdot + \frac{r^n}{n!}\left(\frac{d^nz}{dr^n}\right)_0 + \frac{r^{n+1}}{(n+1)!}\left(\frac{d^{n+1}z}{dr^{n+1}}\right)_{r=\sigma}. \tag{3}$$

Here $\left(\dfrac{d^pz}{dr^p}\right)_0$ means the pth derivative of z with respect to r taken at $r = 0$, and $\left(\dfrac{d^{n+1}z}{dr^{n+1}}\right)_{r=\sigma}$ means the $(n + 1)$th derivative of z with respect to r taken at some value σ of r between 0 and r.

Also, since these derivatives are linear derivatives of z, we have

$$\left(\frac{d^pz}{dr^p}\right)_0 = \frac{d^pf(x, y)}{dr^p}, \quad \text{when } x = a, y = b,$$
$$= \left(l\frac{\partial}{\partial a} + m\frac{\partial}{\partial b}\right)^p f(a, b),$$
$$= \frac{1}{r^p}\left\{(x - a)\frac{\partial}{\partial a} + (y - b)\frac{\partial}{\partial b}\right\}^p f(a, b),$$

since $l = (x - a)/r$, $m = (y - b)/r$, from (1). Hence

$$\frac{r^p}{p!}\left(\frac{d^pz}{dr^p}\right)_0 = \frac{1}{p!}\left\{(x - a)\frac{\partial}{\partial a} + (y - b)\frac{\partial}{\partial b}\right\}^p f(a, b). \tag{4}$$

In like manner let $x = \xi, y = \eta$, when $r = \sigma$; ξ, η being numbers respectively between a and x, b and y. Then

$$\frac{r^{n+1}}{(n+1)!}\left(\frac{d^{n+1}z}{dr^{n+1}}\right)_{r=\sigma} = \frac{1}{(n+1)!}\left\{(x-a)\frac{\partial}{\partial\xi}+(y-b)\frac{\partial}{\partial\eta}\right\}^{n+1} f(\xi,\eta). \quad (5)$$

Substituting the values of (4) and (5) in (3), we have the Law of the Mean Value extended to functions of two variables, or

$$f(x, y) = \sum_{p=0}^{p=n} \frac{1}{p!}\left\{(x-a)\frac{\partial}{\partial a}+(y-b)\frac{\partial}{\partial b}\right\}^{p} f(a, b)$$

$$+\frac{1}{(n+1)!}\left\{(x-a)\frac{\partial}{\partial\xi}+(y-b)\frac{\partial}{\partial\eta}\right\}^{n+1} f(\xi,\eta). \quad (6)$$

207. The geometrical interpretation of § 206 is as follows: Given the ordinate to a surface at a particular point a, b, and the partial derivatives of the ordinate at that point. To find the ordinate at an arbitrary point x, y.

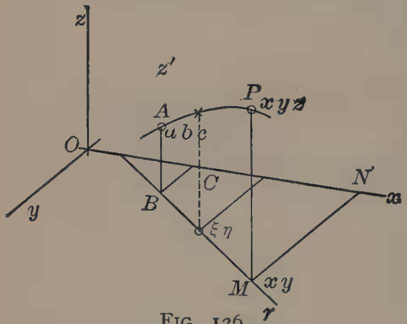


FIG. 126.

Let $z = f(x, y)$ be the equation to a surface on which $A, (a, b, c)$ is the point at which the coordinates and partial derivatives of z are known. Let P be the point on the surface at which x, y are given and z or $f(x, y)$ is required.

Pass a vertical plane through A and P, cutting the surface in the curve AP and the horizontal plane in the straight line BM, whose equation is

$$\frac{x-a}{l} = \frac{y-b}{m} = r.$$

The equation of the curve AP cut out of the surface by this vertical plane is

$$z = f(a + lr,\ b + mr),$$

referred to axes Br, Bz' and coordinates r, z, in its plane rBz' The law of the mean is applied to this function of the variable r, resulting in (3). Then, since these derivatives are linear, they can be expressed in terms of the partial derivatives of z at a, b, and (3) is transformed into (6).

208. Expansion of Functions of Two Variables.—Whenever the function (2), § 206, of the one variable r can be expanded in powers of r by Maclaurin's series as given in Book I, then we can make $n = \infty$ in (6), and we have

$$f(x, y) = \sum_{p=0}^{\infty} \frac{1}{p!} \left\{ (x - a) \frac{\partial}{\partial a} + (y - b) \frac{\partial}{\partial b} \right\}^{p} f(a, b),$$

and the function $f(x, y)$ can be computed in terms of $f(a, b)$ and the partial derivatives at a, b.

209. Functions of Three Variables.—Following exactly the same process as in § 206, for

$$w = f(x, y, z),$$

we have the law of the mean for three variables,

$$f(x, y, z) = \sum_{p=0}^{n} \frac{1}{p!} \left\{ (x-a)\frac{\partial}{\partial a} + (y-b)\frac{\partial}{\partial b} + (z-c)\frac{\partial}{\partial c} \right\}^{p} f(a, b, c)$$

$$+ \frac{1}{(n+1)!} \left\{ (x-a)\frac{\partial}{\partial \xi} + (y-b)\frac{\partial}{\partial \eta} + (z-c)\frac{\partial}{\partial \zeta} \right\}^{n+1} f(\xi, \eta, \zeta), \quad (1)$$

where ξ, η, ζ are the coordinates of some point on the straight-line segment joining the points in space whose coordinates are x, y, z and a, b, c.

Whenever the function of one variable r,

$$f(a + lr, b + mr, c + nr),$$

can be expanded in an infinite series of powers of r by Maclaurin's series, Book I, then we can make $n = \infty$ in (1), and have

$$f(x, y, z) = \sum_{p=0}^{\infty} \frac{1}{p!} \left\{ (x-a)\frac{\partial}{\partial a} + (y-b)\frac{\partial}{\partial b} + (z-c)\frac{\partial}{\partial c} \right\}^{p} f(a, b, c). \quad (2)$$

210. Implicit Functions.—The law of the mean enables us to express the equation of any curve or surface in terms of positive powers of the variables, and permits the study of the curve or surface as though its equation were a polynomial in the variables.

Thus if $z = f(x, y)$ is constant and 0, then $f(x, y) = 0$ is the equation of a curve in the plane xOy. The equation of any such curve can, by (6), § 206, be written in the form

$$0 = \sum_{p=0}^{n} \frac{1}{p!} \left\{ (x - a) \frac{\partial}{\partial a} + (y - b) \frac{\partial}{\partial b} \right\}^{p} f(a, b) + R_n. \quad (1)$$

In like manner, by (1), § 209, the equation to any surface $f(x, y, z) = 0$ can be written

$$0 = \sum_{p=0}^{n} \frac{1}{p!} \left\{ (x-a)\frac{\partial}{\partial a} + (y-b)\frac{\partial}{\partial b} + (z-c)\frac{\partial}{\partial c} \right\}^{p} f(a, b, c) + R_n. \quad (2)$$

R_n being (5), § 206, for equation (1) above, and the corresponding value in (1), § 209, for equation (2).

211. The law of the mean as expressed in this chapter is fundamental in the theory of curves and surfaces. It permits the treatment of implicit equations in symmetrical forms, which is a far-reaching advantage in dealing with general problems whose complexity would otherwise render them almost unintelligible.

A most useful form of the equations for two and three variables is obtained by putting

$$x - a = h, \quad y - b = k, \quad z - c = l,$$

and in the result changing a, b, c into x, y, z.

Thus for two variables

$$f(x + h, y + k) = \sum \frac{1}{r\,!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^r f(x, y). \tag{10}$$

For three variables

$$f(x+h,\ y+k,\ z+l) = \sum \frac{1}{r\,!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^r f(x, y, z). \tag{11}$$

EXERCISES.

1. Show that the equation of any algebraic curve of degree n can be written as either

$$0 = \sum_{r=0}^{n} \frac{1}{r!}\left\{(x - a)\frac{\partial}{\partial a} + (y - b)\frac{\partial}{\partial b}\right\}^r f(a, b), \tag{1}$$

or

$$0 = \sum_{r=0}^{n} \frac{1}{r!}\left(x\frac{\partial}{\partial 0_x} + y\frac{\partial}{\partial 0_y}\right)^r f(0, 0). \tag{2}$$

2. Show that any algebraic surface of nth degree can be written in either of the equations

$$0 = \sum_{r=0}^{n} \frac{1}{r!}\left\{(x - a)\frac{\partial}{\partial a} + (y - b)\frac{\partial}{\partial b} + (z - c)\frac{\partial}{\partial c}\right\}^r f(a, b, c), \tag{1}$$

$$0 = \sum_{r=0}^{n} \frac{1}{r!}\left(x\frac{\partial}{\partial 0_x} + y\frac{\partial}{\partial 0_y} + z\frac{\partial}{\partial 0_z}\right)^r f(0, 0, 0). \tag{2}$$

3. The function $\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^r f(x, y)$ is called a *concomitant* of $f(x, y)$.

Find the concomitants of a homogeneous function $f(x, y)$ of degree n.
In (10), § 211, put $h = gx$, $k = gy$, then

$$f(x + gx, y + gy) = \sum_{r=1}^{n} \frac{g^r}{r!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^r f(x, y).$$

Since f is homogeneous in x and y of degree n,

$$f(x + gx, y + gy) = f\{(1 + g)x, (1 + g)y\} = (1 + g)^n f(x, y).$$

$$\therefore\quad (1 + g)^n f(x, y) = \sum \frac{g^r}{r!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^r f(x, y).$$

This equation is true for all values of g including 0. Therefore, equating like powers of g, we have

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf,$$

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\,\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=n(n-1)f,$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^n f=n!\,f.$$

In the same way, if $f(x, y, z)$ is homogeneous of degree n, we find, by putting $h = gx$, $k = gy$, $l = gz$ in (11), § 211, as above, the concomitants of $f(x, y, z)$,

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)^r f=n(n-1)\ldots(n-r+1)f,$$

for $r = 1, 2, \ldots, n$.

The concomitant functions are important in the theory of curves and surfaces. They are *invariant* under any transformation of rectangular axes, the origin remaining the same.

CHAPTER XXX.

MAXIMUM AND MINIMUM. FUNCTIONS OF SEVERAL VARIABLES.

212. Maxima and Minima Values of a Function of Two Independent Variables.

Definition.—The function $z = f(x, y)$ will be a maximum at $x = a$, $y = b$, when $f(a, b)$ is greater than $f(x, y)$ for *all* values of x and y in the neighborhood of a, b.

In like manner $f(a, b)$ will be a minimum value of $f(x, y)$ when $f(a, b)$ is less than $f(x, y)$ for *all* values of x, y in the neighborhood of a, b.

In symbols, we have $f(a, b)$ a maximum or a minimum value of the function $f(x, y)$ when

$$f(x, y) - f(a, b)$$

is negative or positive, respectively, for all values of x, y in the neighborhood of a, b.

Geometrically interpreted, the point P, Fig. 115, on the surface representing $z = f(x, y)$ is a maximum point when it is higher than *all* other points on the surface in its neighborhood. Also, P is a minimum point on the surface when it is lower than all other points in its neighborhood.

This means that all vertical planes through P cut the surface in curves, each of which has a maximum or a minimum ordinate z at P accordingly.

Also, when P is a maximum point, then any contour line LMN, Fig. 115, cut out of the surface by a horizontal plane passing through the neighborhood of P, below P, must be a small closed curve; and the tangent plane at P is horizontal, having only one point in common with the surface in the neighborhood. Similar remarks apply when P is a minimum point.

When the converse of these conditions holds, the point P will be a maximum or minimum point accordingly.

213. Conditions for Maxima and Minima Values of $f(x, y)$.—Let $z = f(x, y)$, x and y being independent. To find the conditions that z shall be a maximum or a minimum at x, y.

I. Any pair of values x', y' in the neighborhood of x, y can be expressed by

$$x' = x + lr, \quad y' = y + mr,$$

where $l = \cos\theta$, $m = \sin\theta$. Then

$$z = f(x + lr, y + mr)$$

is a function of the one variable r, if θ is constant.

If z is a maximum or a minimum, we must have, by Book I,

$$\frac{dz}{dr} = 0, \quad \frac{d^2z}{dr^2} \text{ negative or positive,}$$

respectively, *for all values of* θ. That is,

$$\frac{dz}{dr} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y} = 0.$$

This must be true for all values of θ. But when $\theta = 0$ and $\theta = \frac{1}{2}\pi$, we have

$$\frac{\partial}{\partial x}f(x, y) = 0 \quad \text{and} \quad \frac{\partial}{\partial y}f(x, y) = 0 \tag{1}$$

respectively. Equations (1) are necessary conditions in order that x, y which satisfy them may give z a maximum or a minimum. But they are not sufficient, for we must in addition have

$$\frac{d^2z}{dr^2} = l^2\frac{\partial^2 f}{\partial x^2} + 2lm\frac{\partial^2 f}{\partial x\,\partial y} + m^2\frac{\partial^2 f}{\partial y^2}, \tag{2}$$

different from 0 and of the same sign for all values of θ. When (2) is negative for all values of θ, then z at x, y is a maximum; and when (2) is positive for all values of θ, then z is a minimum.

Put $$\frac{\partial^2 f}{\partial x^2} = A, \quad \frac{\partial^2 f}{\partial x\,\partial y} = H, \quad \frac{\partial^2 f}{\partial y^2} = B.$$

The quadratic function in l, m (see Ex. 19, § 25),

$$Al^2 + 2Hlm + Bm^2, \tag{3}$$

will keep its sign unchanged for all values of the variables l, m, provided

$$AB - H^2$$

is positive. Then the function (3) has the same sign as A.

(a). Therefore the function $f(x, y)$ is a maximum or a minimum at x, y when

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 = +, \tag{4}$$

and is a maximum or a minimum according as either $\frac{\partial^2 f}{\partial x^2}$ or $\frac{\partial^2 f}{\partial y^2}$ is negative or positive respectively.

(b). If $AB - H^2 = -$, then will (2) have opposite signs when $m = 0$ and $m/l = -A/H$; also when $l = 0$ and $m/l = -H/B$. The function cannot then be either a maximum or a minimum (see Ex. 19, § 25).

(c). If $AB - H^2 = 0$, and A, B, H are not all 0, then the right member of (2) becomes

$$\frac{(lA + mH)^2}{A} \equiv \frac{(mB + lH)^2}{B},$$

and has the same sign as A or B for all values of θ, except when $m/l = -A/H$. Then (2) is 0. This case requires further examination, involving higher derivatives than the second; as also does the case when A, B, H are all 0.

To sum up the conditions, we have $f(x, y)$ a maximum or a minimum at x, y when

$$\begin{array}{lll} & f'_x = 0, & f'_y = 0, \\ & f''_{xx} = \mp \begin{smallmatrix}\text{max.}\\ \text{min.,}\end{smallmatrix} & \begin{vmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{vmatrix} = +. \\ \text{or} & f''_{yy} = \mp \begin{smallmatrix}\text{max.}\\ \text{min.,}\end{smallmatrix} & \end{array}$$

If the determinant is negative, there is neither maximum nor minimum; if zero, the case is uncertain.*

To find the maximum and minimum values of $z = f(x, y)$, we solve $f'_x = 0$, $f'_y = 0$, to find the values of x, y at which the maximum or minimum values may occur, then substitute x, y in the conditions to determine the character of the function there.

The value of the function is obtained by either substituting x, y in $f(x, y)$, or by eliminating x, y between the three equations

$$z = f(x, y), \quad f'_x = 0, \quad f'_y = 0$$

for the maximum or minimum value z.

This method employed for finding the conditions for a maximum or a minimum value of $z = f(x, y)$ has been that which corresponds geometrically to cutting the surface at x, y by vertical planes and determining whether or not *all* these sections have a maximum or a minimum ordinate at x, y.

II. Another way of determining these conditions is directly by the law of mean value. We have

$$f(x', y') - f(x, y) = (x' - x)\frac{\partial f(\xi, \eta)}{\partial \xi} + (y' - y)\frac{\partial f(\xi, \eta)}{\partial \eta}.$$

For all values of x', y' in the neighborhood of x, y we have ξ, η also in the neighborhood of x, y. If the values f'_x, f'_y are different from 0, then the values f'_ξ, f'_η are in the neighborhoods of their limits and have the same signs as those numbers for all values of x', y' in the neighborhood of x, y. Therefore the difference on the left of the equation changes sign when $x' = x$, as y' passes through y, if $f'_y \neq 0$. In like manner this difference changes sign when $y' = y$, as x' passes through x, if $f'_x \neq 0$. Hence it is impossible for $f(x, y)$ to be a maximum or a minimum unless $f'_x = 0$ and $f'_y = 0$.

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* For examples of the uncertain case in which the function may be a maximum, a minimum, or neither, see Exercises 22, 25, at the end of this chapter.

When $f'_x = 0$, $f'_y = 0$, we have

$$f(x', y') - f(x, y) = (x' - x)^2 \frac{\partial^2 f}{\partial \xi^2} + 2(x' - x)(y' - y)\frac{\partial^2 f}{\partial \xi\, \partial \eta} + (y' - y)^2 \frac{\partial^2 f}{\partial \eta^2}.$$

If the member on the right of this equation retains its sign unchanged for all values of x', y' in the neighborhood of x, y, the function will be a maximum or a minimum at x, y. But in this neighborhood the sign of the member on the right is the same as that of its limit,

$$(x' - x)^2 \frac{\partial^2 f}{\partial x^2} + 2(x' - x)(y' - y)\frac{\partial^2 f}{\partial x\, \partial y} + (y' - y)^2 \frac{\partial^2 f}{\partial y^2}.$$

This gives the same conditions as in I, and leads to the same results.

EXAMPLES.

1. Find the maximum value of $z = 3axy - x^3 - y^3$.

This is a surface which cuts the horizontal plane in the folium of Descartes. Here

$$\frac{\partial z}{\partial x} = 3ay - 3x^2, \quad \frac{\partial z}{\partial y} = 3ax - 3y^2, \tag{1}$$

$$\frac{\partial^2 z}{\partial x^2} = -6x, \quad \frac{\partial^2 z}{\partial y^2} = -6y, \quad \frac{\partial^2 z}{\partial x\, \partial y} = 3a. \tag{2}$$

The equations (1) furnish

$$3ay - 3x^2 = 0, \quad 3ax - 3y^2 = 0, \tag{3}$$

for finding the values of x, y at which a maximum or a minimum may occur. Solving (3), we have

$$x = 0, \quad y = 0, \quad \text{and} \quad x = a, \quad y = a.$$

For $x = 0$, $y = 0$,

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^2 = -9a^2,$$

and there can be neither maximum nor minimum at 0, 0.

For $x = a$, $y = a$,

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^2 = +27a^2;$$

and since $\frac{\partial^2 z}{\partial x^2} = -6a$, we have the conditions for a maximum value of z at a, a fulfilled. Hence at a, a the function has a maximum value a^3.

2. Show that $a^3/27$ is a maximum value of

$$(a - x)(a - y)(x + y - a).$$

3. Find the maximum value of $x^2 + xy + y^2 - ax - by$.

Ans. $\frac{1}{3}(ab - a^2 - b^2)$.

4. Show that $\sin x + \sin y + \cos(x + y)$ is a minimum when $x = y = \frac{3}{2}\pi$, a maximum when $x = y = \frac{1}{6}\pi$.

5. Show that the maximum value of

$$(ax + by + c)^2/(x^2 + y^2 + 1) \quad \text{is} \quad a^2 + b^2 + c^2.$$

6. Find the greatest rectangular parallelopiped that can be inscribed in the ellipsoid. That is, find the maximum value of $8xyz$ subject to the condition

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1. \tag{1}$$

Let $u = xyz$. Substituting the value of z in this from (1), we reduce u to a function of two variables,

$$u^2 = x^2y^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right).$$

From $\dfrac{\partial u^2}{\partial x} = 0$, $\dfrac{\partial u^2}{\partial y} = 0$, we find the only values which satisfy the conditions $x = a/\sqrt{3}$, $y = b/\sqrt{3}$. These give $z = c/\sqrt{3}$, and the volume required is $8abc/3\sqrt{3}$.

7. Show that the maximum value of $x^2y^3z^4$, when $2x + 3y + 4z = a$, is $(a/9)^9$.

8. Show that the surface of a rectangular parallelopiped of given volume is least when the solid is a cube.

9. Design a steel cylindrical standpipe of uniform thickness to hold a given volume, which shall require the least amount of material in the construction. [Radius of base = depth.]

10. Design a rectangular tank under the same conditions as Ex. 9. [Base square, depth = ½ side of base.]

11. The function $z = x^2 + xy + y^2 - 5x - 4y + 1$ has a minimum for $x = 2$, $y = 1$.

12. Show that the maximum or minimum value of

$$z = ax^2 + by^2 + 2hxy + 2gx + 2fy + c \tag{1}$$

is

$$z = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \div \begin{vmatrix} a & h \\ h & b \end{vmatrix}.$$

We have

$$\frac{1}{2}\frac{\partial z}{\partial x} = ax + hy + g = 0, \quad \frac{1}{2}\frac{\partial z}{\partial y} = hx + by + f = 0. \tag{2}$$

Multiply the first by x, the second by y, subtract their sum from (1), and we get

$$z = gx + fy + c. \tag{3}$$

Eliminating x and y between (2), (3), the result follows.

The condition shows that when $ab - h^2$ is positive, the above value of z is a maximum or minimum according as the sign of a is negative or positive. If $ab - h^2 = -$, then z is neither maximum nor minimum. We recognize the surface as a paraboloid, elliptic for $ab - h^2$ positive, and hyperbolic when $ab - h^2 = -$.

13. Investigate $z = x^2 + 3y^2 - xy + 3x - 7y + 1$ for maximum and minimum values of z.

14. Investigate max. and min. of $x^4 + y^4 - x^2 + xy - y^2$.

$x = 0$, $y = 0$, max.; $x = y = \pm \frac{1}{2}$, min.; $x = -y = \pm \frac{1}{2}\sqrt{3}$, min.

15. The function $(x - y)^2 - 4y(x - 8)$ has neither maximum nor minimum.

16. The surface $x^2 + 2y^2 - 4x + 4y + 3z + 15 = 0$ has a maximum z-ordinate at the point $(2, -1, -3)$.

17. The function $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ has neither maximum nor minimum for $x = 0$, $y = 0$; but is minimum at $(+\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, +\sqrt{2})$.

18. Show that $\cos x \cos \alpha + \sin x \sin \alpha \cos (y - \beta)$ is a maximum when $x = \alpha, y = \beta$.

19. Show that $x^2 - 6xy^2 + cy^4$ at 0, 0 is minimum if $c > 9$, and is neither maximum nor minimum for other values of c. Hint. Complete the square in x.

20. Show that $(1 + x^2 + y^2)/(1 - ax - by)$ has a maximum and a minimum respectively at

$$\frac{x}{a} = \frac{y}{b} = \frac{1 \pm \sqrt{1 + a^2 + b^2}}{a^2 + b^2}.$$

21. Show that 3, 2 make $x^3y^2(6 - x - y)$ a maximum.

22. Show that a, b make $(2ax - x^2)(2by - y^2)$ a maximum.

23. Show that $3 + 4\sqrt{2}$ is a maximum, $-6 - 4\sqrt{2}$ a minimum, value of

$$y^4 - 8y^3 + 18y^2 - 8y + x^3 - 3x^2 - 3x.$$

214. Maxima and Minima Values of a Function of Three Independent Variables.

Let

$$u = f(x, y, z),$$

$$x_1 - x = lr = h, \quad y_1 - y = mr = k, \quad z_1 - z = nr = g.$$

As before, if u is a maximum or a minimum at x, y, z, we must have

$$u = f(x + lr, \quad y + mr, \quad z + nr),$$

a maximum or a minimum for all values of l, m, n, or

$$\frac{du}{dr} = l\frac{\partial u}{\partial x} + m\frac{\partial u}{\partial y} + n\frac{\partial u}{\partial z},$$

$$= \frac{x_1 - x}{r}\frac{\partial u}{\partial x} + \frac{y_1 - y}{r}\frac{\partial u}{\partial y} + \frac{z_1 - z}{r}\frac{\partial u}{\partial z},$$

must be 0 for all values of l, m, n or of x_1, y_1, z_1 in the neighborhood of x, y, z, or

$$(x_1 - x)\frac{\partial u}{\partial x} + (y_1 - y)\frac{\partial u}{\partial y} + (z_1 - z)\frac{\partial u}{\partial z} = 0.$$

Hence the necessary conditions

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = 0. \tag{1}$$

Now when the relations (1) hold, and for all values of x_1, y_1, z_1 in the neighborhood of x, y, z, we also have

$$\frac{d^2u}{dr^2} = l^2 f''_{xx} + m^2 f''_{yy} + n^2 f''_{zz} + 2lm f''_{x} + 2ln f''_{xz} + 2mn f''_{yz},$$

or, what is the same thing,

$$Ah^2 + Bk^2 + Cg^2 + 2Fkg + 2Ghg + 2Hhk \tag{2}$$

(wherein $A = f''_{xx}$, $B = f''_{yy}$, $C = f''_{zz}$, $F = f''_{yz}$, $G = f''_{xz}$, $H = f''_{xy}$) negative (positive) for all values of h, k, g, then will u be a maximum (minimum).

The condition that (2) shall keep its sign unchanged for all values of h, k, g has been determined in Ex. 20, § 25, where it is shown that when

$$\begin{vmatrix} A & H \\ H & B \end{vmatrix} \quad \text{and} \quad A\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

are both positive (2) has the same sign as A for all values of h, k, g.

Therefore $f(x, y, z)$ is a maximum or a minimum at x, y, z, determined from

$$f_x' = 0, \quad f_y' = 0, \quad f_z' = 0,$$

when we have

$$f_{xx}'' = \mp\,{}^{\text{max.}}_{\text{min.}}, \quad \begin{vmatrix} f_{xx}'' & f_{yx}'' \\ f_{xy}'' & f_{yy}'' \end{vmatrix} = +, \quad \begin{vmatrix} f_{xx}'' & f_{yx}'' & f_{zx}'' \\ f_{xy}'' & f_{yy}'' & f_{zy}'' \\ f_{xz}'' & f_{yz}'' & f_{zz}'' \end{vmatrix} = \mp\,{}^{\text{max.}}_{\text{min.}}$$

The conditions for maximum or minimum can be frequently inferred from the geometrical conditions of a geometrical problem, without having to resort to the complicated tests involving the second derivatives.

EXAMPLES.

1. $$f \equiv x^2 + y^2 + z^2 + x - 2z - xy.$$

$$f_x' = 2x - y + 1 = 0, \quad f_y' = 2y - x = 0, \quad f_z' = 2z - 2 = 0.$$

$$\therefore \quad x = -\tfrac{2}{3}, \quad y = -\tfrac{1}{3}, \quad z = 1, \quad \text{give } f = -\tfrac{4}{3}.$$

Also, $f_{xx}'' = 2$, $f_{yy}'' = 2$, $f_{zz}'' = 2$, $f_{xy}'' = -1$, $f_{xz}'' = 0$, $f_{yz}'' = 0$.

$$\begin{vmatrix} 2 & \bar{1} \\ \bar{1} & 2 \end{vmatrix} = 3, \quad \begin{vmatrix} 2 & \bar{1} & 0 \\ \bar{1} & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6.$$

Therefore $-4/3$ is a minimum value of f.

2. Find the maximum and minimum values of

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy + 2ux + 2vy + 2wz + d.$$

Here

$$\left.\begin{aligned} f_x' &= 2(ax + hy + gz + u) = 0, \\ f_y' &= 2(hx + by + fz + v) = 0, \\ f_z' &= 2(gx + fy + cz + w) = 0. \end{aligned}\right\} \tag{1}$$

Multiply the first by x, the second by y, the third by z. Add together and subtract the result from the function f.

$$\therefore \quad f = ux + vy + wz + d. \tag{2}$$

Eliminating x, y, z between (1) and (2), we have

$$f = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} \div \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

which is a maximum or a minimum according as

$$a = \mp, \quad \begin{vmatrix} a & h \\ h & b \end{vmatrix} = +, \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \mp,$$

the upper and lower signs going together.

3. Find a point such that the sum of the squares of its distances from three given points is a minimum.

Let $x_1, y_1, z_1, \ldots x_3, y_3, z_3$, be the given points. Then

$$f = \Sigma_1^3 [(x - x_r)^2 + (y - y_r)^2 + (z - z_r)^2],$$
$$f'_x = 2\Sigma(x - x_r) = 0 = 3x - \Sigma x_r,$$
$$f'_y = 2\Sigma(y - y_r) = 0 = 3y - \Sigma y_r,$$
$$f'_z = 2\Sigma(z - z_r) = 0 = 3z - \Sigma z_r.$$

$\therefore \quad x = \frac{1}{3}(x_1 + x_2 + x_3), \quad y = \frac{1}{3}(y_1 + y_2 + y_3), \quad z = \frac{1}{3}(z_1 + z_2 + z_3).$

The point is therefore the *centroid* of the three given points.

$f''_{xx} = f''_{yy} = f''_{zz} = 6$, $f''_{xy} = f''_{xz} = f''_{yz} = 0$. Show that the solution is a minimum.

Extend the problem to the case of n given points.

4. If $w = ax^2 + byx + dz^2 + lxy + myz$, show that $x = y = z = 0$ gives neither a maximum nor a minimum.

215. Maximum and Minimum for an Implicit Function of Three Variables.—To find the maximum or minimum values of z in

$$f(x, y, z) = 0.$$

Since the total differentials of f are 0, we have

$$df = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)f = 0, \tag{1}$$

$$d^2f = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)^2 f + \frac{\partial f}{\partial x}d^2x + \frac{\partial f}{\partial y}d^2y + \frac{\partial f}{\partial z}d^2z = 0. \tag{2}$$

Also, at a maximum or a minimum value of z we must have

$$dz = -\frac{\dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy}{\dfrac{\partial f}{\partial z}} = 0$$

for all values of dy and dx. It is therefore necessary that

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} \neq 0. \tag{3}$$

Substituting these values in (2), we have at the values of x, y which satisfy (3), and make $dz = 0$,

$$d^2z = -\frac{\left(dx\dfrac{\partial}{\partial x} + dy\dfrac{\partial}{\partial y}\right)^2 f}{\dfrac{\partial f}{\partial z}},$$

$$= -\frac{dx^2 f''_{xx} + 2dy\,dx\,f''_{xy} + dy^2 f''_{yy}}{f'_z}.$$

In order that this shall retain its sign for all values of dy and dx, we must have

$$f''_{xx}f''_{yy} - (f''_{xy})^2 = +. \tag{4}$$

Then the sign of d^2z is that of $f'_z f''_{xx}$. (See Ex. 19, p. 31.)

Hence z will be a maximum (minimum) at x, y, z, determined from

$$f'_x = 0, \quad f'_y = 0, \quad f = 0,$$

when $f'_z f''_{xx}$ is positive (negative), provided (4) is true.

EXAMPLES.

1. Find the maximum and minimum of z in

$$2x^2 + 5y^2 + z^2 - 4xy - 2x - 4y - \tfrac{1}{2} = 0.$$

$$f'_x = 4x - 4y - 2 = 0, \quad f'_y = 10y - 4x - 4 = 0,$$

give

$$x = \tfrac{3}{2}, \quad y = 1, \quad z = \pm 2.$$

$$f'_z = 2z = \pm 4, \quad f''_{xx} = 4, \quad f''_{xx} f''_{yy} - (f''_{xy})^2 = 24.$$

z is therefore a maximum and a minimum at $\frac{3}{2}$, 1.

2. Show that z in $z^3 + 3x^2 - 4xy + yz = 0$ has neither a maximum nor a minimum at $x = -\frac{3}{64}, y = -\frac{9}{128}, \quad z = -\frac{3}{16}$.

216. Conditional Maximum and Minimum. — Consider the determination of the maximum or minimum value of $z = f(x, y)$, when x and y are subject to the condition $\phi(x, y) = 0$.

Geometrically illustrated, $z = f(x, y)$ and $\phi(x, y) = 0$ are the equations of the line of intersection of the surface $z = f$ and the vertical cylinder $\phi = 0$. We seek the highest and lowest points of this curve.

Since, at a maximum or minimum value of z,

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0, \tag{1}$$

also

$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0, \tag{2}$$

we have, eliminating dy, dx, the equation

$$f'_x \phi'_y - f'_y \phi'_x = 0 \tag{3}$$

to be satisfied by x, y at which a maximum or minimum occurs. Equation (3) together with $\phi = 0$ determines x and y for which a maximum or minimum may occur.

Usually the conditions of the problem serve to discriminate between a maximum, minimum, or inflexion at the critical values of x, y.

The test of the second derivative, however, can be applied as follows: We have

$$d^2z = f''_{xx} dx^2 + 2f''_{xy} dx\,dy + f''_{yy} dy^2 + f'_x d^2x + f'_y d^2y, \tag{4}$$

which must keep its sign unchanged for all values of x, y satisfying $\phi = 0$ in the neighborhood of the x, y also satisfying (3). But we also have

$$\phi''_{xx} dx^2 + 2\phi''_{xy} dx\,dy + \phi''_{yy} dy^2 + \phi'_x d^2x + \phi'_y d^2y = 0. \tag{5}$$

To eliminate the differentials from (4), (5), multiply (4) by ϕ'_y, (5) by f'_y, and subtract, having regard for (3). In the result substitute for dy/dx from (2).

$$d^2z = \frac{(dx)^2}{(\phi'_y)^3}\left\{\begin{vmatrix}\phi'_y, & f'_y\\ \phi''_{xx} & f''_{xx}\end{vmatrix}\phi'^2_y - 2\begin{vmatrix}\phi'_y, & f'_y\\ \phi''_{xy} & f''_{xy}\end{vmatrix}\phi'_x\phi'_y + \begin{vmatrix}\phi'_y, & f'_y\\ \phi''_{yy} & f''_{yy}\end{vmatrix}\phi'^2_x\right\}. \quad (6)$$

When this is negative (positive) we have a maximum (minimum) value of z. The form of the test (6) is too complicated to be very useful, and it is usually omitted.

EXAMPLES.

1. Find the minimum value of $x^2 + y^2$ when x and y are subject to the condition $ax + by + d = 0$.

Condition (3) gives $bx = ay$. Therefore, at

$$x = -\frac{ad}{a^2+b^2}, \quad y = -\frac{bd}{a^2+b^2},$$

we have

$$x^2+y^2 = \frac{d^2}{a^2+b^2},$$

which can be shown to be a minimum by (6). Otherwise we see at once from the geometrical interpretations that this value of $x^2 + y^2$ must be a minimum.

First. $\sqrt{x^2+y^2}$ is the distance from the origin, of the point x, y which is on the straight line $ax + by + d = 0$, and this is least when it is the perpendicular from the origin to the straight line, which was found above.

Second. $z = x^2 + y^2$ is the paraboloid of revolution. The vertical plane $ax + by + d = 0$ cuts it in a parabola, whose vertex we have found above, and which is the lowest point on the curve.

2. Determine the axes of the conic $ax^2 + by^2 + 2hxy = 1$.

Here the origin is in the center, and the semi-axes are the greatest and least distances of a point on the curve from the origin. We have to find the maximum and minimum values of $x^2 + y^2$, subject to the above condition of x, y being on the conic.

Let $u = x^2 + y^2$ and $\phi = ax^2 + by^2 + 2hxy - 1 = 0$.

Condition (3) gives

$$\frac{x}{y} = \frac{ax+hy}{by+hx}.$$

Multiply both sides by x/y and compound the proportion, and we get

$$(a - u^{-1})x + hy = 0,$$
$$hx + (b - u^{-1})y = 0.$$

Eliminating x and y, there results

$$\begin{vmatrix} a - u^{-1}, & h \\ h\quad, & b - u^{-1}\end{vmatrix} \equiv \frac{1}{u^2} - (a+b)\frac{1}{u} + ab - h^2 = 0$$

for determining the maximum and minimum values of u.

217. The whole question of *conditional* maximum and minimum is most satisfactorily treated by the method of undetermined multipliers of Lagrange.

The process is best illustrated by taking an example sufficiently general to include all cases that are likely to occur and at the same time to point out the general treatment for any case that can occur.

To find the maximum and minimum values of

$$u \equiv f(x, y, z, w) \tag{1}$$

when the variables x, y, z, w are subject to the conditions

$$\phi(x, y, z, w) = 0, \tag{2}$$

$$\psi(x, y, z, w) = 0. \tag{3}$$

Since, at a maximum or minimum value of u, we must have $du = 0$, the conditions furnish

$$\left.\begin{array}{l} f'_x\,dx + f'_y\,dy + f'_z\,dz + f'_w\,dw = 0, \\ \phi'_x\,dx + \phi'_y\,dy + \phi'_z\,dz + \phi'_w\,dw = 0, \\ \psi'_x\,dx + \psi'_y\,dy + \psi'_z\,dz + \psi'_w\,dw = 0. \end{array}\right\} \tag{4}$$

Multiply the second of these by λ, the third by μ, λ and μ being arbitrary numbers. Add the three equations.

$$\begin{aligned} &(f'_x + \lambda\phi'_x + \mu\psi'_x)dx + (f'_y + \lambda\phi'_y + \mu\psi'_y)dy + \\ &(f'_z + \lambda\phi'_z + \mu\psi'_z)dz + (f'_w + \lambda\phi'_w + \mu\psi'_w)dw = 0. \end{aligned} \tag{5}$$

Since λ and μ are perfectly arbitrary, we can assign to them values which will make the coefficients of dx and dy vanish; moreover, since equations (2) and (3) connect four variables, we can take two of them, say z and w, independent, and therefore dz and dw are arbitrary. Consequently, in (5), after assigning λ and μ as above, we must have the coefficients of dz and dw equal to 0. Therefore

$$\left.\begin{array}{l} f'_x + \lambda\phi'_x + \mu\psi'_x = 0, \\ f'_y + \lambda\phi'_y + \mu\psi'_y = 0, \\ f'_z + \lambda\phi'_z + \mu\psi'_z = 0, \\ f'_w + \lambda\phi'_w + \mu\psi'_w = 0. \end{array}\right\} \tag{6}$$

The six equations (2), (3), (6) enable us to determine x, y, z, w, λ, μ, which furnish the maxima and minima values of u.

The discrimination between a maximum and a minimum by means of the higher derivatives is too complicated for our investigation. In ordinary problems this discrimination can generally be made through the conditions of the problem proposed.

minimum

EXAMPLES.

1. Find the maximum value of $u = x^2 + y^2 + z^2$ when x, y, z are subject to the condition

$$\phi = ax + by + cz + d = 0.$$

Here we have, as in equations (6),

$$\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 = 2x + \lambda a,$$

$$\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 = 2y + \lambda b,$$

$$\frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 = 2z + \lambda c.$$

Multiply by a, b, c and add. Also, transpose and square. Then

$$2(ax + by + cz) + (a^2 + b^2 + c^2)\lambda \equiv -2d + (a^2 + b^2 + c^2)\lambda = 0,$$
$$4(x^2 + y^2 + z^2) - (a^2 + b^2 + c^2)\lambda^2 \equiv 4u - (a^2 + b^2 + c^2)\lambda^2 = 0.$$
$$\therefore \quad \sqrt{u} = \frac{d}{\sqrt{a^2 + b^2 + c^2}}.$$

The problem is to find the perpendicular distance from the origin to a plane.

2. Find when $u = x^2 + y^2 + z^2$ is a maximum or minimum, x, y, z being subject to the two conditions

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1, \quad lx + my + nz = 0.$$

Geometrically interpreted: Find the axes of a central plane section of an ellipsoid.

Equations (6) give

$$2x + \lambda\frac{2x}{a^2} + \mu l = 0.$$
$$2y + \lambda\frac{2y}{b^2} + \mu m = 0,$$
$$2z + \lambda\frac{2z}{c^2} + \mu n = 0.$$

Multiply by x, y, z and add. We get $\lambda = -u$. Therefore

$$lx = \frac{\mu a^2 l^2}{2(u - a^2)}, \quad my = \frac{\mu b^2 m^2}{2(u - b^2)}, \quad nz = \frac{\mu c^2 n^2}{2(u - c^2)}.$$

Hence the required values of u are the roots of the quadratic

$$\frac{a^2 l^2}{u - a^2} + \frac{b^2 m^2}{u - b^2} + \frac{c^2 n^2}{u - c^2} = 0.$$

3. Find the maximum and minimum values of

$$u = a^2x^2 + b^2y^2 + c^2z^2,$$

x, y, z being subject to the conditions

$$x^2 + y^2 + z^2 = 1, \quad lx + my + nz = 0.$$

The required values are the roots of the quadratic

$$l^2/(u - a^2) + m^2/(u - b^2) + n^2/(u - c^2) = 0.$$

4. Find the maximum and minimum values of

$$u = x^2 + y^2 + z^2$$

when x, y, z are subject to the condition

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy = 1.$$

Geometrically interpreted: Find the axes of a central conicoid.

The conditions (6) give

$$x + (ax + hy + gz)\lambda = 0,$$
$$y + (hx + by + fz)\lambda = 0,$$
$$z + (gx + fy + cz)\lambda = 0.$$

Multiply by x, y, z and add. $\therefore \quad \lambda = -u.$

Eliminating x, y, z from the above equations,

$$\begin{vmatrix} a - u^{-1}, & h, & g \\ h, & b - u^{-1}, & f \\ g, & f, & c - u^{-1} \end{vmatrix} = 0.$$

The three real roots of this cubic, see Ex. 17, § 25, furnish the squares of the semi-axes of the conicoid.

5. Show how to determine the maxima and minima values of $x^2 + y^2 + z^2$ subject to the conditions

$$(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2,$$
$$lx + my + nz = 0.$$

EXERCISES.

1. Show that the area of a quadrilateral of four given sides is greatest when it is inscribable in a circle.

2. Also, show that the area of a quadrilateral with three given sides and the fourth side arbitrary is greatest when the figure is inscribable in a circle.

3. Given the vertical angle of a triangle and its area, find when its base is least.

4. Divide a number a into three parts x, y, z such that $x^my^nz^p$ may be a maximum.

Ans. $\dfrac{x}{m} = \dfrac{y}{n} = \dfrac{z}{p} = \dfrac{a}{m+n+p}$.

5. Find the maximum value xy subject to the condition $x^2/a^2 + y^2/b^2 = 1$. This finds the greatest rectangle that can be inscribed in a given ellipse.

6. Find a maximum value of xy subject to $ax + by = c$, and interpret the result geometrically.

7. Divide a into three parts x, y, z, such that $xy/2 + xz/3 + yz/4$ shall be a maximum.

Ans. $x/21 = y/20 = z/6 = a/47$.

8. Find the maximum value of xyz subject to the condition

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

by the method of § 216.

9. Show that $x + y + z$ subject to $a/x + b/y + c/z = 1$ is a minimum when

$$x/\sqrt{a} = y/\sqrt{b} = z/\sqrt{c} = \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

10. Find a point such that the sum of the squares of its distances from the corners of a tetrahedron shall be least.

11. If each angle of a triangle is less than 120°, find a point such that the sum of its distances from the vertices shall be least. [The sides must subtend 120° at the point.]

12. Determine a point in the plane of a triangle such that the sum of the squares of its distances from the sides a, b, c is least. Δ being the area of the triangle.

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{2\Delta}{a^2 + b^2 + c^2}.$$

13. Circular sectors are taken off the corners of a triangle. Show how to leave the greatest area with a given perimeter. [The radii of the sectors are equal.]

14. In a given sphere inscribe a rectangular parallelopiped whose surface is greatest; also whose volume is greatest. [Cube.]

15. Find the shortest distance from the origin to the straight line.

$$\left.\begin{array}{l} l_1x + m_1y + n_1z = p_1, \\ l_2x + m_2y + n_2z = p_2. \end{array}\right\}$$

The equations of the planes being in the normal form.

We have, if $u^2 = x^2 + y^2 + z^2$,

$$\begin{aligned} 2x + l_1\lambda + l_2\mu &= 0,\\ 2y + m_1\lambda + m_2\mu &= 0,\\ 2z + n_1\lambda + n_2\mu &= 0. \end{aligned}$$

Multiply these by x, y, z in order and add. Multiply by l_1, m_1, n_1 in order and add. Multiply by l_2, m_2, n_2 in order and add. Whence the equations

$$\begin{aligned} 2u^2 + p_1\lambda + p_2\mu &= 0,\\ 2p_1 + \lambda + \cos\theta\,\mu &= 0,\\ 2p_2 + \cos\theta\,\lambda + \mu &= 0. \end{aligned}$$

Since $l_1^2 + m_1^2 + n_1^2 = l_2^2 + m_2^2 + n_2^2 = 1$, $l_1l_2 + m_1m_2 + n_1n_2 = \cos\theta$, where θ is the angle between the normals to the planes. Eliminating λ and μ, we have

$$\begin{vmatrix} u^2 & p_1 & p_2 \\ p_1 & 1 & \cos\theta \\ p_2 & \cos\theta & 1 \end{vmatrix} = 0,$$

or
$$u^2 \sin^2\theta = p_1^2 + p_2^2 - 2p_1p_2\cos\theta,$$

which result is easily verified geometrically as being the perpendicular from the origin to the straight line.

16. A given volume of metal, v, is to be made into a rectangular box; the sides and bottom are to be of a given thickness a, and there is no top.

Find the shape of the vessel so that it may have a maximum capacity.

If x, y, z are the external length, breadth, depth,

$$x = y = a + \sqrt{\frac{v - a^3}{3a}}; \quad z = \tfrac{1}{2}x.$$

17. Find a point such that the sum of the squares of its distances from the faces of a tetrahedron shall be least. If V is the volume of the solid, x, y, z, w the perpendicular distances of the point from the faces whose areas are A, B, C, D, then

$$\frac{x}{A} = \frac{y}{B} = \frac{z}{C} = \frac{w}{D} = \frac{3V}{A^2 + B^2 + C^2 + D^2}.$$

18. Of all the triangular pyramids having a given triangle for base and a given altitude above that base, find that one which has the least surface.

The surface is $\frac{1}{2}(a + b + c)\sqrt{r^2 + h^2}$, where a, b, c are the sides of the base, r the radius of the circle inscribed in the base, h the given altitude.

19. Show that the maximum of $(ax + by + cz)e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}$ is given by

$$\frac{a}{\alpha^2 x} = \frac{b}{\beta^2 y} = \frac{c}{\gamma^2 z} = \sqrt{2\left(\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}\right)}.$$

20. Show that the highest and lowest points on a curve whose equations are

$$\phi(x, y, z) = 0, \qquad \psi(x, y, z) = 0, \tag{1}$$

are determined from these equations and

$$\phi_x' + \lambda\psi_x' = 0, \quad \phi_y' + \lambda\psi_y' = 0. \tag{2}$$

21. Show that the maximum and minimum values of $r^2 = x^2 + y^2 + z^2$, where x, y, z are subject to the two conditions

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy = 1, \quad lx + my + nz = 0,$$

are given by the roots of the quadratic,

$$\begin{vmatrix} a - r^{-2} & h & g & l \\ h & b - r^{-2} & f & m \\ g & f & c - r^{-2} & n \\ l & m & n & 0 \end{vmatrix} = 0.$$

Geometrically, this finds the axes of any central plane section of a conicoid with origin at the center. It also solves the problem of finding the principal radii of curvature of a surface at any point.

The following four exercises are given to illustrate the uncertain case of maximum and minimum conditions.

22. Investigate $z = 2x^2 - 3xy^2 + y^4 = (y^2 - x)(y^2 - 2x)$.

At 0, 0 we have $z'_x = z'_y = z''_{xy} = z''_{yy} = 0$, $z''_{xx} = 4$. The conditions $z''_{xx}\, z''_{yy} - (z''_{xy})^2 = 0$ makes the case uncertain. The function z vanishes along each of the parabolæ $y^2 = x$, $y^2 = 2x$. It is positive for all values x, y in the plane $z = 0$, except between the two parabolæ, where it is negative. The function is neither a maximum nor a minimum at 0, 0, since it has positive and negative values in the neighborhood of that point. In fact z is negative all along $y^2 = 3x/2$ except at 0, 0.

23. $$z = a^2y^2 - 2ax^2y + x^4 + y^4.$$

At 0, 0 the case is uncertain. Put $y = mx$, then

$$z = x^2[(1 + m^4)x^2 - 2amx + a^2m^2].$$

When x or y is 0 the function is positive. For all values of m the quadratic factor in the brace is positive.

Hence z is a minimum at 0, 0.

24. $$z = y^2 - xy^2 - 2x^2y + x^4.$$

As in 23, the condition is uncertain at 0, 0. Put $y = mx$. Then

$$z = x^2[x^2 - m(m + 2)x + m^2].$$

The function is positive when x or y is 0. For any value of m not arbitrarily small z is positive for all arbitrarily small values of x. But since

$$m^2(\tfrac{3}{4}m^2 - \tfrac{1}{2}m - 1)$$

is negative for all arbitrarily small values of m, the quadratic function of x in the brace has two small positive roots for each such value of m. Between each pair of these arbitrarily small roots the quadratic factor, and therefore z, is negative. The function is neither a maximum nor a minimum at 0, 0. In fact along the curve $x^2 = y$ the function is $z = -x^5$.

25. Consider the function z defined by the equation

$$(z - a)^2 + (\sqrt{x^2 + y^2} - a)^2 = a^2,$$

or

$$z = a - \sqrt{2a\rho - \rho^2},$$

wherein the positive value of the radical is taken and $\rho^2 = x^2 + y^2$. This is the lower half of the surface generated by revolving the circle $(x - a)^2 + y^2 = a^2$ about the y-axis.

Here

$$\frac{\partial z}{\partial x} = -\frac{x}{\rho^{\frac{3}{2}}}\,\frac{a - \rho}{\sqrt{2a - \rho}}, \quad \frac{\partial z}{\partial y} = -\frac{y}{\rho^{\frac{3}{2}}}\,\frac{a - \rho}{\sqrt{2a - \rho}}.$$

At all points satisfying $x^2 + y^2 = \rho^2 = a^2$ these derivatives are 0. Also at such points

$$z''_{xx} = x^2/a^3, \quad z''_{yy} = y^2/a^3, \quad z''_{xy} = xy/a^3.$$

$$\therefore \quad z''_{xx}\, z''_{yy} - (z''_{xy})^2 = 0.$$

The function z is 0 at each point x, y satisfying $x^2 + y^2 = a^2$, and is positive for every other x, y. It is neither a maximum nor a minimum, nor does it change sign in the neighborhood of any x, y in $x^2 + y^2 = a^2$ We shall see later that the plane $z = 0$ is a *singular* tangent plane to the surface.

CHAPTER XXXI.

APPLICATION TO PLANE CURVES.

I. ORDINARY POINTS.

218. We have seen that when the equation of a curve is given in the explicit form $y = f(x)$, and the ordinate is one-valued, or two-valued in such a way that the branches can be separated, the curve can be investigated by means of the derivatives of y with respect to x, or through the law of the mean, as given in Book I, for functions of one variable.

In the same way, when the equation of the curve is given in the implicit form $F(x,y) = 0$, we can investigate the curve through the partial derivatives and the law of the mean for functions of two variables. This amounts, geometrically, to considering the surface $z = F(x, y)$, whose intersection with the plane $z = 0$ is the curve we wish to investigate.*

219. Ordinary Point.—If $F(x, y) = 0$ is the equation to a curve, then any point x, y at which we do not have both

$$\frac{\partial F}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} = 0$$

is called a *single* point on the curve, or a point of *ordinary position*, or simply an *ordinary* point.

By the law of the mean,

$$F(x, y) = F(a, b) + (x - a)\frac{\partial F}{\partial \xi} + (y - b)\frac{\partial F}{\partial \eta}.$$

If $F(x, y) = 0$, and a, b is an ordinary point on this curve, then $F(a, b) = 0$. Hence

$$0 = (x - a)\frac{\partial F}{\partial \xi} + (y - b)\frac{\partial F}{\partial \eta}.$$

From this we derive for $x(=)a, y(=)b$,

$$\frac{dy}{dx} = -\frac{\partial F}{\partial x}\bigg/\frac{\partial F}{\partial y}.$$

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* For convenience of notation we shall generally write the explicit equation to a curve in the form $y = f(x)$, and the implicit equation as $F(x, y) = 0$.

Therefore the curve $F(x, y) = 0$ and the straight line

$$(x - a)\frac{\partial F}{\partial a} + (y - b)\frac{\partial F}{\partial b} = 0 \tag{1}$$

have a contact of the first order at a, b, or (1) is the equation of the tangent to the curve at a, b.

We propose to deduce the equation to the tangent at length, in order to lead up to the general methods which are to follow.

Let $F(x, y) = 0$ be the equation to a curve, then

$$0 = F(x, y) + (X - x)\frac{\partial F}{\partial x} + (Y - y)\frac{\partial F}{\partial y} + \frac{1}{2!}\left\{(X - x)\frac{\partial}{\partial \xi} + (Y - y)\frac{\partial}{\partial \eta}\right\}^2 F \tag{2}$$

is the equation to the curve in the form of the law of the mean. The straight line

$$\frac{X - x}{l} = \frac{Y - y}{m} = r \tag{3}$$

intersects this curve in points whose distances from x, y are the roots of the equation in r,

$$0 = F(x, y) + r\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right)F + \frac{r^2}{2}\left(l\frac{\partial}{\partial \xi} + m\frac{\partial}{\partial \eta}\right)^2 F. \tag{4}$$

If the point x, y is on the curve, this is one point of intersection, and one root of (4) is 0, for $F(x, y) = 0$.

If in addition we have

$$l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} = 0, \tag{5}$$

then two roots of (4) are 0, and the line (3) cuts the curve in two coincident points at x, y, and is by definition a tangent to the curve at x, y.

Eliminating l, m between the condition of tangency (5) and the equation to the straight line (3), we have the equation to the tangent at x, y,

$$(X - x)\frac{\partial F}{\partial x} + (Y - y)\frac{\partial F}{\partial y} = 0, \tag{6}$$

the current coordinates being X, Y.

The corresponding equation to the normal at x, y is

$$\frac{X - x}{\dfrac{\partial F}{\partial x}} = \frac{Y - y}{\dfrac{\partial F}{\partial y}}. \tag{7}$$

EXAMPLES.

1. Use Ex. 3, § 211, to show that if $F(x, y) = c$ is the equation to a curve, in which $F(x, y)$ is homogeneous of degree n, then the length of the perpendicular from the origin on the tangent is

$$p = \frac{nc}{\sqrt{F_x'^2 + F_y'^2}}.$$

2. If $F(x, y) \equiv u_n + u_{n-1} + \ldots + u_1 + u_0 = 0$ is the equation of a curve of nth degree, in which u_r is the homogeneous part of degree r, show that the equation of the tangent at x, y is

$$X\frac{\partial F}{\partial x} + Y\frac{\partial F}{\partial y} + u_{n-1} + 2u_{n-2} + \ldots + nu_0 = 0.$$

If X, Y is a fixed point, this is a curve of the $(n - 1)$th degree in x, y which intersects $F(x, y) = 0$ in $n(n - 1)$ points, real or imaginary. These points of intersection are the points of contact of the $n(n - 1)$ tangents which can be drawn from any point X, Y to a curve $F = 0$ of the nth degree.

3. If X, Y be a fixed point, the equation of the normal through X, Y to $F = 0$ at x, y is

$$(X - x)\frac{\partial F}{\partial y} = (Y - y)\frac{\partial F}{\partial x}.$$

This is of the nth degree in x, y, which intersects $F = 0$ in n^2 points, real or imaginary, the normals at which to $F = 0$ all pass through X, Y. There can then, in general, be drawn n^2 normals to a given curve of the nth degree from any given point.

4. Show that the points on the ellipse $x^2/a^2 + y^2/b^2 = 1$ at which the normals pass through a given point α, β are determined by the intersection of the hyperbola

$$xy(a^2 - b^2) = \alpha a^2 y - \beta b^2 x$$

with the ellipse.

5. If $F(x, y) = 0$ is a conic, show that its equation can always be written

$$0 = F(a, b) + \left\{(x - a)\frac{\partial}{\partial a} + (y - b)\frac{\partial}{\partial b}\right\}F + \tfrac{1}{2}\left\{(x - a)\frac{\partial}{\partial a} + (y - b)\frac{\partial}{\partial b}\right\}^2 F. \quad (1)$$

(a). Show that the straight line whose equation is

$$\frac{x - a}{l} = \frac{y - b}{m} = r, \quad (2)$$

where $l = \cos\theta$, $m = \sin\theta$, cuts the curve in two points whose distances from a, b are the roots of the quadratic

$$0 = F(a, b) + r\left(l\frac{\partial}{\partial a} + m\frac{\partial}{\partial b}\right)F + \tfrac{1}{2}r^2\left(l\frac{\partial}{\partial a} + m\frac{\partial}{\partial b}\right)^2 F. \quad (3)$$

(b). Show that

$$(x - a)\frac{\partial F}{\partial a} + (y - b)\frac{\partial F}{\partial b} = 0 \quad (4)$$

is the equation of a secant of which a, b is the middle point of the chord.

(c). Show that the equations

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0,$$

solved simultaneously, give the coordinates of the center of the conic.

(d). Show that

$$\frac{x - a}{l} = \frac{y - b}{m} \quad \text{and} \quad l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} = 0$$

are the equations of a pair of conjugate diameters of the conic $F = 0$, whose center is a, b.

6. If $k^2 < 1$, show that the tangent to $x^2/a^2 + y^2/b^2 = k^2$ cuts off a constant area from $x^2/a^2 + y^2/b^2 = 1$.

7. In Ex. 5, show how to determine the axes and their directions in the conic $F = 0$, by finding the maximum and minimum values of r in the quadratic (3), as a function of θ, the center of the conic being a, b.

220. The Inflexional Tangent.—At an ordinary point x, y on the curve $F(x, y) = 0$, the straight line

$$\frac{X - x}{l} = \frac{Y - y}{m} = 0 \tag{1}$$

cuts the curve in points whose distances from x, y are the roots of the equation in r,

$$0 = r\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right)F + \tfrac{1}{2}r^2\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right)^2 F + \tfrac{1}{6}r^3\left(l\frac{\partial}{\partial \xi} + m\frac{\partial}{\partial \eta}\right)^3 F. \tag{2}$$

If we have

$$l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} = 0, \tag{3}$$

the line (1) cuts the curve in two coincident points at x, y, and is tangent to the curve there.

If, in addition to (3), l and m satisfy

$$\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right)^2 F = 0, \tag{4}$$

then the line cuts the curve $F = 0$ in three coincident points at x, y, provided

$$\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right)^3 F \neq 0. \tag{5}$$

In this case the line (1) has a contact of the second order with $F = 0$ at x, y, and this point is an ordinary point of inflexion. This means that the value of $l/m = \tan\theta$ in (3) must be one of the roots of the quadratic in l/m (4).

Eliminating l and m between (3) and (4), we have a condition that x, y may be a point of inflexion,

$$F''_{xx} F'^2_y - 2F''_{xy} F'_x F'_y + F''_{yy} F'^2_x = 0. \tag{6}$$

To find an ordinary point of inflexion on $F = 0$, solve (6) and $F = 0$ for x and y. If the values of x, y thus determined do not make both F'_x and F'_y vanish, and do satisfy (5), the point is an ordinary point of inflexion.

The solution of equations (6) and $F = 0$ is generally difficult.

In general, if x, y is an ordinary point satisfying $F = 0$, and

$$\left(\frac{\partial F}{\partial y}\frac{\partial}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial}{\partial y}\right)^r F = 0,$$

$r = 2, 3, \ldots, n - 1$, and

$$\left(\frac{\partial F}{\partial y}\frac{\partial}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial}{\partial y}\right)^n F \neq 0,$$

then when n is odd we have a point of inflexion at which the tangent cuts the curve in n coincident points at x, y. When n is even x, y is called a point of *undulation* and the curve there does not cross the tangent but is concave or convex at the contact.

The conditions for concavity, convexity, or inflexion at an ordinary point on $F = 0$ can be determined as in Book I. For, differentiating $F = 0$ with respect to x as independent variable,

$$0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx},$$

$$0 = \left(\frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y}\right)^2 F + \frac{\partial F}{\partial y}\frac{d^2y}{dx^2}.$$

At an ordinary point $\partial_x F \neq 0$ or $\partial_y F \neq 0$. Hence the curve is convex, concave, or inflects at x, y according as

$$\frac{d^2y}{dx^2} = -\frac{\left(\frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y}\right)^2 F}{\frac{\partial F}{\partial y}} = -\frac{\left(\frac{\partial F}{\partial y}\frac{\partial}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial}{\partial y}\right)^2 F}{\left(\frac{\partial F}{\partial y}\right)^3}$$

is positive, negative, or zero.

EXAMPLES.

1. Show that the origin is a point of inflexion on

$$a^3y = bxy + cx^3 + dx^4.$$

2. Show that $x = b$, $y = 2b^3/a^2$ is an inflexion on

$$x^3 - 3bx^2 + a^2y = 0.$$

3. Show that the cubical parabola $y^2 = (x - a)^2(x - b)$ has points of inflexion determined by $3x + a = 4b$.

Hint. Solve the conditional equation for $(x - a)/(y - b)$.

4. If $y^2 = f(x)$ be the equation to a curve, prove that the abscissæ of its points of inflexion satisfy

$$2f(x)f''(x) = \{f'(x)\}^2.$$

II. SINGULAR POINTS.

221. If at any point x, y on a curve $F(x, y) = 0$

$$\frac{\partial F}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} = 0,$$

the point x, y is called a *singular point*.

Since $\frac{dy}{dx} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial y}$, the direction of a curve at a singular point is indeterminate.

222. Double Point.—If at a singular point the second partial derivatives of F are not all 0, we shall have

$$0 = (X - x)^2 \frac{\partial^2 F}{\partial \xi^2} + 2(X - x)(Y - y)\frac{\partial^2 F}{\partial \xi\, \partial \eta} + (Y - y)^2 \frac{\partial^2 F}{\partial \eta^2}.$$

Divide through by $(X - x)^2$ and let $X(=)x$. Then

$$0 = \frac{\partial^2 F}{\partial x^2} + 2\frac{\partial^2 F}{\partial x\, \partial y}\left(\frac{dy}{dx}\right) + \frac{\partial^2 F}{\partial y^2}\left(\frac{dy}{dx}\right)^2.$$

This quadratic furnishes, in general, two directions to the curve at x, y. Such a point is called a *double point.* The two straight lines

$$(X - x)^2 \frac{\partial^2 F}{\partial x^2} + 2(X - x)(Y - y)\frac{\partial^2 F}{\partial x\, \partial y} + (Y - y)^2 \frac{\partial^2 F}{\partial y^2} = 0$$

pass through the point x, y and have the same directions there as the curve, and are therefore the two tangents to the curve at the double point.

The coordinates of a double point on $F(x, y) = 0$ must satisfy the equations

$$F = 0, \quad F'_x = 0, \quad F'_y = 0. \tag{1}$$

The slopes of the tangents there are the roots t_1 and t_2 of the quadratic

$$t^2 F''_{yy} + 2t F''_{xy} + F''_{xx} = 0. \tag{2}$$

(A). *Node.* If the roots of the quadratic (2) are real and different, then

$$F''_{xx} F''_{yy} - F''^2_{xy} = -, \tag{3}$$

the curve has two distinct tangents at x, y, and the point is called a *node.* The curve cuts and crosses itself at a node.

(B). *Conjugate.* If the roots of the quadratic in t (2) are imaginary, or

$$F''_{xx} F''_{yy} - F''^2_{xy} = +, \tag{4}$$

the point is a *conjugate*, or *isolated* point of the curve. The direction of the curve there is wholly indeterminate. There are no other points in the neighborhood of a conjugate point that are on the curve. For the equation to the curve can be written

$$0 = \left\{ (X - x)\frac{\partial}{\partial x} + (Y - y)\frac{\partial}{\partial y} \right\}^2 F$$
$$+ \tfrac{1}{3}\left\{ (X - x)\frac{\partial}{\partial \xi} + (Y - y)\frac{\partial}{\partial \eta} \right\}^3 F.$$

For arbitrarily small values of $X - x$ and $Y - y$ the sign of the second member is that of the first term, and (4) is the condition that this term shall keep its sign unchanged. Therefore the equation cannot be satisfied for X, Y in the neighborhood of x, y.

(C). *Cusp-Conjugate.* If the roots of (2) are equal, or

$$F''_{xx}F''_{yy} - F''^2_{xy} = 0, \tag{5}$$

the point may be either a *conjugate* point or a *cusp*. The curve has one determinate direction there and a double tangent. Equation (5) assumes that F_{xy} F''_{xx}, F''_{yy} are not independently 0. Further consideration of the cusp-conjugate class is postponed.

ILLUSTRATIONS.

1. The following example, taken from Lacroix, serves to illustrate the distinction and connection between the different kinds of double points.

(a). Let $$y^2 = (x - a)(x - b)(x - c), \tag{1}$$
where a, b, c are positive numbers, and $a < b < c$.

The curve is real, finite, two-valued, and symmetrical with respect to Ox for $a < x < b$. It does not exist for $x < a$ or $b < x < c$; it is finite and symmetrical with respect to Ox for all finite values of $x > c$. The ordinate is ∞ when $x = \infty$. The curve consists of a closed loop from a to b, and an infinite branch from c on. The curve is shown in Fig. 127.

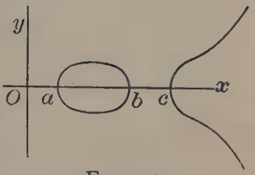


FIG. 127.

(b). Let c converge to b.

Then the loop and open branch tend to come together, and in the limit unite in

$$y^2 = (x - a)(x - b)^2, \tag{2}$$

giving at b a node. (See Fig. 128.)

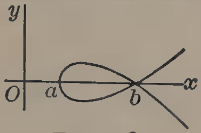


FIG. 128.

(c). Let b converge to a in (1). The closed oval continually diminishes, shrinking to the point a.

In the limit we have

$$y^2 = (x - a)^2(x - c), \tag{3}$$

which consists of a single isolated or conjugate point $x = a$, and an open branch for $x > c$. (Fig. 129.)

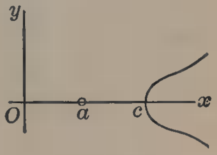


FIG. 129.

(d). Let c and b both converge to a. The oval shrinks to a, and the open branch elongates to a also, resulting in

$$y^2 = (x - a)^3, \tag{4}$$

which has a cusp at a. (Fig. 130.)

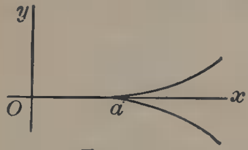


FIG. 130.

2. A clear idea of the meaning of singular points on a curve is obtained when we consider the surface $z = F(x, y)$, which for any constant value of z is a curve cut out of the surface by a horizontal plane.

For example, using (1), Ex. 1, we have the surface

$$z = (x - a)(x - b)(x - c) - y^2,$$

which is symmetrical with respect to the xOz plane, and cuts the xOz plane in the cubic parabola

$$z = (x - a)(x - b)(x - c),$$

and the horizontal plane in the curve

$$y^2 = (x - a)(x - b)(x - c).$$

A moving horizontal plane cuts the surface in curves of the same family. For example, *DD* is an open branched curve; *BB* is a curve with a node as in Fig. 128; *AA* is a curve with a closed oval and one open branch as in Fig. 127; so also is *CC*.

As the horizontal cutting plane rises until it reaches a maximum point *T* on the surface the closed oval shrinks until it becomes the point of contact of the horizontal tangent plane, which plane cuts the surface again in the open branch *T*. The point of touch *T* is a conjugate of the curve *TT* and part of the intersection of the surface by the plane. If the cutting plane be raised higher, to a position *J*, the oval and conjugate point disappear altogether and the section is only the open branch *J*.

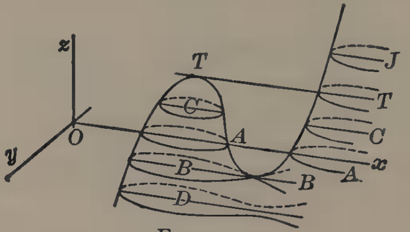


FIG. 131.

Observe that the tangent plane at the node of *BB* is also horizontal, but the ordinate to the surface is there neither a maximum nor a minimum.

The node of *BB* is a *saddle* point on the surface.

To illustrate the cusp, consider the surface

$$z = (x - a)^3 - y^2.$$

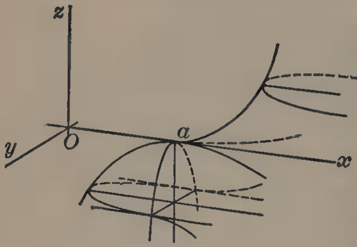


FIG. 132.

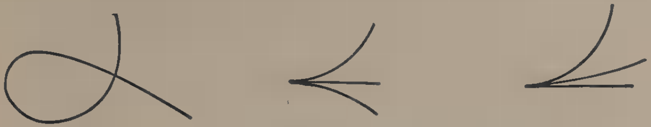
This cuts xOz in $z = (x - a)^3$, and the horizontal plane in $y^2 = (x - a)^3$. All planes parallel to yOz cut the surface in ordinary parabolæ. All sections of the surface by horizontal planes are open branched curves, none having cusps except that one in xOy. All horizontal sections for z negative have inflexions in the plane $x = a$, and their tangents there are parallel to Ox. The horizontal sections above xOy have no inflexions. As the plane of the horizontal section below xOy rises, the inflexional tangents unite in the *unique* double tangent at the cusp in the plane xOy.

3. The above considerations will always enable us to discriminate between a conjugate point and a cusp of the first species,* when the singular point is of the cusp-conjugate class under condition (5). For, let $F(x, y) = 0$ have a point of this class, and let $F(x, y) = 0$ be the equation of the curve referred to the singular point as origin and the tangent there as x-axis. The point is a *cusp of the first species* if $F(x, 0)$ changes sign as x passes through 0. If $F(x, 0)$ does not change sign as x passes through 0, the point is either a conjugate or a cusp of the second species. If in the neighborhood of such a point no real values of x, y satisfy the equation, the conjugate point is identified. Also, the conjugate points on $F = 0$ are the values of x, y which make $z = F$ a maximum or a minimum.

---

* A cusp is of the *first species* when the branches of the curve lie on opposite sides of the tangent there. If both branches lie on the same side of the tangent, the cusp is of the *second species*.

The only forms that double points on an algebraic curve can have, besides the conjugate point, are nodes and cusps. (See Fig. 133.)



Node. Cusp, first species. Cusp, second species.

FIG. 133.

In fact, all other singular points of algebraic curves are but combinations of these, together with inflexions.

EXAMPLES.

1. Show that the origin is a node of $y^2(a^2 + x^2) = x^2(a^2 - x^2)$, and that the tangents bisect the angles between the axes.

2. Show that the origin is a cusp in $ay^2 = x^3$.

3. Find the singular point on $y^3 = x^2(x + a)$. [Cusp.]

4. Investigate $b(x^2 + y^2) = x^3$ at the origin.

5. Investigate $x^3 - 3axy + y^3 = 0$ at the origin.

6. Find the double point of $(bx - cy)^2 = (x - a)^5$, and draw the curve there. [$x = a, \quad y = ab/c$. Cusp.]

7. The curve $(y - c)^2 = (x - a)^4(x - b)$ has a cusp at a, c, if $a \geqq b$; conjugate if $a < b$.

8. Investigate $y^2 = x(x + a)^2$ and $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ for singular points.

9. Investigate at the origin the curve

$$F \equiv ay^2 - 2xy^2 + 3yx^2 - ax^3 + by^3 + x^4 + y^5 = 0.$$

Here $F'_x = 0$, $F'_y = 0$, $F''^2_{xy} - F''_{xx}F''_{yy} = 0$, at the origin, and the third partial derivatives are not all 0. The origin is a point of the cusp-conjugate class, and $y^2 = 0$ is the double tangent.

Since $F(x, 0) \equiv -ax^3 + x^4$ changes sign as x passes through 0, the origin is a cusp of the first kind.

223. Triple Point.—If x, y satisfy the equations

$$F = F'_x = F'_y = F''_{xx} = F''_{yy} = F''_{xy} = 0, \tag{1}$$

and do not make all the third partial derivatives of F vanish. Then we have at any point X, Y on the curve

$$0 = \left\{ (X - x)\frac{\partial}{\partial \xi} + (Y - y)\frac{\partial}{\partial \eta} \right\}^3 F.$$

Divide by $(X - x)^3$ and make $X (=) x$. We have the cubic in t for finding the three directions of the curve at x, y,

$$0 = F'''_{xxx} + 3tF'''_{xxy} + 3t^2F'''_{xyy} + t^3F'''_{yyy}. \tag{2}$$

The solution of this gives, in general, three values of $t \equiv dy/dx$, furnishing the three directions in which the curve passes through x, y, which is a *triple point* on the curve. The equation of the three tangents at x, y is

$$\left\{ (X-x)\frac{\partial}{\partial x} + (Y-y)\frac{\partial}{\partial y} \right\}^3 F = 0.$$

Some forms of triple points are shown in Fig. 134.

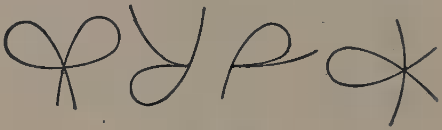


FIG. 134.

EXAMPLES.

1. Show that $x^4 = (x^2 - y^2)y$ has a triple point at the origin.
2. Investigate at O the curve $x^4 - 3axy^2 + 2ay^3 = 0$.

224. Higher Singularities.—In general, if

$$\frac{\partial^r F}{\partial x^p\, \partial y^q} = 0$$

for all values of $p + q = r$, and $r = 0, 1, 2, \ldots, n-1$, then the curve $F = 0$ has an n-ple point at x, y, and in general passes through the point n times.

The equation of the n tangents there is

$$\left\{ (X-x)\frac{\partial}{\partial x} + (Y-y)\frac{\partial}{\partial y} \right\}^n F = 0.$$

Their slopes are the roots of

$$\left(\frac{\partial}{\partial x} + t\frac{\partial}{\partial y}\right)^n F = 0.$$

Examples of multiple points are shown in Fig. 135.

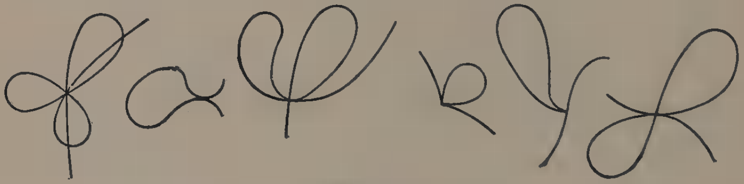


FIG. 135.

EXAMPLES.

1. Investigate $x^5 + y^5 = 5ax^2y^2$, at 0, 0,
2. Investigate $(y - x^2)^2 = x^5$, at 0, 0.
3. In $x^5 + bx^4 - a^3y^2 = 0$, the origin is a double cusp.
4. Determine the tangents at the origin to

$$y^2 = x^2(1 - x^2). \qquad [x \pm y = 0.]$$

5. Show that $x^4 - 3axy + y^4 = 0$ touches the axes at the origin.

6. Investigate $x^4 - ax^2y + by^3 = 0$ at 0, 0.

7. Show that 0, 0 is a conjugate point on

$$ay^2 - x^3 + bx^2 = 0$$

if a and b are like signed, and a node when not.

8. Show that the origin is a conjugate point on

$$y^2(x^2 - a^2) = x^4, \quad \text{and a cusp on} \quad (y - x^2)^2 = x^3.$$

9. Investigate $(y - x^2)^2 = x^n$ at 0, 0, for $n \lessgtr 4$.

10. Investigate $(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1$, where it cuts the axes.

11. Find the double points on

$$x^4 - 4ax^3 + 4a^2x^2 - b^2y^2 + 2b^3y - a^4 - b^4 = 0.$$

12. Also on $x^4 - 2ax^2y - axy^2 + a^2y^2 = 0$.

13. Find and classify the singular points on

$$x^4 - 2ax^2y - axy^2 + a^2y^2 = 0$$

when $a = 1, \quad a > 1, \quad a < 1$.

14. Show that no curve of the second or third degree in x and y can have a cusp of the second species.

Show that if $F(x, y) = 0$ is any equation of the third degree, having a point of the cusp-conjugate class at the origin and the x-axis as tangent, the origin is a cusp or conjugate point according as $F(x, 0)$ does or does not change sign as x passes through 0, that is, according as the lowest power of x is *odd* or *even*.

15. If $F(x, y) = 0$ is any curve of the fourth degree, having at the origin a double point of the cusp-conjugate class, and the tangent there as x-axis, then the origin is a cusp of the first species if the lowest power of x in $F(x, 0)$ is odd; otherwise, it is a cusp of the second kind or a conjugate point according as the co-factor of x^2 in $F(x, mx)$ has real or imaginary roots for arbitrarily small values of m.

16. Show that the origin is a cusp of the second kind in

$$x^4 + y^4 - ay^3 - 2ax^2y + axy^2 + a^2y^2 = 0;$$

is a conjugate point in

$$x^4 + y^4 - ay^3 - ax^2y + axy^2 + a^2y^2 = 0;$$

and a cusp of the first kind in

$$x^3 + y^4 - ay^3 - bx^2y + cxy^2 + a^2y^2 = 0.$$

225. Homogeneous Coordinates.—The study of homogeneous functions is very much simpler than that of heterogeneous functions, owing to the symmetry of the results. This is exemplified in the concomitants. It is therefore of great advantage, in the study of curves, to make the equations homogeneous by the introduction of a third variable. While we do not propose to follow up this method, it is so necessary and so universally employed in the higher analysis that it is mentioned here in order to give a geometrical interpretation to the meaning of the process and to illustrate what has been said about the study of surfaces facilitating the study of curves.

In the present chapter we have been really studying a curve as the section of a surface by the plane $z = 0$. If now we make the equation to any curve $F(x, y) = 0$ homogeneous in x, y, z, by writing the equation

$$F\left(\frac{x}{z}, \frac{y}{z}\right) = 0, \tag{1}$$

and clearing of fractions, then we have the homogeneous equation in three variables x, y, z,

$$F_1(x, y, z) = 0. \tag{2}$$

$F_1 = 0$ becomes $F = 0$ when we make $z = 1$.

But $F_1 = 0$ being homogeneous in x, y, z, it is the equation of a *cone* with vertex at the origin, and which cuts the horizontal plane $z = 1$ in the curve $F = 0$, which curve is the subject of investigation.

Consequently any investigation of $F_1 = 0$ carried on for a homogeneous function in x, y, z is applicable to the curve $F = 0$ when in the results of that investigation we make $z = 1$.

III. CURVE TRACING.

226. In the tracing of algebraic curves, the following remarks are important.

(I). If the origin be taken on a curve of the nth degree, at an ordinary point, the straight line $y = mx$ can meet the curve in only $n - 1$ other points.

If a curve has a singular point of multiplicity m, and this be taken as origin, the line $y = mx$ can meet the curve in only $n - m$ other points.

Therefore, if any curve of the nth degree has at the origin a singularity of multiplicity $n - 2$, the line $y = mx$ can meet it in only two other points besides the origin, and by assigning different values to m we can plot the curve by points conveniently.

(II). If any curve has a rectilinear asymptote, and we take the y-axis parallel to this asymptote, we lower the degree of the equation in y by 1. If there be m parallel asymptotes, and we take the y-axis parallel to them, we lower the degree of the equation in y by m. If the degree of the equation in y can thus be made quadratic or linear in y, then by assigning different values to x, the curve can be plotted by points conveniently.

(III). In any algebraic equation of a curve $F = 0$, when the origin is on the curve, the coefficients of the terms in x, y are the respective partial derivatives of the function F at 0, 0. Therefore the homogeneous part of the equation of lowest degree equated to 0 is the equation of the tangents at the origin. The origin is a singular point whose multiplicity is that of the degree of the lowest terms; it is an ordinary point if this be 1.

(IV). **The Analytical Polygon.**—Newton designed the following method of separating the branches of an algebraic curve at a singular point, and tracing the curve in the neighborhood of that point. The method also determines the manner in which the curve passes off to ∞.

Let $F(x, y)$ be any polynomial in x and y which contains no constant term. Then

$$F(x, y) \equiv \Sigma C_r x^p y^q = 0$$

is the equation of a curve passing through the origin.

Corresponding to each term $C_r x^p y^q$, plot a point with reference to axes Op, Oq, having as abscissa and ordinate the exponents p and q of x and y respectively. Thus locating points $A_1, \ldots, A_{10}$, draw the *simple* polygon $A_1A_3A_5A_6A_8A_{10}A_1$ in such a manner that no point shall lie outside the polygon.

Such a polygon is determined by sticking pins in the points and stretching a string around the system of pins so as to *include* them all.

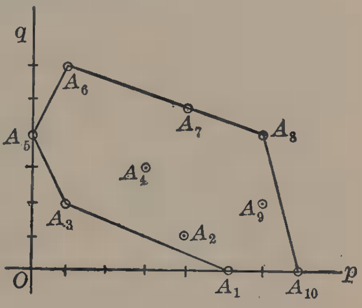


FIG. 136.

The properties of the polygon are :*

(1). Any part of the equation $F = 0$, corresponding to terms which are on a side of the polygon cutting the positive parts of the axes Op, Oq, and such that no point of the polygon lies *between that side and the origin*, when equated to 0 is a curve passing through the *origin* in the same way as does $F = 0$.

Thus, if we strike out of $F = 0$ all terms except those corresponding to terms on the side A_3A_5, we have left a simple curve which passes through the origin in the same way as $F = 0$. In like manner, if we strike out all terms save those corresponding to points on the side A_1A_3, we have another simple curve passing through the origin in the same way as does $F = 0$, and so on.

(2). Any part of the equation $F = 0$ corresponding to points which lie on a side of the polygon cutting the positive parts of the axes Op, Oq, and such that no point of the polygon lies on the *opposite side of this line from the origin*, when equated to 0 gives a simple curve which passes off to *infinity* in the same way as does $F = 0$.

Thus the part of $F = 0$ corresponding to the side A_6A_8 gives such a curve. Again, the part corresponding to A_8A_{10} gives another such curve.

(3). Any side of the polygon which cuts the positive part of one axis and the negative part of the other merely gives one of the axes Ox or Oy as the direction of an asymptote to $F = 0$, and these are more simply determined by equating to 0 the coefficients of the highest powers of x and of y in $F = 0$. Such a side is A_6A_5.

(4). Any side of the polygon which is coincident with one of the axes Oq, Op, as A_1A_{10}, merely gives the points of intersection of $F = 0$ with Ox or Oy accordingly.

---

* For a demonstration of these properties see Appendix, Note 12.

(5). Any side of the polygon which is parallel to one of the axes Op, Oq gives rectilinear asymptotes parallel to an axis, or the axis as a tangent to the curve according as the side falls under conditions (2) or (1).

EXAMPLES.

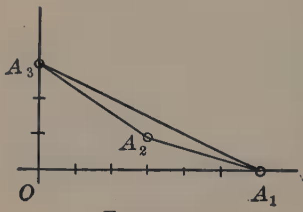


FIG. 137.

1. Trace $x^6 + 2a^2x^3y - b^3y^3 = 0$.

Numbering the terms in the order in which they occur, we have A_1, A_2, A_3, in the polygon corresponding to the terms of the equation.

The curve passes through O in the same way as does the curve

$$x^6+2a^2x^3y \equiv x^3(x^3+2a^2y)=0,$$

corresponding to A_1A_2, or as shown in Fig. 138.

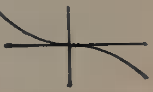


FIG. 138.

Also, the curve passes through O in the same way as does

$$2a^2x^3y - b^3y^3 \equiv y(2a^2x^3 - by^2) = 0,$$

corresponding to A_2A_3, as shown in Fig. 139.

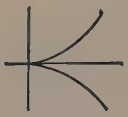


FIG. 139.

The curve passes off to ∞ in the same way as does the curve

$$x^6 - b^3y^3 = 0, \quad \text{or} \quad x^2 = by,$$

corresponding to A_1A_3, Fig. 140.

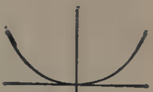


FIG. 140.

The form of the curve is therefore as in Fig. 141.

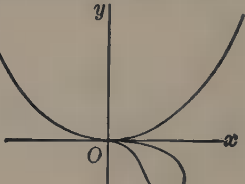


FIG. 141.

Trace the following curves:

2. $x^4 - 2ax^2y - axy^2 + a^2y^2 = 0$.
3. $x^4 - ax^2y + axy^2 + a^2y^2 = 0$.
4. $ay^2 - xy^2 - 2yx^2 + ax^4 - x^2 = 0$ for $a = 1$, $a > 1$, $a < 1$.
5. $x^4 - a^2xy + b^2y^2 = 0$.
6. $y = x(x^2 - 1)$.
7. $x^4 - 3axy^2 + 2ay^3 = 0$.
8. $x^5-2a^3x^2+5a^3xy-2a^3y^2+y^5=0$.
9. $x^4 + a^2xy - y^4 = 0$.
10. $a^2(x^2 - y^2) + x^4 + y^4 = 0$.
11. $a^2(x^2+y^2)-2a(x-y)^3+x^4+y^4=0$.
12. $a(y^2 - x^2)(y - 2x) = y^4$.
13. $a(y - x)^2(y + x) = y^4 + x^4$.
14. $x^4 - axy^2 + y^4 = 0$.
15. $ax(y - x)^2 = y^4$.
16. $x^4 - a^2xy + b^2y^2 = 0$.
17. Trace $x^5 - ax^3y - axy^3 + a^2y^3 = 0$, near the origin.
18. $x^4 - a^2xy^2 = ay^3$.
19. $x^4 - a^2xy = ay^3$.
20. $x^4 + ax^2y = ay^3$.
21. $x(y - x)^2 = b^2y$.
22. $x^5 + y^5 = 5ax^3y$.
23. $(x - 3)y^3 = (y - 1)x^3$.
24. $(x - 2)y^2 = 4x$.
25. $(x - 1)(x - 2)y^2 = x^2$.
26. $(y-x)(y - 4x)(y + 2x)= 8ax^2$.
27. $(y - x)^2(y + x)(y + 2x) = 16a^4$.

IV. ENVELOPES.

227. Differentiation of functions of several variables affords a method of treating the envelopes of curves, which in general simplifies that problem considerably and gives a new geometrical interpretation of the envelope.

For example, we can supply the missing proof, in § 104, that the envelope is tangent to each member of the curve family. When x, y moves along a curve of the family

$$F(x, y, \alpha) = 0, \tag{1}$$

α is constant, and we have on differentiation

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0. \tag{2}$$

But if x, y moves along the envelope, α is variable, and on differentiation of (1)

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial \alpha}d\alpha = 0. \tag{3}$$

Also, on the envelope $\dfrac{\partial F}{\partial \alpha} = 0$. Therefore $\dfrac{dy}{dx}$, from (2) and (3), are the same at a point x, y common to the curve and its envelope.

228. Again, let α, β, γ be variable parameters in the equation

$$F(x, y, \alpha, \beta, \gamma) = 0, \tag{1}$$

where α, β, γ are connected by the two relations

$$\phi(\alpha, \beta, \gamma) = 0, \qquad (2) \qquad\qquad \psi(\alpha, \beta, \gamma) = 0. \qquad (3)$$

We require the envelope of the family of curves (1) when α, β, γ vary. Obviously, if we could solve equations (2), (3) with respect to two of the parameters and substitute in (1), or, what is the same thing, eliminate two of the parameters between equations (1), (2), (3), we could reduce the equation to the family of a single parameter and proceed as in Book I. This is not in general possible, and it is generally simpler to follow the process below.

Differentiating (1), (2), (3), the parameters being the variables,

$$\frac{\partial F}{\partial \alpha}d\alpha + \frac{\partial F}{\partial \beta}d\beta + \frac{\partial F}{\partial \gamma}d\gamma = 0,$$

$$\frac{\partial \phi}{\partial \alpha}d\alpha + \frac{\partial \phi}{\partial \beta}d\beta + \frac{\partial \phi}{\partial \gamma}d\gamma = 0,$$

$$\frac{\partial \psi}{\partial \alpha}d\alpha + \frac{\partial \psi}{\partial \beta}d\beta + \frac{\partial \psi}{\partial \gamma}d\gamma = 0.$$

Multiply the second of these by λ, the third by μ, and add. Determine λ and μ so that the coefficients of $d\alpha$ and $d\beta$ are zero. Then if we take dx as the independent variable parameter, the differentials $d\beta$, $d\gamma$ are arbitrary and we can assign them so that the remainder of the equation shall be zero. Then

$$\frac{\partial F}{\partial \alpha} + \lambda \frac{\partial \phi}{\partial \alpha} + \mu \frac{\partial \psi}{\partial \alpha} = 0, \tag{4}$$

$$\frac{\partial F}{\partial \beta} + \lambda \frac{\partial \phi}{\partial \beta} + \mu \frac{\partial \psi}{\partial \beta} = 0, \tag{5}$$

$$\frac{\partial F}{\partial \gamma} + \lambda \frac{\partial \phi}{\partial \gamma} + \mu \frac{\partial \psi}{\partial \gamma} = 0. \tag{6}$$

The envelope is the result obtained by eliminating α, β, γ, λ, μ between the six numbered equations.

If we have only two parameters and one equation of condition, the particular treatment is obvious; as is also the treatment of the general case when we have n variable parameters connected by $n - 1$ equations of condition.

229. We can get a concrete geometrical intuition of the relation of curves of a family and their envelope, by letting z be a variable parameter and considering

$$F(x, y, z) = 0$$

as the equation of a surface in space. Then the curves of the family are the projections on the horizontal plane xOy of horizontal plane sections of the surface, obtained by varying $z = \alpha$.

EXAMPLES.

1. Find the envelope of a line of given length, l, whose ends move on two fixed rectangular axes.

We have to find the envelope of

$$x/a + y/b = 1 \quad \text{when} \quad a^2 + b^2 = l^2.$$

$$\therefore \quad x/a^2 = \lambda a, \quad y/b^2 = \lambda b.$$

Hence $\qquad \lambda = a^{-2}, \quad \text{and} \quad a = (l^2x)^{\frac{1}{3}}, \quad b = (l^2y)^{\frac{1}{3}},$

and the envelope is $\qquad x^{\frac{2}{3}} + y^{\frac{2}{3}} = l^{\frac{2}{3}}.$

2. Find the envelope of concentric and coaxial ellipses of constant area.

Here $\qquad x^2/a^2 + y^2/b^2 = 1 \quad \text{and} \quad ab = c.$

$$\therefore \quad x^2/a^3 = \lambda b, \quad y^2/b^3 = \lambda a. \quad \therefore \quad 2c\lambda = 1.$$

The required envelope is the equilateral hyperbola $2xy = c$.

3. Find the envelope of the normals to the ellipse.

Here $\qquad a^2x/\alpha - b^2y/\beta = a^2 - b^2 \quad \text{and} \quad \alpha^2/a^2 + \beta^2/b^2 = 1.$

$$\therefore \quad a^2x/\alpha^2 = \lambda\alpha/a^2, \quad b^2y/\beta^2 = -\lambda\beta/b^2. \quad \therefore \quad \lambda = a^2 - b^2.$$

Hence $\qquad \frac{\alpha}{a} = \left(\frac{ax}{a^2 - b^2}\right)^{\frac{1}{3}}, \quad \frac{\beta}{b} = -\left(\frac{by}{a^2 - b^2}\right)^{\frac{1}{3}},$

give the required envelope

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

4. Show that the envelope of $x/a + y/b = 1$, where a and b are connected by $a^m + b^m = c^m$, is $x^{\frac{m}{m+1}} + y^{\frac{m}{m+1}} = c^{\frac{m}{m+1}}$.

5. Show that the envelope of $x/l + y/m = 1$, where the variable parameters l, m are connected by the linear relation $l/a + m/b = 1$, is the parabola

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1.$$

6. Show that if a straight line always cuts off a constant area from two fixed intersecting straight lines, it envelops an hyperbola.

7. Show that the envelope of a line which moves in such a manner that the sum of the squares of its distances from n fixed points x_r, y_r is a constant k^2, is the locus

$$\begin{vmatrix} \Sigma x_r^2 - k^2, & \Sigma x_r y_r \quad, & \Sigma x_r, & x \\ \Sigma x_r y_r \quad, & \Sigma y_r^2 - k^2, & \Sigma y_r, & y \\ \Sigma x_r \quad, & \Sigma y_r \quad, & n \quad, & 1 \\ x \quad, & y \quad, & 1 \quad, & 0 \end{vmatrix} = 0.$$

Let the line be $lx + my + p = 0$. Then

$$k^2 = l^2\Sigma x_r^2 + m^2\Sigma y_r^2 + np^2 + 2mp\,\Sigma y_r + 2lp\,\Sigma x_r + 2lm\,\Sigma x_r y_r,$$
$$= al^2 + bm^2 + cp^2 + 2fmp + 2glp + 2hlm.$$

Also, $l^2 + m^2 = 1$, l and m being direction cosines of the line.
Hence we have

$$al + hm + gp + \lambda l + \tfrac{1}{2}\mu x = 0,$$
$$hl + bm + fp + \lambda m + \tfrac{1}{2}\mu y = 0,$$
$$gl + fm + cp + 0 + \tfrac{1}{2}\mu = 0.$$

Multiply by l, m, p in order and add. $\therefore\ \lambda = -k^2$.
Eliminating l, m, p, μ between the equations

$$(a - k^2)l + hm + gp + \tfrac{1}{2}\mu x = 0,$$
$$hl + (b - k^2)m + fp + \tfrac{1}{2}\mu y = 0,$$
$$gl + fm + cp + \tfrac{1}{2}\mu = 0,$$
$$xl + ym + p + 0 = 0,$$

we have the result.

8. Show that the envelope of a straight line which moves in such a manner that the sum of its distances from n points x_r, y_r is equal to a constant k, is a circle whose center is the centroid of the fixed points and whose radius is one nth the distance k.

Let $lx + my + p = 0$ be the line. Then $l^2 + m^2 = 1$, and

$$k = l\Sigma x_r + m\Sigma y_r + np,$$
$$= al + bm + cp.$$

Here we have

$$a + \lambda x + 2\mu l = 0,$$
$$b + \lambda y + 2\mu m = 0,$$
$$c + \lambda + 0 = 0.$$

$\therefore\ \lambda = -c = -n$. Multiply these three equations by l, m, p in order and add. Hence $k + 2\mu = 0$.

The equations $a - nx = kl$, $b - ny = km$, squared and added, give the envelope

$$\left(x - \frac{\Sigma x_r}{n}\right)^2 + \left(y - \frac{\Sigma y_r}{n}\right)^2 = \left(\frac{k}{n}\right)^2.$$

9. Find the envelope of a right line when the sum of the squares of its distances from two fixed points is constant, and also when the product of these distances is constant.

10. A point on a right line moves uniformly along a fixed right line, while the moving line revolves with a uniform angular velocity. Show that the envelope is a cycloid.

11. Show that the envelope of the ellipses $x^2/a^2 + y^3/b^2 = 1$, when $a^2 + b^2 = k^2$, is a square whose side is $k\sqrt{2}$.

12. Show that the envelope of line $xa^m + yb^m = c^{m+1}$, when $a^n + b^n = d^n$, is

$$x^{\frac{n}{n-m}} + y^{\frac{n}{n-m}} = \left(\frac{c^{m+1}}{d^m}\right)^{\frac{n}{n-m}}.$$

13. Find the envelope of the family of parabolæ which pass through the origin, have their axes parallel to Oy and their vertices on the ellipse $x^2/a^2 + y^2/b^2 = x$.
[*A parabola.*]

14. The ends of a straight line of constant length a describe respectively the circles $(x \pm c)^2 + y^2 = a^2$. Show that the envelope of the curve described by the mid-point of the line, c being a variable parameter, is

$$4(x^2 + y^2 - \tfrac{3}{4}a^2)x^2 + a^2y^2 = 0.$$

15. Find the envelope of a family of circles having as diameters the chords of a given circle drawn through a fixed point on its circumference. [*A cardioid.*]

16. In Ex. 14 show that the area of each curve of the family is $\frac{3}{4}\pi a^2$ when $c > \frac{1}{2}a$. Also, show that the entire area of the envelope is $\frac{1}{2}a^2\left[\frac{4}{3}\pi - \sqrt{3}\right]$.

PART VI.

APPLICATION TO SURFACES.

CHAPTER XXXII.

STUDY OF THE FORM OF A SURFACE AT A POINT.

230. We shall in the present chapter use $f(x, y)$ and $F(x, y, z)$, when abbreviated into f and F, to mean a function of *two* and *three* variables respectively.

The functions immediately under consideration are

$$z = f(x, y) \quad \text{and} \quad F(x, y, z) = 0.$$

The first expresses z explicitly as a function of x and y, and is to be regarded as the solution of the implicit function $F = 0$ with respect to z.

It is to be observed generally that since

$$F \equiv f - z,$$

results obtained from the investigation of $F = 0$ are translated into those for $z = f$ by

$$\frac{\partial^{p+q} F}{\partial x^p \, \partial y^q} = \frac{\partial^{p+q} f}{\partial x^p \, \partial y^q}, \quad \frac{\partial F}{\partial z} = -1, \quad \frac{\partial^{p+1} F}{\partial z^{p+1}} = 0.$$

231. In the present article we recall a few fundamental principles of solid analytical geometry which will be needed subsequently.*

I. *The Plane.* The equation of the first degree in x, y, z can always be represented by a plane.

We repeat the proof of this from geometry as follows:

Let $$Ax + By + Cz + D = 0, \tag{1}$$

A, B, C, D being any constants. Assign to x and y any values x_1, y_1 whatever, and compute z_1, so that x_1, y_1, z_1 satisfy (1). In like manner assign arbitrarily x_2, y_2, and compute z_2 so that x_2, y_2, z_2 also satisfy (1).

Represent, as previously, x, y, z by a point in space with respect to coordinate axes Ox, Oy, Oz. Then, whatever be the numbers m and n, the point whose coordinates are

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* For a more detailed discussion, see any solid analytical geometry.

$$x' = \frac{mx_1 + nx_2}{m + n}, \quad y' = \frac{my_1 + ny_2}{m + n}, \quad z' = \frac{mz_1 + nz_2}{m + n},$$

is a point on the straight line through the points x_1, y_1, z_1 and x_2, y_2, z_2, which divides the segment between these points in the ratio of m to n.

By varying m and n we can make x', y', z' the coordinates of any point whatever on this straight line. But the point x', y', z' must be on the surface (1), since, on substitution, these values satisfy (1). Therefore, whatever be the two points whose coordinates satisfy (1), the straight line through these points must lie wholly in the locus representing (1). This is Euclid's definition of a plane surface.

The intercept of the plane on the axis Oz is $-D/C$. Therefore, when $C = 0$, the intercept is ∞, or the plane is parallel to Oz. Hence (1) becomes

$$Ax + By + D = 0,$$

the equation of a plane parallel to Oz, cutting the plane xOy in the straight line whose equations are $z = 0$, $Ax + By + D = 0$.

We use orthogonal coordinates unless otherwise specially mentioned. If l, m, n are the direction cosines of the perpendicular from the origin on the plane and p is the length of that perpendicular, the equation of the plane can be written in the useful form

$$lx + my + nz - p = 0, \tag{2}$$

where

$$l^2 + m^2 + n^2 = 1.$$

II. *The Straight Line.* Since the intersection of any two planes is a straight line, the *equations* of a straight line are the *simultaneous* equations

$$\left.\begin{aligned} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0. \end{aligned}\right\} \tag{3}$$

The equations (3) of a straight line can always be transformed into the symmetrical form

$$\frac{x - a}{l} = \frac{y - b}{m} = \frac{z - c}{n} = \lambda, \tag{4}$$

where a, b, c is a point on the line; l, m, n, the direction cosines of the line; and λ is the distance between the points x, y, z and a, b, c.

III. *The Cylinder.* A cylinder is any surface which is generated by a straight line moving always parallel to a fixed straight line and intersecting a given curve. The moving straight line is called the *element* or *generator*, and the fixed curve the *directrix* of the cylinder.

With reference to space of three dimensions and rectangular coordinates, any equation

$$f(x, y) = 0 \tag{5}$$

is the equation of a cylinder generated by a straight line moving parallel to Oz and intersecting the plane xOy in the curve $f(x, y) = 0$. For $f(x, y) = 0$ is nothing more than the equation

$$f(x, y, z) = 0$$

in three variables, in which the coefficients of z are zero, and which is therefore satisfied by any x, y on the curve $f(x, y) = 0$ in xOy and any finite value of z whatever.

In like manner $f(y, z) = 0$, $f(x, z) = 0$ are cylinders parallel to the Ox, Oy axes respectively.

IV. *The Cone.* A cone is a surface generated by a straight line passing through a fixed point, called the *vertex*, and moving according to *any* given law, such as intersecting a given curve called the *directrix* or *base* of the cone.

Any *homogeneous* equation of the nth degree in x, y, z, such as

$$F(x, y, z) = 0, \tag{6}$$

is the equation of a cone having the *origin as vertex*.

Let α, β, γ be any values of x, y, z satisfying (6). Then, since (6) is homogeneous, $k\alpha, k\beta, k\gamma$ will also satisfy (6), and we shall have

$$F(kx, ky, kz) = k^n F(x, y, z) = 0$$

whatever be the assigned number k. The coordinates of any point whatever on the straight line through the origin and α, β, γ can be represented by $k\alpha, k\beta, k\gamma$. Therefore all points of this straight line satisfy (6). When the point α, β, γ describes any curve, the straight line through O and α, β, γ generates a surface whose equation is (6), and this is by definition a cone.

If we translate the axes to the new origin $-a, -b, -c$, by writing $x - a, y - b, z - c$, for x, y, z in (6), we have

$$F(x - a, y - b, z - c) = 0, \tag{7}$$

a *homogeneous* equation in $x - a$, $y - b$, $z - c$, which is the equation of a cone whose vertex is a, b, c.

232. General Definition of a Surface.—If $F(x, y, z)$ is a continuous function of the independent variables x, y, z, and is partially differentiable with respect to these variables, we shall define the assemblage of points whose coordinates x, y, z satisfy

$$F(x, y, z) = 0 \tag{1}$$

as a *surface*, and call (1) the equation of the surface.

233. The General Equation of a Surface.—Let $F(x, y, z) = 0$ be the equation of any surface.

Then, by the law of the mean, we can write

$$F(x, y, z) = F(x', y', z') + \sum_{r=1} \frac{1}{r!} \left\{ (x - x') \frac{\partial}{\partial x'} + (y - y') \frac{\partial}{\partial y'} + (z - z') \frac{\partial}{\partial z'} \right\}^r F,$$

in which the summation can be stopped at any term we choose, provided we write ξ, η, ζ instead of x', y', z' in the last term, where ξ, η, ζ is a point on the straight line between x, y, z and x', y', z'. We can therefore always write the equation to any surface in the standard form

$$F(x', y', z') + \sum_{r=1} \frac{1}{r!}\left\{(x-x')\frac{\partial}{\partial x'} + (y-y')\frac{\partial}{\partial y'} + (z-z')\frac{\partial}{\partial z'}\right\}^r F = 0. \quad (1)$$

This enables us to study the function as a rational integral function of x, y, z.

If the equation of the surface be given in the explicit form $z = f(x, y)$, then in like manner, by the law of the mean, we have for the equation to the surface

$$z = f(x', y') + \sum_{r=1} \frac{1}{r!}\left\{(x-x')\frac{\partial}{\partial x'} + (y-y')\frac{\partial}{\partial y'}\right\}^r f, \qquad (2)$$

in which the summation stops at any term we choose, provided in the last term we write ξ for x' and η for y'; ξ, η being a point on the line joining x, y to x', y'.

234. Tangent Line to a Surface.—A tangent straight line to a surface at a point A on the surface is defined to be the limiting position of a secant straight line AB passing through a second point B on the surface, when B converges to A as a limit along a curve on the surface passing through A in a definite way.

To find the condition that the straight line

$$\frac{x-x'}{l} = \frac{y-y'}{m} = \frac{z-z'}{n} = \lambda \qquad (1)$$

shall be tangent to the surface $F(x, y, z) = 0$.

The equation of a surface in implicit form is, § 233, (1),

$$F(x', y', z') + (x-x')\frac{\partial F}{\partial x'} + (y-y')\frac{\partial F}{\partial y'} + (z-z')\frac{\partial F}{\partial z'} + R = 0. \quad (2)$$

Substitute $l\lambda$, $m\lambda$, $n\lambda$ for $x-x'$, $y-y'$, $z-z'$, from (1) in (2). We have the equation in λ,

$$0 = F(x', y', z') + \left(l\frac{\partial F}{\partial x'} + m\frac{\partial F}{\partial y'} + n\frac{\partial F}{\partial z'}\right)\lambda + R, \qquad (3)$$

for determining distances from x', y', z' to the points in which (1) intersects the surface (2). If $F(x', y', z') = 0$, or x', y', z' is on the surface, one root of (3) is 0. If in addition

$$l\frac{\partial F}{\partial x'} + m\frac{\partial F}{\partial y'} + n\frac{\partial F}{\partial z'} = 0, \qquad (4)$$

two values of λ are 0, or two points in which the secant (1) cuts the surface (2) coincide in x', y', z', and the line will be tangent to the surface at x', y', z', and have the direction determined by l, m, n.

Observe that in conditions (4) and $l^2 + m^2 + n^2 = 1$ we have only two relations to be satisfied by the three numbers l, m, n, and therefore there are an indefinite number of tangent lines to a surface at a point x', y', z'.

If the equation to the surface be in the explicit form $z = f(x, y)$, or

$$z = z' + (x - x')\frac{\partial f}{\partial x'} + (y - y')\frac{\partial f}{\partial y'} + R, \tag{5}$$

then, as before, the straight line (1) meets the surface (5) in x', y', z' when $z = z'$ and other points whose distances from x', y', z' are the roots of the equation in λ,

$$0 = \left(l\frac{\partial f}{\partial x'} + m\frac{\partial f}{\partial y'} - n\right)\lambda + R.$$

The condition of tangency is that a second point of intersection shall coincide with x', y', z', or

$$l\frac{\partial f}{\partial x'} + m\frac{\partial f}{\partial y'} - n = 0.$$

235. Tangent Plane to a Surface.—When the locus of the tangent lines at a point on a surface is a plane, that plane is called the tangent plane to the surface at that point. The point is called the point of contact.

Tangent plane to $F(x, y, z) = 0$ at x', y', z'.

The straight line

$$\frac{x - x'}{l} = \frac{y - y'}{m} = \frac{z - z'}{n} \tag{1}$$

is tangent to the surface $F = 0$ at x', y', z' when $F(x', y', z') = 0$ and

$$l\frac{\partial F}{\partial x'} + m\frac{\partial F}{\partial y'} + n\frac{\partial F}{\partial z'} = 0. \tag{2}$$

If now at x', y', z' the derivatives

$$\frac{\partial F}{\partial x'},\quad \frac{\partial F}{\partial y'},\quad \frac{\partial F}{\partial z'}$$

are not all 0, we obtain the locus of the tangent lines to $F = 0$ at x', y', z' by eliminating l, m, n between (1) and (2). Therefore this locus is

$$(x - x')\frac{\partial F}{\partial x'} + (y - y')\frac{\partial F}{\partial y'} + (z - z')\frac{\partial F}{\partial z'} = 0. \tag{3}$$

Equation (3) is of the first degree in x, y, z, and therefore is a plane tangent to $F = 0$ at x', y', z'.

Tangent plane to $z = f(x, y)$.

Eliminating l, m, n between (1) and

$$l\frac{\partial f}{\partial x'} + m\frac{\partial f}{\partial y'} - n = 0, \tag{4}$$

we have

$$z - z' = (x - x')\frac{\partial f}{\partial x'} + (y - y')\frac{\partial f}{\partial y'}, \tag{5}$$

as the tangent plane to $z = f$ at x', y'.

236. Definition of an Ordinary Point on a Surface.—We have just seen that when at any point on a surface $F = 0$ the first partial derivatives,

$$\frac{\partial F}{\partial x}, \quad \frac{\partial F}{\partial y}, \quad \frac{\partial F}{\partial z},$$

are not all zero, the surface has at that point a unique determinate tangent plane. Such a point is called a *point of ordinary position*, or simply an *ordinary* point.

On the contrary, if at x, y, z we have

$$\partial_x F = 0, \quad \partial_y F = 0, \quad \partial_z F = 0,$$

the point is called a *singular* point on the surface. We shall see presently that the surface does not have a unique determinate tangent plane at a singular point.

EXAMPLES.

1. Find the conditions that the tangent plane to $z = f(x, y)$ shall be parallel to xOy.

Ans. $\partial_x f = \partial_y f = 0$.

2. Find the conditions that the tangent plane to $F(x, y, z) = 0$ shall be horizontal.

Ans. $\partial_x F = \partial_y F = 0, \quad \partial_z F \neq 0$.

3. Show that the tangent plane at x', y', z' to the sphere $x^2 + y^2 + z^2 = a^2$ is

$$xx' + yy' + zz' = a^2.$$

4. Find the tangent plane to the central conicoid $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$.

Ans. $\frac{xx'}{a} + \frac{yy'}{b} + \frac{zz'}{c} = 1$.

5. Show that the tangent plane to the paraboloid $ax^2 + by^2 = 2z$ at x', y', z' is

$$axx' + byy' = z + z'.$$

6. Show that the tangent plane to the cone $F(x, y, z) = 0$, having the origin as vertex, is

$$x\partial_{x'}F + y\partial_{y'}F + z\partial_{z'}F = 0.$$

This follows directly from the fact that F is homogeneous, and therefore the tangent plane is

$$x\frac{\partial F}{\partial x'} + y\frac{\partial F}{\partial z'} + z\frac{\partial F}{\partial z'} = x'\frac{\partial F}{\partial x'} + y'\frac{\partial F}{\partial y'} + z'\frac{\partial F}{\partial z'} = nF(x', y', z') = 0,$$

where n is the degree of the cone.

7. Find the equation to the tangent plane at any point of the surface $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$, and show that the sum of the squares of the intercepts on the axes made by the tangent plane is constant.

8. Prove that the tetrahedron formed by the coordinate planes and any tangent plane to the surface $xyz = a^3$ is of constant volume.

9. Show that the equation of the tangent plane to the conicoid

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy + 2ux + 2vy + 2wz + d = 0,$$

at x', y', z', is

$$(ax' + hy' + gz' + u)x + (hx' + by' + fz' + v)y +$$
$$(gx' + fy' + cz' + w)z + ux' + vy' + wz' + d = 0.$$

10. Show that

$$\sum_0^2 \frac{1}{r!}\left[(X - x)\frac{\partial}{\partial x} + (Y - y)\frac{\partial}{\partial y} + (Z - z)\frac{\partial}{\partial z}\right]^r F = 0$$

is the general equation of any conicoid, and that

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

are the equations of the center of the surface.

11. Show that the plane

$$(x - \alpha)\frac{\partial F}{\partial \alpha} + (y - \beta)\frac{\partial F}{\partial \beta} + (z - \gamma)\frac{\partial F}{\partial \gamma} = 0$$

cuts the conicoid $F = 0$ in a conic whose center is α, β, γ, and therefore this is the tangent plane when α, β, γ is on the surface.

12. Show that the locus of the points of contact of all tangent planes to the surface $F = 0$, which pass through a fixed point α, β, γ, is the intersection of $F = 0$ with the surface

$$(\alpha - x)\frac{\partial F}{\partial x} + (\beta - y)\frac{\partial F}{\partial y} + (\quad - z)\frac{\partial F}{\partial z} = 0.$$

13. This surface is of degree $n - 1$ when $F = 0$ is of degree n.

For, let $F = U_n + \ldots + U_1 + U_0$, where U_r is the homogeneous part of degree r. Then, as in two variables, we have the concomitant

$$x\frac{\partial U_r}{\partial x} + y\frac{\partial U_r}{\partial y} + z\frac{\partial U_r}{\partial z} = rU_r.$$

Therefore the tangent plane at x, y, z may be written

$$X\frac{\partial F}{\partial x} + Y\frac{\partial F}{\partial y} + Z\frac{\partial F}{\partial z} = nU_n + (n - 1)U_{n-1} + \ldots + U_1, \text{ or}$$

$$X\frac{\partial F}{\partial x} + Y\frac{\partial F}{\partial y} + Z\frac{\partial F}{\partial z} + U_{n-1} + 2U_{n-2} + \ldots + (n - 1)U_1 + nU_0 = 0,$$

since

$$U_n + \ldots + U_0 = 0.$$

14. Find the condition that the plane $lx + my + nz = 0$ shall be tangent to the cone

$$F \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy = 0$$

at

$$x', y', z'.$$

The equation to the tangent plane at x', y', z' is, Ex. 6,

$$x\frac{\partial F}{\partial x'} + y\frac{\partial F}{\partial y'} + z\frac{\partial F}{\partial z'} = 0.$$

In order that this shall be identical with $lx + my + nz = 0$, the coefficients of x, y, z must be proportional.

$$\therefore \quad \frac{\partial F}{\partial x'}\Big/ l = \frac{\partial F}{\partial y'}\Big/ m = \frac{\partial F}{\partial z'}\Big/ n = \lambda, \text{ say.}$$

Hence

$$\begin{aligned} ax' + hy' + gz' &= l\lambda, \\ hx' + by' + fz' &= m\lambda, \\ gx' + fy' + cz' &= n\lambda, \\ lx' + my' + nz' &= 0. \end{aligned}$$

In order that these equations shall be consistent we have

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0,$$

the required condition.

15. Generalize Ex. 14 by finding the conditions that the plane may cut, be tangent to, or not cut the cone except in the vertex.

Eliminating z, the horizontal projection of the intersection of the plane and cone is two straight lines

$$(an^2 + cl^2 - 2gln)x^2 + 2(hn^2 + clm - gmn - fln)xy + (bn^2 + cm^2 - 2fmn)y^2 = 0.$$

These will be real and different, coincident or imaginary, according as

$$(an^2 + cl^2 - 2gln)(bn^2 + cm^2 - 2fmn) - (hn^2 + clm - gmn - fln)^2,$$

which can be written as the determinant

$$\Delta \equiv \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix},$$

is negative, zero, or positive, respectively.

16. Show that the projections of the two lines in 15 can be written

$$\frac{\partial \Delta}{\partial b} x^2 - \frac{\partial \Delta}{\partial h} xy + \frac{\partial \Delta}{\partial a} y^2 = 0,$$

with similar equations for the projections on the other two coordinate planes.

17. Show that Δ in Ex. 15 can be written

$$l^2 \frac{\partial D}{\partial a} + m^2 \frac{\partial D}{\partial b} + n^2 \frac{\partial D}{\partial c} + ln \frac{\partial D}{\partial g} + mn \frac{\partial D}{\partial f} + lm \frac{\partial D}{\partial h} = -\Delta,$$

where

$$D \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

237. Conventional Abbreviations for the Partial Derivatives.— The elementary study of a surface is usually confined to those properties which depend only on the first and second derivatives, that is, on the quadratic part of the equation to the surface when the equation is expressed by the law of the mean.

This being the case, it is of great convenience in printing and writing to have compact symbols for the first and second partial derivatives. These derivatives being the coefficients of the first and second powers of x, y, z in the equation, it is customary to represent them by the same letters as are conventionally employed as the coefficients of the terms in the general equation of the second degree in three variables.

We shall hereafter frequently write:

When $$F(x, y, z) = 0,$$

$$L \equiv \frac{\partial F}{\partial x}, \qquad M \equiv \frac{\partial F}{\partial y}, \qquad N \equiv \frac{\partial F}{\partial z},$$

$$A \equiv \frac{\partial^2 F}{\partial x^2}, \qquad B \equiv \frac{\partial^2 F}{\partial y^2}, \qquad C \equiv \frac{\partial^2 F}{\partial z^2},$$

$$F \equiv \frac{\partial^2 F}{\partial y\,\partial z}, \qquad G \equiv \frac{\partial^2 F}{\partial x\,\partial z}, \qquad H \equiv \frac{\partial^2 F}{\partial x\,\partial y}.$$

When $$z = f(x, y),$$

$$p \equiv \frac{\partial f}{\partial x}, \quad q \equiv \frac{\partial f}{\partial y}, \quad r \equiv \frac{\partial^2 f}{\partial x^2}, \quad s \equiv \frac{\partial^2 f}{\partial x\,\partial y}, \quad t \equiv \frac{\partial^2 f}{\partial y^2}.$$

238. Inflexional Tangents at an Ordinary Point.—We have seen, §§ 234, 235, that there are an indefinite number of tangent lines to a surface at an ordinary point, lying in the tangent plane and passing through the point of contact. If the second partial derivatives of $F = 0$ are not all 0, there are two of these tangent lines that are of particular interest.

(1). Let $z = f(x, y)$ be the equation of a surface.

The straight line

$$\frac{X - x}{l} = \frac{Y - y}{m} = \frac{Z - z}{n} = \lambda \tag{1}$$

cuts the surface $z = f$ in points whose distances from the point x, y, z on the surface are the roots of the equation in λ

$$0 = \left(l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} - n\right) + \frac{\lambda}{2}\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right)^2 f + R = 0. \tag{2}$$

If $$l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} - n = 0, \tag{3}$$

we have seen that (1) is tangent to $z = f$ at x, y, z.

If in addition we have l, m, n satisfying the condition

$$\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right)^2 f \equiv l^2\frac{\partial^2 f}{\partial x^2} + 2lm\frac{\partial^2 f}{\partial x\,\partial y} + m^2\frac{\partial^2 f}{\partial y^2} = 0, \tag{4}$$

two roots of (2) are 0, and the line (1) cuts the surface in three coincident points at x, y, z.

The conditions

$$\begin{aligned} l^2 + m^2 \ + n^2 &= 1, \\ pl + qm \ - n &= 0, \\ rl^2 + 2slm + tm^2 &= 0, \end{aligned}$$

determine two straight lines, in the tangent plane, tangent to the surface $z = f$ at the point of contact. Each cuts the surface in three coincident points there.

These are called the inflexional tangents at x, y, z. They are real and distinct, coincident, or imaginary, according as the quadratic condition

$$rl^2 + 2slm + tm^2 = 0,$$

in l/m, has real and different, double, or imaginary roots, or according as

$$rt - s^2 \equiv \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\, \partial y}\right)^2 \tag{5}$$

is negative, zero, or positive.

Since any straight line, such as (1), cuts any surface of the nth degree in n points, the straight lines in any plane cut the curve of section of a surface of degree n in n points. Therefore a plane cuts a surface of the nth degree in a plane curve of degree n.

The tangent plane to a surface of degree n cuts the surface in a curve of degree n passing through the point of contact. But each of the inflexional tangents to the surface cuts this curve in three coincident points at the point of contact. Each is therefore tangent to the curve of section at the point of contact of the tangent plane, which is therefore a singular point on the curve of section. This point is a node, conjugate point, or cusp according to the value of condition (5). Compare singular points, plane curves.

Eliminating l, m, n between (1), (3), (4), we have for the equations of the inflexional tangents at x, y, z

$$Z - z = (X - x)p + (Y - y)q,$$
$$(X - x)^2 r + 2(X - x)(Y - y)s + (Y - y)^2 t = 0.$$

The second is the equation of two vertical planes cutting the first, the tangent plane, in the inflexional tangents.

(2). If the equation of the surface is $F = 0$, then the straight line (1) cuts the surface in points whose distances from x, y, z are the roots of the equation in λ,

$$F + \lambda\left(l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z}\right) + \frac{\lambda^2}{2}\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y} + n\frac{\partial}{\partial z}\right)^2 F + R = 0.$$

If x, y, z is on the surface, or $F(x, y, z) = 0$, and

$$Ll + Mm + Nn = 0,$$

the line (1) is tangent at x, y, z. If in addition l, m, n satisfy the condition

$$\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y} + n\frac{\partial}{\partial z}\right)^2 F = 0,$$

the line (1) cuts the surface in three coincident points at x, y, z. The conditions

$$l^2 + m^2 + n^2 = 1, \tag{6}$$

$$Ll + Mm + Nn = 0, \tag{7}$$

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gln + 2Hlm = 0, \tag{8}$$

determine the directions of the two inflexional tangents.

Eliminating l, m, n between (1), (7), (8), we have the equations of the inflexional tangents at x, y, z,

$$\left\{ (X - x)\frac{\partial}{\partial x} + (Y - y)\frac{\partial}{\partial y} + (Z - z)\frac{\partial}{\partial z} \right\} F = 0, \tag{9}$$

$$\left\{ (X - x)\frac{\partial}{\partial x} + (Y - y)\frac{\partial}{\partial y} + (Z - z)\frac{\partial}{\partial z} \right\}^2 F = 0. \tag{10}$$

The first is the tangent plane, which cuts the second, a cone of the second degree with vertex x, y, z, in the two inflexional tangents.

These tangents will be real and different, coincident, or imaginary, according as the plane (9) *cuts* the cone (10), is tangent to it, or passes through the vertex without cutting it elsewhere. That is, according as the determinant (see Ex. 15, § 236)

$$\begin{vmatrix} A & H & G & L \\ H & B & F & M \\ G & F & C & N \\ L & M & N & 0 \end{vmatrix} \tag{11}$$

is negative, zero, or positive.

239. Should the second partial derivatives also be separately 0 at x, y, z, and r the order of the first partial derivatives thereafter which do not all vanish at x, y, z, then there will be at x, y, z on the surface r inflexional tangents, which are the r straight lines in which the tangent plane at x, y, z cuts the r planes

$$\left\{ (X - x)\frac{\partial}{\partial x} + (Y - y)\frac{\partial}{\partial y} \right\}^r f = 0,$$

or the cone of the rth degree,

$$\left\{ (X - x)\frac{\partial}{\partial x} + (Y - y)\frac{\partial}{\partial y} + (Z - z)\frac{\partial}{\partial z} \right\}^r F = 0.$$

These r inflexional tangents to the surface are the r tangents to the curve cut out of the surface by the tangent plane at the point of contact, which point is an r-ple singular point on the curve of section.

EXAMPLES.

1. Show that the inflexional tangents at any point x', y', z' on the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$, lie wholly on the surface and are therefore the two right-line generators passing through the point. Show that their equations are

$$b\,\frac{x - x'}{bx'z' \pm acy'} = a\,\frac{y - y'}{ay'z' \mp bcx'} = \frac{z - z'}{c^2 + z'^2}.$$

2. Show that the inflexional tangents at a point x, y, z on the hyperbolic paraboloid $x^2/a^2 - y^2/b^2 = 2z$ lie wholly on the surface, and that their equations are

$$\frac{X-x}{a} = \frac{Y-y}{\pm b} = \frac{Z-z}{\dfrac{x}{a} \mp \dfrac{y}{b}},$$

the upper signs going together and the lower together.

3. Show that the inflexional tangents to the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy = 0$$

are coincident with the generator through the point of contact.

4. Show that at a point on a surface at which any one of the coordinates is a maximum or a minimum the inflexional tangents are imaginary.

240. The Normal to a Surface at an Ordinary Point.—The straight line perpendicular to the tangent plane at the point of contact is called the *normal* to the surface at that point.

Since the equation to the tangent plane at x, y, z is

$$(X-x)\frac{\partial F}{\partial x} + (Y-y)\frac{\partial F}{\partial y} + (Z-z)\frac{\partial F}{\partial z} = 0,$$

or

$$Z - z = (X-x)\frac{\partial f}{\partial x} + (Y-y)\frac{\partial f}{\partial y},$$

the coefficients of X, Y, Z are proportional to the direction cosines of the normal, and we have for the equation to the normal at x, y, z

$$\frac{X-x}{\dfrac{\partial F}{\partial x}} = \frac{Y-y}{\dfrac{\partial F}{\partial y}} = \frac{Z-z}{\dfrac{\partial F}{\partial z}},$$

or

$$\frac{X-x}{\dfrac{\partial f}{\partial x}} = \frac{Y-y}{\dfrac{\partial f}{\partial y}} = \frac{Z-z}{-1}.$$

EXAMPLES.

1. Show that the normal at x, y, z to $xyz = a^3$ is

$$Xx - x^2 = Yy - y^2 = Zz - z^2.$$

2. Find the equations of the normal to the central conicoid $ax^2 + by^2 + cz^2 = 1$.

$$\frac{X-x}{ax} = \frac{Y-y}{by} = \frac{Z-z}{cz}.$$

3. Show that the normal to the paraboloid $ax^2 + by^2 = 2z$ has for its equations

$$\frac{X-x}{ax} = \frac{Y-y}{by} = z - Z.$$

241. Study of the Form of a Surface at an Ordinary Point.—We may study the form of a surface at an ordinary point by examining it (1) with respect to the *tangent plane*, (2) with respect to the *conicoid of curvature*, (3) with respect to the plane *sections parallel* *to the tangent plane*, (4) with respect to the plane *sections through the normal.*

242. With respect to the tangent plane:

(1). Let $z = f(x, y)$. Then the equation of the surface is

$$Z - z - (X - x)\frac{\partial f}{\partial x} - (Y - y)\frac{\partial f}{\partial y} = \frac{1}{2}\left\{ (X - x)\frac{\partial}{\partial \xi} + (Y - y)\frac{\partial}{\partial \eta} \right\}^2 f.$$

Let X, Y, Z_1 be a point in the tangent plane in the neighborhood of the point of contact x, y, z. Then the difference between the ordinate to the surface and the ordinate to the tangent plane is

$$Z - Z_1 = \frac{1}{2}\left\{ (X - x)\frac{\partial}{\partial \xi} + (Y - y)\frac{\partial}{\partial \eta} \right\}^2 f.$$

This difference is positive for all values of X, Y in the neighborhood of x, y when

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}$$

are positive (Ex. 19, § 25). Then in the neighborhood of the point of contact the surface lies wholly above the tangent plane, and is said to be *convex* there.

In like manner $Z - Z_1$ is negative throughout the neighborhood when $rt - s^2$ is positive and r is negative at the point of contact. Then the surface in the neighborhood of the point of contact lies wholly below the tangent plane and is said to be *concave* there.

(2). Let $F(x, y, z) = 0$. In the same way we have the equation of the surface,

$$(X-x)F'_x + (Y-y)F'_y + (Z-z)F'_z =$$
$$-\frac{1}{2}\left\{ (X-x)\frac{\partial}{\partial \xi} + (Y-y)\frac{\partial}{\partial \eta} + (Z-z)\frac{\partial}{\partial \zeta} \right\}^2 F,$$

and for that of the tangent plane at x, y, z,

$$(X - x)F'_x + (Y - y)F'_y + (Z_1 - z)F'_z = 0.$$

On subtraction,

$$(Z_1 - Z)F'_z = \frac{1}{2}\left\{ (X-x)\frac{\partial}{\partial \xi} + (Y-y)\frac{\partial}{\partial \eta} + (Z-z)\frac{\partial}{\partial \zeta} \right\}^2 F.$$

Therefore, at x, y, z, by Ex. 20, § 25, when

$$\begin{vmatrix} A & H \\ H & B \end{vmatrix} \quad \text{and} \quad A\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

are positive, the surface is convex when A and F'_z are unlike signed, concave when A and F'_z are like signed.

Observe that a surface is concave or convex at a point when the inflexional tangents there are imaginary, and conversely. When a surface is either concave or convex at a point, its form is said to be *synclastic* there. When the inflexional tangents are real and different the surface does not lie wholly on one side of the tangent plane in the neighborhood of the point of contact, but cuts the tangent plane in a curve having a node at the point of contact and tangent to the inflexional tangents. At such a point the form of the surface is said to be *anticlastic*, and the surface lies partly on one side and partly on the other side of the tangent plane in the neighborhood of the contact.

The conditions that a surface may be synclastic or anticlastic at a point are, (11), § 238,

$$\begin{vmatrix} A & H & G & L \\ H & B & F & M \\ G & F & C & N \\ L & M & N & 0 \end{vmatrix} \begin{array}{l} = + \textit{ synclastic}, \\ \\ = - \textit{ anticlastic}. \end{array}$$

The hyperboloid of one sheet and the hyperbolic paraboloid are the simplest examples of anticlastic surfaces, these being anticlastic at every point of the surfaces. The surface generated by the revolution of a circle about an external axis in its plane generates a *torus*. This surface is anticlastic or synclastic at a point according as the point is nearer or further from the axis of revolution than the center of the circle.

243. With Respect to the Conicoid of Curvature.

(1). The explicit equation $z = f(x, y)$, or

$$Z = z + (X-x)\frac{\partial f}{\partial x} + (Y-y)\frac{\partial f}{\partial y} + \frac{1}{2}\left\{(X-x)\frac{\partial}{\partial \xi} + (Y-y)\frac{\partial}{\partial \eta}\right\}^2 f,$$

shows that in the neighborhood of x, y, z the surface differs arbitrarily little from the paraboloid

$$Z = z + (X-x)\frac{\partial f}{\partial x} + (Y-y)\frac{\partial f}{\partial y} + \frac{1}{2}\left\{(X-x)\frac{\partial}{\partial x} + (Y-y)\frac{\partial}{\partial y}\right\}^2 f.$$

This is called the paraboloid of curvature of the surface at x, y, z. It has the same first and second derivatives at x, y, z as has the surface $z = f$, and therefore, at that point, has, in common with the surface, all those properties which are dependent on these derivatives.

Obviously, the surface is synclastic or anticlastic according as the paraboloid is elliptic or hyperbolic.

From analytical geometry, the discriminating quadratic of the paraboloid

$$rx^2 + ty^2 + 2sxy + 2px + 2qy - 2z + k = 0$$

is

$$\lambda^2 - (r + t)\lambda + (rt - s^2) = 0.$$

This gives the elliptic or hyperbolic form according as $rt - s^2$ is positive or negative.

(2). In the same way, the implicit equation $F(x, y, z) = 0$, or

$$(X - x)\frac{\partial F}{\partial x} + (Y - y)\frac{\partial F}{\partial y} + (Z - z)\frac{\partial F}{\partial z}$$
$$+ \frac{1}{2}\left\{(X - x)\frac{\partial}{\partial x} + (Y - y)\frac{\partial}{\partial y} + Z - z)\frac{\partial}{\partial z}\right\}^2 F = R,$$

shows that in the neighborhood of x, y, z the surface differs arbitrarily little from the conicoid of curvature whose equation is the same as the left member of the equation above when equated to 0. The form of the surface at x, y, z is the same as that of the conicoid of curvature there, and they have the same properties there as far as these properties are dependent on the first and second derivatives of F.

The discrimination of the conicoid can be made through the discriminating cubic (see Ex. 17, p. 30)

$$\begin{vmatrix} A - \lambda, & H & , G \\ H & , B - \lambda, & F \\ G & , F & , C - \lambda \end{vmatrix} = 0,$$

and the four determinants

$$\left\|\begin{matrix} A & H & G & L \\ H & B & F & M \\ G & F & C & N \end{matrix}\right\|,$$

as in analytical geometry.*

244. The Indicatrix of a Surface.—At an ordinary point x, y, z on a surface, at which the second derivatives are not all 0, a section of the surface by a plane parallel to and arbitrarily near the tangent plane differs arbitrarily little from the section of the conicoid of curvature made by this plane. Such a plane section of the conicoid of curvature is called the *indicatrix* of the surface at x, y, z.

Points on a surface are said to be *circular* (umbilic), *elliptic*, *parabolic*, or *hyperbolic* according as the indicatrix is a circle, ellipse, parabola (two parallel lines), or hyperbola (two cutting lines).

245. Equation to Surface when the Tangent Plane and Normal are the z-plane and z-axis.—If the equation is $z = f(x, y)$, then since $z = 0$, $p = 0$, $q = 0$ at the origin, the equation is

$$2z = rx^2 + 2sxy + ty^2 + 2R.$$

The equation of the indicatrix at the origin is

$$z = rx^2 + 2sxy + ty^2,$$

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* See Frost's, Charles Smith's, or Salmon's Analytical Geometry.

z being an arbitrarily small constant. This is an ellipse or hyperbola according as $rt - s^2$ is positive or negative, giving the synclastic or anticlastic form of the surface there accordingly.

246. Singular Points on Surfaces.—If, at a point x, y, z on a surface $F = 0$, we have independently

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0, \tag{1}$$

the point is said to be a *singular* point.

If the second derivatives are not all zero, then all the straight lines whose direction cosines l, m, n satisfy the relation

$$\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y} + n\frac{\partial}{\partial z}\right)^2 F = 0 \tag{2}$$

will cut the surface in three coincident points at x, y, z, and are called tangent lines. Eliminating l, m, n by means of the equation to the line and (2), we obtain the locus of the tangent lines at x, y, z,

$$\left\{(X - x)\frac{\partial}{\partial x} + (Y - y)\frac{\partial}{\partial y} + (Z - z)\frac{\partial}{\partial z}\right\}^2 F = 0. \tag{3}$$

This is the equation of a cone of the second degree, with vertex x, y, z, which is tangent to the surface $F = 0$ at the point x, y, z. The form of the surface at x, y, z is therefore the same as that of this cone. Such a point is called a conical point on the surface.

When this cone degenerates into two planes, then all the tangent lines to the surface at x, y, z lie in one or the other of two planes. The point is then called a *nodal* point. The condition for a nodal point is that equation (3) shall break up into two linear factors, or

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0. \tag{4}$$

A line on the surface $F = 0$ at all points of which (4) is satisfied is called a *nodal line* on the surface. Such a line is geometrically defined by the surface folding over and cutting itself in a nodal line, in the same way that a curve cuts itself in a nodal point.

If r is the order of the first partial derivatives which are not all zero, then the surface has a conical point at x, y, z of order r, and a tangent cone there of the rth degree whose equation is

$$\left\{(X - x)\frac{\partial}{\partial x} + (Y - y)\frac{\partial}{\partial y} + (Z - z)\frac{\partial}{\partial z}\right\}^r F = 0. \tag{5}$$

247. A *singular tangent plane* is a plane which is tangent to a surface all along a line on the surface. For example, a torus laid on a plane is tangent to it all along a circle. The torus has two singular tangent planes. All planes tangent to a cylinder or cone are singular.

EXERCISES.

1. The tangent plane to $yx^2 = a^2z$ at x_1, y_1, z_1 is

$$2xx_1y_1 + yx_1^2 - a^2z = 2a^2z_1.$$

Find the equation to the normal there.

2. The tangent plane to $z(x^2 + y^2) = 2kxy$ at x_1, y_1, z_1 is

$$2x(x_1z_1 - ky_1) + 2y(y_1z_1 - kx_1) + z(x_1^2 + y_1^2) - 2kx_1y_1 = 0.$$

The tangent plane meets the surface in a straight line, and an ellipse whose projection on the xOy plane is the circle

$$(x^2 + y^2)(x_1^2 - y_1^2) + (x_1^2 + y_1^2)(yy_1 - xx_1) = 0.$$

Show that the z-axis is a nodal line.

3. The tangent plane to $a^2y^2 = x^2(c^2 - z^2)$ at x_1, y_1, z_1 is

$$xx_1(c^2 - z_1^2) - a^2yy_1 - zz_1x_1^2 + x_1^2z_1^2 = 0.$$

At any point on Oz, $F_x' = F_y' = F_z' = 0$, show that at any such point there are two tangent planes

$$\frac{y}{x} = \pm\sqrt{\frac{c^2 - z_1^2}{a^2}}.$$

4. Show that the tangent plane at x_1, y_1, z_1 to

$$x^3 + y^3 + z^3 - 3xyz = a^3$$

is
$$x(x_1^2 - y_1z_1) + y(y_1^2 - x_1z_1) + z(z_1^2 - x_1y_1) = a^3.$$

5. The tangent plane at x_1, y_1, z_1 to $x^my^nz^p = a$ is

$$\frac{m}{x_1}x + \frac{n}{y_1}y + \frac{p}{z'}z = m + n + p.$$

6. Show that $(2a, 2a, 2a)$ is a conical point on

$$xyz - a(x^2 + y^2 + z^2) + 4a^3 = 0,$$

and find the tangent cone at the point.

Ans. $x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0$

7. Show that the surface

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)^2 - 3\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) - \frac{z^2}{c^2} + \frac{1}{4} = 0$$

has two conical points.

The tangent cone at 0, 0, 0 is $3x^2/a^2 + 3y^2/b^2 + z^2/c^2 = 0$.

8. Determine the nature of the surface

$$ay^2 + bz^2 + x(x^2 + y^2 + z^2) = 0$$

at the origin.

The origin is a singular point, the tangent cone there is $ay^2 + bz^2 = 0$. If a and b are like signed, the origin is a cuspal point around the x-axis.

9. A surface is generated by the revolution of a parabola $z^2 = 4mx$ about an ordinate through the focus; find the nature of the points where it meets the axis of revolution.

Hint. The equation of the surface can be written

$$16m^2(x^2+y^2) = (z^2-4m^2)^2.$$

The two right-angled circular cones $x^2+y^2=(z\pm 2m)^2$ are tangent to the surface at the singular points.

10. If tangent planes are drawn at every point of the surface

$$a(yz+zx+xy)=xyz,$$

where it is cut by a sphere whose center is the origin, show that the sum of the intercepts on the axes will be constant.

11. Show that the general equation of surfaces of revolution having Oz for axis is

$$x^2+y^2=f(z).$$

Thence show that the normal to the surface at any point intersects the axis of revolution.

12. Show that at all points of the line which separates the synclastic from the anticlastic parts of a surface the inflexional tangents must coincide.

13. The equation of an anchor-ring or torus is

$$(x^2+y^2+z^2+c^2-a^2)^2 = 4c^2(x^2+y^2).$$

Show that the tangent plane at x', y', z', is

$$(r-c)(xx'+yy')+rzz' = r[a^2+c(r-c)],$$

where $r^2 \equiv x'^2+y'^2$.

The tangent plane at any point on the circle $x^2+y^2=(c-a)^2$ cuts the surface in a figure 8 curve whose form is given by the equation

$$(y^2+z^2)^2-4acy^2+4c(c-a)z^2=0.$$

14. When the tangent plane passes through the origin it cuts two circles out of the torus which intersect in the two points of contact.

15. Show that the cylinder $x^2+y^2=c^2$ cuts the torus in two parts, one of which is synclastic, the other is anticlastic.

CHAPTER XXXIII.

CURVATURE OF SURFACES.

248. Normal Sections. Radius of Curvature.—The normal section of a surface at a point is the curve cut on the surface by a plane passing through the normal to the surface at the point.

To find the radius of curvature of a normal section.

Let the tangent plane and normal at an ordinary point on the surface be taken as the z-plane and z-axis respectively. Then the equation to the surface can be written

$$z = \tfrac{1}{2}(rx^2 + 2sxy + ty^2) + R, \qquad (1)$$

since at the origin $z = 0$, $p = 0$, $q = 0$.

Cut the surface by a plane passing through Oz and making an angle θ with Ox. At every point of this plane let

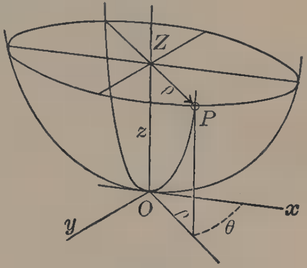

FIG. 142.

$$x = \rho\cos\theta, \quad y = \rho\sin\theta.$$

$$\therefore \quad z = \tfrac{1}{2}\rho^2(r\cos^2\theta + 2s\cos\theta\sin\theta + t\sin^2\theta) + T,$$

where T contains as a factor a higher power of ρ than ρ^2.

The radius of curvature R of this normal section PO is, by Newton's method, § 101, Ex. 4, given by

$$\frac{1}{R} = \mathcal{L}_{\rho(=)0}\frac{2z}{\rho^2}, \qquad (2)$$

$$= r\cos^2\theta + 2s\cos\theta\sin\theta + t\sin^2\theta,$$

$$= \tfrac{1}{2}(r + t) + \tfrac{1}{2}(r - t)\cos 2\theta + s\sin 2\theta. \qquad (3)$$

The directions of the normal sections in which the radius of curvature is a maximum or a minimum are given by the equation

$$\tan 2\theta = \frac{2s}{r - t}. \qquad (4)$$

If α is the least positive value of θ satisfying (4), the general solution is $\tfrac{1}{2}n\pi + \alpha$, showing that the normal sections of maximum and minimum curvature are at right angles. These sections are called the *principal sections* of the surface at the point considered. Their radii of curvature at the point are called the *principal radii of curvature.*

If the principal sections be taken for the planes xOz, yOz, the expression for the radius of curvature of any section will be

$$\frac{1}{R} = r\cos^2\theta + t\sin^2\theta, \tag{5}$$

since then $s = 0$, by (4).

Let R_1 and R_2 be the radii of the principal sections.

Then when $\theta = 0$, $R_1^{-1} = r$; $\theta = \frac{1}{2}\pi$, $R_2^{-1} = t$, in (5).

$$\therefore \quad \frac{1}{R} = \frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2}. \qquad (6)$$

Also, if R' is the radius of curvature of a normal section perpendicular to that of R, then

$$\frac{1}{R'} = \frac{\sin^2\theta}{R_1} + \frac{\cos^2\theta}{R_2}. \qquad (6)$$

$$\therefore \quad \frac{1}{R} + \frac{1}{K'} = \frac{1}{R_1} + \frac{1}{R_2}. \qquad (7)$$

The sum of the reciprocals of the radii of curvature of normal sections at right angles is constant. This is Euler's Theorem.

249. Meunier's Theorem. — To find the relation between the radii of curvature of a normal section and an oblique section passing through the same tangent line.

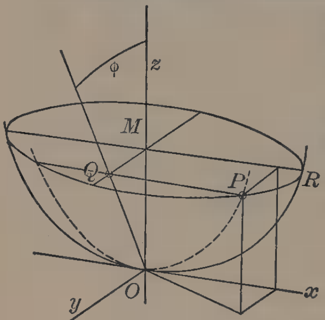

FIG. 143.

Take xOz as the normal plane, and let the oblique plane $xPOQ$ make the angle ϕ with xOz.

Then the equation of the surface is

$$2z = rx^2 + 2sxy + ty^2 + \frac{1}{3}\left(x\frac{\partial}{\partial\xi} + y\frac{\partial}{\partial\eta}\right)^3 f.$$

At any point P in the oblique section $y = z\tan\phi$.

$$\therefore \quad \frac{2z}{x^2} = r + 2s\frac{z}{x}\tan\phi + t\frac{z^2}{x^2}\tan^2\phi + \frac{x}{3}\left(\frac{\partial}{\partial\xi} + \frac{z}{x}\tan\phi\frac{\partial}{\partial\eta}\right)^3 f.$$

But since Ox is tangent to the curve OP at 0,

$$\mathop{\mathcal{L}}_{x(=)0} \frac{z\sec\phi}{x} = 0 = \mathcal{L}\frac{z}{x}.$$

$$\therefore \quad \mathcal{L}\frac{2z}{x^2} = \left(\frac{\partial^2 f}{\partial x^2}\right)_{x=0}$$

as P converges to O along PO. Also, in the xOz section, if $MR \equiv x_0$, we have $y = 0$, and

$$\mathcal{L}\frac{2z}{x_0^2} = \left(\frac{\partial^2 f}{\partial x^2}\right)_{x=0}.$$

Let R_0, R be the radii of the normal and oblique sections. Then, for $z(=)0$,

$$R = \pounds \frac{x^2}{2z} \cos\phi, \quad R_0 = \pounds \frac{x_0^{\,2}}{2z}.$$

Hence $$R = R_0 \cos\phi.$$

This is Meunier's theorem.

250. Observe, in the equation to the surface (1), § 248, the equation of the indicatrix is

$$2z = rx^2 + 2sxy + ty^2. \tag{1}$$

The principal sections of the surface at O pass through the axes of the indicatrix conic, whose equation is

$$2z = rx^2 + ty^2 \tag{2}$$

when xOz and yOz are the principal planes.

Equation (1) shows that the radius of curvature of a normal section varies as the square of the corresponding central radius vector of the indicatrix. All the theorems in central conics which can be expressed by homogeneous equations in terms of the radii and axes furnish corresponding theorems in curvature of surfaces.

We shall adopt the convention that the radius of curvature of a normal section of a surface is positive or negative according as the center of curvature of the section is above or below the tangent plane.

When the indicatrix is an ellipse the principal radii have like signs, and have opposite signs when the indicatrix is the hyperbola. The inflexional tangents are the asymptotes of the indicatrix.

251. At any point of a surface to find the radius of curvature of a normal section passing through a given tangent line at the point.

Let $F = 0$ be the equation of the surface. Let P be the given point x, y, z, and l, m, n the direction cosines of the tangent line there. Let Q be another point X, Y, Z on the surface and in the normal section.

Let QR be the perpendicular from Q on the tangent line PR.

Then for R, the radius of curvature of the section, we have

$$R = \pounds \frac{PR^2}{2QR} = \pounds \frac{PQ^2}{2QR}\left(\frac{PR}{PQ}\right)^2 = \pounds \frac{PQ^2}{2QR}.$$

The tangent plane at P is

$$(X - x)L + (Y - y)M + (Z - z)N = 0.$$

The distance of Q from this plane is

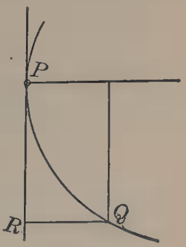

FIG. 144.

$$QR = \frac{1}{\kappa}\{(X - x)L + (Y - y)M + (Z - z)N\},$$

where $$\kappa^2 = L^2 + M^2 + N^2.$$

Also, Q being a point on the surface,

$$(X-x)L+(Y-y)M+(Z-z)N$$
$$+\frac{1}{2}\left\{(X-x)\frac{\partial}{\partial x}+(Y-y)\frac{\partial}{\partial y}+(Z-z)\frac{\partial}{\partial z}\right\}^2 F+T=0.$$

$$\therefore\quad \frac{\kappa}{R}=\pounds\,\frac{\left\{(X-x)\dfrac{\partial}{\partial x}+(Y-y)\dfrac{\partial}{\partial y}+(Z-z)\dfrac{\partial}{\partial z}\right\}^2 F+2T}{(X-x)^2+(Y-y)^2+(Z-z)^2},$$

$$=Al^2+Bm^2+Cn^2+2Fmn+2Gln+2Hlm, \tag{1}$$

since $\pounds T=0$ for $Q(=)P$, and

$$\frac{X-x}{l}=\frac{Y-y}{m}=\frac{Z-z}{n}=\lambda$$

is the equation of the tangent PR. The derivatives L, A, etc., of course being taken at P.

252. If the equation of the surface be $f(x,y)-z\equiv F=0$, then since $L=p$, $M=q$, $N=-1$, $C=F=G=0$, (1), § 251, becomes

$$\frac{1}{R}=\frac{rl^2+2slm+tm^2}{\sqrt{1+p^2+q^2}}. \tag{1}$$

253. To Find the Principal Radii at Any Point on a Surface.—We have only to find the maximum and minimum values of R in (1), § 251, § 252.

I. In (1), § 251, let l, m, n vary subject to the two conditions

$$lL+mM+nN=0,\quad l^2+m^2+n^2=1.$$

Then, by the method of § 217,

$$\begin{aligned}Al+Hm+Gn+\lambda L+\mu l&=0,\\ Hl+Bm+Fn+\lambda M+\mu m&=0,\\ Gl+Fm+Cn+\lambda N+\mu n&=0.\end{aligned}$$

Multiply by l, m, n, respectively and add. $\therefore\quad \mu=-\kappa/R$.

$$\begin{aligned}\therefore\quad (A-\kappa/R)l+{}&Hm+{}&&Gn+\lambda L=0,\\ Hl+(B-\kappa/R)m+{}&Fm+(C-\kappa/R)n+\lambda N&&=0,\end{aligned}$$

Eliminating l, m, n, λ, we get the quadratic

$$\begin{vmatrix}A-\kappa/R, & H, & G, & L\\ H, & B-\kappa/R, & F, & M\\ G, & F, & C-\kappa/R, & N\\ L, & M, & N, & 0\end{vmatrix}=0,$$

the roots of which are the principal radii of curvature at the point at which the derivatives are taken.

II. If $z = f(x, y)$ be the equation to the surface, then in (1), § 252, we have l, m, n subject to the two conditions

$$pl + qm - n = 0, \quad \text{and} \quad l^2 + m^2 + n^2 = 1,$$

which reduce to the single condition

$$(1 + p^2)l^2 + 2pqlm + (1 + q^2)m^2 = 1.$$

Applying the general method for finding the maximum and minimum values to (1), § 252,

$$rl + sm + \lambda[(1 + p^2)l + pqm] = 0,$$
$$sl + tm + \lambda[pql + (1 + q^2)m] = 0.$$

Multiply respectively by l and m and add. Whence $\lambda = -\kappa/R$. Eliminating l and m from

$$[rR - (1 + p^2)\kappa]l + (sR - pq\kappa)m = 0,$$
$$(sR - pq\kappa)l + [tR - (1 + q^2)\kappa]m = 0,$$

there results the quadratic

$$[rR - (1 + p^2)\kappa][tR - (1 + q^2)\kappa] - (sR - pq\kappa)^2 = 0,$$

or

$$(rt - s^2)R^2 - [r(1 + q^2) + t(1 + p^2) - 2pqs]\kappa R + \kappa^4 = 0,$$

for finding the radii of principal curvature. In this equation

$$\kappa^2 = 1 + p^2 + q^2.$$

The problem of finding the directions of the principal sections and the magnitude of the principal radii of curvature is the same as that of finding the direction and magnitude of the principal axes of a section of the conicoid

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gxz + 2Hxy = 1,$$

made by the plane $Lx + My + Nz = 0$.

254. To Determine the Umbilics on a Surface.—At an umbilic the radius of normal curvature is the same for all normal sections. Consequently equation (1), § 251, for any three particular tangent lines will furnish the conditions which must exist at an umbilic.

Through any umbilic pass three planes parallel to the coordinate planes cutting the tangent plane there in three tangent lines whose direction cosines are $l_1, m_1, 0$; $l_2, 0, n_2$; $0, m_3, n_3$, respectively. Then equating the corresponding values of κ/R in (1), § 251,

$$Al_1^2 + Bm_1^2 + 2Hl_1m_1 = Al_2^2 + Cn_2^2 + 2Gl_2n_2 = Bm_3^2 + Cn_3^2 + 2Fm_3n_3.$$

Also, since these three tangent lines are parallel to the tangent plane, the equations

$$Ll_1 + Mm_1 = Ll_2 + Nn_2 = Mm_3 + Nn_3 = 0$$

give

$$l_1^2 = \frac{M^2}{L^2 + M^2}, \quad m_1^2 = \frac{L^2}{L^2 + M^2},$$

and l_1, m_1 have opposite signs. The same equations give like values for l_2, n_2, etc. On substitution we obtain the conditions which must exist at an umbilic,

$$\frac{AM^2 + BL^2 - 2HLM}{L^2 + M^2} = \frac{AN^2 + CL^2 - 2GLN}{L^2 + N^2}$$
$$= \frac{BN^2 + CM^2 - 2FMN}{M^2 + N^2}. \tag{1}$$

These two equations in x, y, z, together with the equation to the surface, give the points at which umbilics occur.

If the equation of the surface is $f(x, y) - z = 0$, results are correspondingly simplified and the conditions which must exist at an umbilic are immediately obtained from the fact that κ/R is constant for all values of l, m, n, satisfying the identical equations

$$\frac{\kappa}{R} = rl^2 + 2slm + tm^2,$$

$$1 = (1 + p^2)l^2 + 2pq\,lm + (1 + q^2)m^2.$$

Whence results, from proportionality of the constants,

$$\frac{\kappa}{R} = \frac{r}{1 + p^2} = \frac{s}{pq} = \frac{t}{1 + q^2}. \tag{2}$$

255. Equations (2), § 254, are very simply obtained by seeking the point on the surface $z = f(x, y)$ at which the sphere

$$\phi(x, y, z) \equiv (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 - \rho^2 = 0$$

osculates the surface $z = f$. The first and second partial derivatives of z in ϕ are the same as those for f at the point of osculation. Differentiating $\phi = 0$ partially with respect to x and y, we get

$$x - \alpha + (z - \gamma)\,p = 0,$$
$$y - \beta + (z - \gamma)\,q = 0,$$
$$1 + p^2 + (z - \gamma)\,r = 0,$$
$$1 + q^2 + (z - \gamma)\,t = 0,$$
$$pq + (z - \gamma)\,s = 0.$$

$$\therefore \quad -(z - \gamma) = \frac{pq}{s} = \frac{1 + p^2}{r} = \frac{1 + q^2}{t}.$$

Also, $R = -(z - \gamma)\sqrt{1 + p^2 + q^2}$, since the direction secant of the normal with the z-axis is $-(1 + p^2 + q^2)^{\frac{1}{2}}$.

256. Measure of Curvature of a Surface.—The measure of curvature of a surface is an extension of the measure of curvature of a curve in a plane, as follows:

The measure of entire curvature of a curve in a plane is the amount of bending. Let P_1 and P_2 be two points on a curve whose distances, measured along the curve, from a fixed point are s_1 and s_2. Let ϕ_1 and ϕ_2 be the angles which the tangents at P_1, P_2 make with a fixed line in the plane of the curve. Then the whole change of direction of the curve between P_1 and P_2 is the angle $\phi_2 - \phi_1$. This angle is also the angle through which the normal has turned as a point P passes from P_1 to P_2 along the curve.

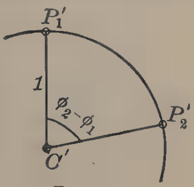

FIG. 145.

This angle between the normals is called the entire curvature of the curve for the portion P_1P_2. It can also be measured on a standard circle of radius r, as the angle between two radii parallel to the normals to the curve at P_1, P_2. If $P_1'P_2'$ be the subtended arc in the standard circle (Fig. 145), the whole curvature of P_1P_2 is proportional to $P_1'P_2'$, or

$$\phi_2 - \phi_1 = \frac{s_2' - s_1'}{r}.$$

If the standard circle be taken with unit radius, the entire curvature of P_1P_2 is measured by the arc $s_2' - s_1'$ on the unit circle.

The *mean* curvature, or *average* curvature, of P_1P_2 is the entire curvature divided by the length of the curve P_1P_2,

$$\frac{\phi_2 - \phi_1}{s_2 - s_1} = \frac{s_2' - s_1'}{s_2 - s_1},$$

or, is the quotient of the corresponding arc on the unit circle divided by the length of curve P_1P_2.

The *specific curvature* of a curve, or the measure of curvature of a curve at a point P, is the limit of the mean curvature, as the length of the arc converges to zero. It is therefore the derivative of ϕ with respect to s. But since $ds = Rd\phi$, where R is the radius of curvature of the curve at a point, we have for the specific curvature

$$\frac{d\phi}{ds} = \frac{1}{R}.$$

The curvature of a curve at a point is therefore properly measured by the reciprocal of the radius of curvature.

To extend this to surfaces, we measure a *solid* or *conical* angle by describing a sphere with the vertex of a cone as center and radius r. Then the measure of the solid angle ω is defined to be the area of the surface cut out of the sphere by the cone, divided by the *square* on the radius, or

$$\omega = \frac{A}{R^2}.$$

The unit solid angle, called the *steradian*, is that solid angle which cuts out an area A equal to the square on the radius. In particular, if we take as a standard sphere one of unit radius, then

$$\omega = A,$$

or, the area subtended is the measure of the solid angle.

Definition.—The entire curvature of any given portion of a curved surface is measured by the area enclosed on a sphere surface, of unit radius, by a cone whose vertex is the center of the sphere and whose generating lines are parallel to the normals to the surface at every point of the boundary of the given portion of the surface.

Horograph.—The curve traced on the surface of a sphere of unit radius by a line through the center moving so as to be always parallel to a normal to a surface at the boundary of a given portion of the surface is called the *horograph* of the given portion of the surface.

Mean or average curvature of any surface. The mean or average curvature of any portion of a surface is the entire curvature (area of the horograph), divided by the area of the given portion of the surface. If S be the area of the given portion and ω the entire curvature, the mean curvature is

$$\frac{\omega}{S}.$$

Specific Curvature of a surface, or curvature of a surface at a point. The specific curvature of a surface at any point, or, as we briefly say, the curvature of a surface at a point on the surface, is the limit of the average curvature of a portion of the surface containing the point, as the area of that portion converges to 0. In symbols, the curvature at a point is

$$\frac{d\omega}{dS}.$$

Gauss' Theorem. The curvature of a surface at any point is equal to the reciprocal of the product of the principal radii of curvature of the surface at the point, or

$$\frac{d\omega}{dS} = \frac{1}{R_1 R_2}.$$

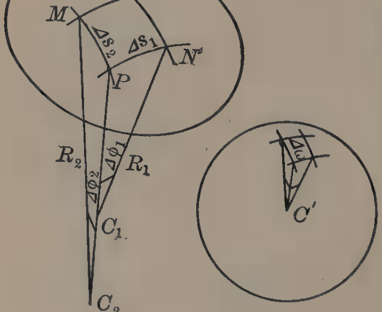

FIG. 146.

Let S be any portion of a surface containing a point P.

Draw the principal normal sections $PM = \Delta s_2$, $PN = \Delta s_1$.

$$\therefore \Delta S = \Delta s_1 \cdot \Delta s_2 = R_1' \Delta\phi_1 \cdot R_2' \Delta\phi_2$$

$$\Delta\omega = \Delta\sigma_1 \cdot \Delta\sigma_2 = \Delta\phi_1 \cdot \Delta\phi_2.$$

$\Delta\sigma_1$, $\Delta\sigma_2$ being the arcs of the horograph corresponding to Δs_1, Δs_2, on the surface.

$$\therefore \quad \frac{\Delta\omega}{\Delta S} = \frac{1}{R_1' R_2'}.$$

In the limit

$$\frac{d\omega}{dS} = \frac{1}{R_1 R_2}.$$

EXERCISES.

1. Find the principal radii of curvature at the origin for the surface

$$2z = 6x^2 - 5xy - 6y^2. \qquad \textit{Ans. } \tfrac{2}{13}, -\tfrac{2}{13}.$$

2. A surface is formed by the revolution of a parabola about its directrix; show that the principal radii of curvature at any point are in the constant ratio 1 : 2.

3. Find the principal radii of curvature, at x, y, z, of the surface

$$y \cos \frac{z}{a} - x \sin \frac{z}{a} = 0. \qquad \textit{Ans. } \pm \frac{x^2 + y^2 + a^2}{a}.$$

4. Show that at all points on the curve in which the planes $z = \pm \dfrac{a-b}{2ab}$ cut the hyperbolic paraboloid $2z = ax^2 - by^2$ the radii of principal curvature of the latter surface are equal and opposite. This curve is also the locus of points at which the right-line generators are at right angles.

5. Show from (6), § 248, that the mean curvature of all the normal sections of a surface at a point is

$$\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

6. Show that at every point on the revolute generated by a catenary revolving about its axis, the principal radii of curvature are equal and opposite.

7. Show that at every point on a sphere the specific curvature is constant and positive.

8. Show that at every point of a plane the specific curvature is constant and 0.

9. Show that at every point on the revolute generated by the tractrix revolving about its asymptote, the specific curvature is constant and negative. This surface is called the pseudo-sphere.

10. If the plane curve given by the equations

$$x/a = \cos\theta + \log \tan \tfrac{1}{2}\theta, \quad y/a = \sin\theta,$$

revolves about Ox, the surface generated has its specific curvature constant.

11. If R_1, R_2 are the principal radii of curvature at any point of the ellipsoid on the line of intersection with a given concentric sphere, prove that

$$\frac{(R_1 R_2)^{\frac{1}{2}}}{R_1 + R_2} = \text{const.}$$

12. Prove that the specific curvature at any point of the elliptic or hyperbolic paraboloid $y^2/b + z^2/c = x$ varies as $(p/z)^4$, p being the perpendicular from the origin on the tangent plane.

13. In the helicoid $y = x \tan (z/a)$ show that the principal radii of curvature, at every point at the intersection of the helicoid with a coaxial cylinder, are constant and equal in magnitude, opposite in sign.

14. Prove that the specific curvature at every point of the elliptic paraboloid $2z = x^2/a + y^2/b$, where it is cut by the cylinder $x^2/a^2 + y^2/b^2 = 1$, is $(4ab)^{-1}$.

15. Prove that the principal curvatures are equal and opposite in the surface $x^2(y - z) + ayz = 0$ where it is met by the cone $(x^2 + 6yz)yz = (y - z)^4$.

16. The principal radii of curvature at the points of the surface

$$a^2x^2 = z^2(x^2 + y^2), \quad \text{where} \quad x = y = z,$$

are given by

$$2R^2 + 2\sqrt{3}\,aR - 9a^2 = 0.$$

17. Prove that the radius of curvature of the surface $x^m + y^m + z^m = a^m$ at an umbilic is $3^{\frac{m-2}{2m}} a/(m - 1)$.

18. Show that $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ is an umbilic on the surface

$$x^3/a + y^3/b + z^3/c = k^2.$$

19. Show that $x = y = z = (abc)^{\frac{1}{3}}$ is an umbilic on the surface $xyz = abc$ and the curvature there is $\frac{1}{3}(abc)^{-\frac{1}{3}}$.

20. Find the umbilici on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Ans. The four real umbilics are $x^2 = \frac{a^2(a^2 - b^2)}{a^2 - c^2}$, $z^2 = \frac{c^2(b^2 - c^2)}{a^2 - c^2}$.

21. At an ordinary point on a surface the locus of the centers of curvature of all plane sections is a fixed surface, whose equation referred to the tangent plane as z-plane and the principal planes as the x- and y-planes, is

$$(x^2 + y^2 + z^2)\left(\frac{x^2}{R_1} + \frac{y^2}{R_2}\right) = z(x^2 + y^2).$$

22. Show that an umbilicus on the surface

$$(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} + (z/c)^{\frac{2}{3}} = 1$$

is given by

$$\frac{1}{a}\left(\frac{x}{a}\right)^{-\frac{2}{3}} = \frac{1}{b}\left(\frac{y}{b}\right)^{-\frac{2}{3}} = \frac{1}{c}\left(\frac{z}{c}\right)^{-\frac{2}{3}}.$$

23. If $F = 0$ is the equation of a conicoid, show that the tangent cone to the surface drawn from the vertex α, β, γ touches a surface along a plane curve which is the intersection of $F = 0$ and the plane

$$(x - \alpha)\frac{\partial F}{\partial \alpha} + (y - \beta)\frac{\partial F}{\partial \beta} + (z - \gamma)\frac{\partial F}{\partial \gamma} + F(\alpha, \beta, \gamma) = 0.$$

24. Find the quadratic equation for determining the principal radii of curvature at any point of the surface

$$\phi(x) + \psi(y) + \chi(z) = 0,$$

and find the condition that the principal curvatures may be equal and opposite.

25. Show that the cylinder

$$(a^2 + c^2)b^2x^2 + (b^2 + c^2)a^2y^2 = (a^2 + b^2)a^2b^2$$

cuts the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ in a curve at each point of which the principal curvatures of the hyperboloid are equal and opposite.

26. Show that the principal radii of curvature are equal and opposite at every point in which the plane $x = a$ cuts the surface

$$x(x^2 + y^2 + z^2) = 2a(x^2 + y^2).$$

27. In the surface in Ex. 24 show that the point which satisfies

$$\phi''(x) = \psi''(y) = \chi''(z)$$

is an umbilic.

28. Find the umbilici on the surface $2z = x^2/a + y^2/b$.

Ans. $x = 0$, $y = -\sqrt{(ab - b^2)}$, $z = \frac{1}{2}(a - b)$, if $a > b$.

29. Show that $z = f(x, y)$ is generated by a straight line if at all points

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2.$$

This is also the condition that the inflexional tangents at each point of the surface shall be coincident. Such a surface is called a *torse* or developable surface.

CHAPTER XXXIV.

CURVES IN SPACE.

257. General Equations.—A curve in space is generally defined as the intersection of two surfaces. A curve will in general have for its equations

$$\phi_1(x, y, z) = 0, \quad \phi_2(x, y, z) = 0. \qquad (1)$$

If between these two equations we eliminate successively x, y, z, we obtain the projecting cylinders of the curve on the coordinate planes, respectively,

$$\psi_1(y, z) = 0, \quad \psi_2(x, z) = 0, \quad \psi_3(x, y) = 0.$$

Any two of these can be taken as the equations of the curve.

258. A curve in space is also determined when the coordinates of any point on the curve are given as functions of some fourth variable, such as t,

$$x = \phi(t), \quad y = \psi(t), \quad z = \chi(t). \qquad (2)$$

The elimination of t between these equations two and two give the projecting cylinders of the curve.

259. Equations of the Tangent to a Curve at a Point.—If the equations of the line are (1), the equations of the tangent line to (1) at x, y, z are the equations to the tangent planes to $\phi_1 = 0$, $\phi_2 = 0$, taken simultaneously, or

$$\left.\begin{aligned} (X - x)\frac{\partial \phi_1}{\partial x} + (Y - y)\frac{\partial \phi_1}{\partial y} + (Z - z)\frac{\partial \phi_1}{\partial z} = 0, \\ (X - x)\frac{\partial \phi_2}{\partial x} + (Y - y)\frac{\partial \phi_2}{\partial y} + (Z - z)\frac{\partial \phi_2}{\partial z} = 0. \end{aligned}\right\} \qquad (1)$$

Since the tangent line is perpendicular to the normals to these planes, the direction cosines l, m, n of the tangent line are given by

$$\frac{l}{M_1N_2 - M_2N_1} = \frac{m}{N_1L_2 - N_2L_1} = \frac{n}{L_1M_2 - L_2M_1} = \frac{1}{\kappa},$$

where

$$\kappa^2 = (M_1N_2 - M_2N_1)^2 + (N_1L_2 - N_2L_1)^2 + (L_1M_2 - L_2M_1)^2.$$

L_1, M_1, N_1 being the first partial derivatives of ϕ_1 at x, y, z, and similarly L_2, M_2, N_2 are those of ϕ_2.

260. If s is the length of a curve measured from a fixed point to x, y, z, then the direction cosines of the tangent to the curve at x, y, z are

$$l = \frac{dx}{ds}, \quad m = \frac{dy}{ds}, \quad n = \frac{dz}{ds},$$

and the equations of the tangent are

$$\frac{X-x}{\frac{dx}{ds}} = \frac{Y-y}{\frac{dy}{ds}} = \frac{Z-z}{\frac{dz}{ds}}. \tag{2}$$

If the equations to the curve be given by (2), § 258, then

$\frac{dx}{ds} = \phi'(t)\frac{dt}{ds}$, etc., and the equations (2) become

$$\frac{X-x}{\phi'(t)} = \frac{Y-y}{\psi'(t)} = \frac{Z-z}{\chi'(t)}. \tag{3}$$

In general the equations to the tangent are

$$\frac{X-x}{dx} = \frac{Y-y}{dy} = \frac{Z-z}{dz}, \tag{4}$$

without specifying the independent variable.

261. The Equation to the Normal Plane to a Curve at x, y, z is

$$(X-x)\frac{dx}{ds} + (Y-y)\frac{dy}{ds} + (Z-z)\frac{dz}{ds} = 0, \tag{1}$$

the normal plane being defined as the plane which is perpendicular to the tangent at the point of contact.

Regardless of the independent variable, (1) becomes

$$(X-x)dx + (Y-y)dy + (Z-z)dz = 0. \tag{2}$$

EXAMPLES.

1. Find the tangent line to the central plane section of an ellipsoid.

The equations of the curve are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad Ax + By + Cz = 0.$$

The equations of the tangent at x, y, z are

$$\frac{X-x}{C\frac{y}{b^2} - B\frac{z}{c^2}} = \frac{Y-y}{A\frac{z}{c^2} - C\frac{x}{a^2}} = \frac{Z-z}{B\frac{x}{a^2} - A\frac{y}{b^2}}.$$

2. Trace the curve (the helix)

$$x = a\cos t, \quad y = a\sin t, \quad z = bt.$$

Show that the tangent makes a constant angle with the x, y plane, and that the curve is a line drawn on a circular cylinder of revolution cutting all the elements at a constant angle.

3. Find the highest and lowest points on the curve of intersection of the surfaces

$$2z = ax^2 + by^2, \quad Ax + By + Cz + D = 0,$$

from the fact that at these points the tangent to the curve must be horizontal.

4. Show that at every point of a line of steepest slope on any surface $F = 0$ we must have

$$\frac{\partial F}{\partial x} dy - \frac{\partial F}{\partial y} dx = 0.$$

5. Show that the lines of steepest slope on the right conoid $x = y f(z)$ are cut out by the cylinders $x^2 + y^2 = r^2$, r being an arbitrary radius.

262. Osculating Plane.—If P, Q, R be three points on a curve, these three points determine a plane. The limiting position of this plane when P, Q, R converge to one point as a limit is called the *osculating* plane of the curve at that point.

The coordinates x, y, z of any point on a curve are functions of the length, s, of the curve measured from some fixed point to x, y, z. Therefore, if s_1 be the length to x_1, y_1, z_1,

$$x_1 = x + (s_1 - s)\frac{dx}{ds} + \tfrac{1}{2}(s_1 - s)^2\frac{d^2x}{ds^2} + \tfrac{1}{6}(s_1 - s)^3\frac{d^3x}{d\sigma^3},$$

where σ is the length to some point between x, y, z and x_1, y_1, z_1.

Put $\delta s \equiv s_1 - s$, $x' \equiv D_s x$, $x'' \equiv D_s^2 x$, etc., then

$$x_1 = x + \delta s \cdot x' + \tfrac{1}{2}\delta s^2 \cdot x'' + \tfrac{1}{6}\delta s^3 \cdot x'''_\sigma.$$

Let P, Q, R be x, y, z; x_1, y_1, z_1; x_2, y_2, z_2. Then

$$x_1 = x + \delta s \cdot x'_\sigma, \quad y_1 = y + \delta s \cdot y'_\sigma, \quad z_1 = z + \delta s \cdot z'_\sigma. \tag{1}$$

$$\left.\begin{aligned} x_2 &= x + k\delta s \cdot x' + \tfrac{1}{2}k^2\delta s^2 \cdot x''_{\sigma'}, \\ y_2 &= y + k\delta s \cdot y' + \tfrac{1}{2}k^2\delta s^2 \cdot y''_{\sigma'}, \\ z_2 &= z + k\delta s \cdot z' + \tfrac{1}{2}k^2\delta s^2 \cdot z''_{\sigma'}. \end{aligned}\right\} \tag{2}$$

The equation to the plane through P can be written

$$A(X - x) + B(Y - y) + C(Z - z) = 0. \tag{3}$$

If this passes through Q and R, then

$$\left.\begin{aligned} A(x_1 - x) + B(y_1 - y) + C(z_1 - z) &= 0, \\ A(x_2 - x) + B(y_2 - y) + C(z_2 - z) &= 0. \end{aligned}\right\} \tag{4}$$

Substitute the values of the coordinates from (1) and (2) in (4). Divide by δs, δs^2, and let $\delta s (=) 0$.

$$\therefore \quad \left.\begin{aligned} Ax' + By' + Cz' &= 0, \\ Ax'' + By'' + Cz'' &= 0. \end{aligned}\right\} \tag{5}$$

Eliminating A, B, C between (3) and (5), we have the equation to the osculating plane at x, y, z,

$$\begin{vmatrix} X - x, & Y - y, & Z - z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0. \tag{6}$$

Or, regardless of the independent variable,

$$\begin{vmatrix} X - x, & Y - y, & Z - z \\ dx & dy & dz \\ d^2x & d^2y & d^2z \end{vmatrix} = 0. \tag{7}$$

263. To Find the Condition that a Curve may be a Plane Curve.—If a curve is a plane curve, the coordinates of any point must satisfy a linear relation

$$Ax + By + Cz + D = 0,$$

where A, B, C, D are constants. Differentiating,

$$Adx + Bdy + Cdz = 0,$$
$$Ad^2x + Bd^2y + Cd^2z = 0,$$
$$Ad^3x + Bd^3y + Cd^3z = 0.$$

Eliminating A, B, C, we have the condition

$$\begin{vmatrix} dx, & dy, & dz \\ d^2x, & d^2y, & d^2z \\ d^3x, & d^3y, & d^3z \end{vmatrix} = 0,$$

which must be satisfied at all points on the curve.

264. Equations of the Principal Normal.—The principal normal to a curve at a point is the intersection of the osculating plane and the normal plane at the point.

Let l, m, n be the direction cosines of the principal normal at x, y, z. Then, since this line lies in the normal and osculating planes,

$$l\begin{vmatrix} y' & z' \\ y'' & z'' \end{vmatrix} + m\begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix} + n\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix} = 0,$$

$$lx' + my' + nz' = 0.$$

These conditions are satisfied by $l = x''$, $m = y''$, $n = z''$, since

$$x''\begin{vmatrix} y' & z' \\ y'' & z'' \end{vmatrix} + y''\begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix} + z''\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix} \equiv \begin{vmatrix} x'' & y'' & z'' \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0.$$

Also differentiating $x'^2 + y'^2 + z'^2 = 1$,

$$\therefore \quad x'x'' + y'y'' + z'z'' = 0.$$

Therefore the equations of the principal normal are

$$\frac{X - x}{x''} = \frac{Y - y}{y''} = \frac{Z - z}{z''}, \tag{1}$$

or

$$\frac{X - x}{d^2x} = \frac{Y - y}{d^2y} = \frac{Z - z}{d^2z}. \tag{2}$$

265. The Binormal.—The binormal to a curve at a point is the straight line perpendicular to the osculating plane at the point.

Its equations are therefore, from (6), § 262,

$$\frac{X - x}{y' z'' - y'' z'} = \frac{Y - y}{z' x'' - z'' x'} = \frac{Z - z}{x' y'' - x'' y'}. \qquad (4)$$

Dividing through by ds^3, the equations can be written without specifying the independent variable.

266. The Circle of Curvature.—The circle of curvature at a given point of a space curve is the limiting position of the circle passing through three points on the curve when the three points converge to the given point.

Clearly, the circle of curvature lies in the osculating plane and is the osculating circle of the curve.

To find the radius of curvature. Let α, β, γ be the coordinates of the center, and ρ the radius of the circle of curvature at x, y, z. Then

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = \rho^2.$$

Let x, y, z vary on the circle. Differentiate twice with respect to s. Then

$$(x - \alpha)x'' + (y - \beta)y'' + (z - \gamma)\, z'' + x'^2 + y'^2 + z'^2 = 0.$$

But $x'^2 + y'^2 + z'^2 = 1$. Also, the line through x, y, z and α, β, γ is the principal normal, whose direction cosines, by (1), § 264, are

$$l = \frac{x''}{\sqrt{x''^2 + y''^2 + z''^2}},$$

with similar values for m and n. Since

$$x - \alpha = l\rho, \quad y - \beta = m\rho, \quad z - \gamma = n\rho,$$

$$\therefore \quad \frac{1}{\rho^2} = x''^2 + y''^2 + z''^2.$$

The center of the circle is $\alpha = x - l\rho$, etc.

267. The direction cosines of the binormal are

$$l = \rho(y'z'' - z'y''), \quad m = \rho(z'x'' - x'z''), \quad n = \rho(x'y'' - y'x'').$$

For, by (4), § 265,

$$\frac{l}{y'z'' - z'y''} = \frac{m}{z'x'' - x'z''} = \frac{n}{x'y'' - y'x''}. \qquad (1)$$

Also differentiating $x'^2 + y'^2 + z'^2 = 1$,

$$\therefore \quad x'x'' + y'y'' + z'z'' = 0.$$

The sum of the squares of the denominators in (1) is

$$(x'^2 + y'^2 + z'^2)(x''^2 + y''^2 + z''^2) - (x'x'' + y'y'' + z'z'') = 1/\rho^2.$$

Hence the results stated.

268. Tortuosity. Measure of Twist.

Definition.—The measure of torsion or twist of a space curve is the rate per unit length of curve at which the osculating plane turns around the tangent to the curve, as the point of contact moves along the curve.

If the osculating plane turns through the angle $\Delta\tau$ as the point of contact P moves to Q through the arc Δs, the measure of torsion at P is

$$\frac{d\tau}{ds} = \mathcal{L}\frac{\Delta\tau}{\Delta s},$$

when $\Delta s (=) 0$. The number $\sigma = D_\tau s$ is sometimes called the radius of torsion.

Let l_1, m_1, n_1; l_2, m_2, n_2, be the direction cosines of two planes including an angle θ. Then

$$\sin^2\theta = (m_1n_2 - n_1m_2)^2 + (n_1l_2 - l_1n_2)^2 + (l_1m_2 - m_1l_2)^2.$$

Let l, m, n be the direction cosines of the osculating plane at P, and $l + \Delta l$, $m + \Delta m$, $n + \Delta n$ those at Q.

Let $\Delta\tau$ be the angle between these planes. Then

$$\sin^2\Delta\tau = (m\Delta n - n\Delta m)^2 + (n\Delta l - l\Delta n)^2 + (l\Delta m - m\Delta l)^2.$$

Divide by Δs^2 and write

$$\frac{\sin^2\Delta\tau}{\Delta s^2} = \frac{\sin^2\Delta\tau}{\Delta\tau^2}\left(\frac{\Delta\tau}{\Delta s}\right)^2.$$

Let $\Delta s (=) 0$. Then, in the limit,

$$\left(\frac{d\tau}{ds}\right)^2 = \left(m\frac{dn}{ds} - n\frac{dm}{ds}\right)^2 + \left(n\frac{dl}{ds} - l\frac{dn}{ds}\right)^2 + \left(l\frac{dm}{ds} - m\frac{dl}{ds}\right)^2. \quad (1)$$

Since $l^2 + m^2 + n^2 = 1$, $\therefore\ l\frac{dl}{ds} + m\frac{dm}{ds} + n\frac{dn}{ds} = 0.$

Square this last equation and subtract from (1).

$$\therefore \left(\frac{d\tau}{ds}\right)^2 = \left(\frac{dl}{ds}\right)^2 + \left(\frac{dm}{ds}\right)^2 + \left(\frac{dn}{ds}\right)^2. \quad (2)$$

269. The measure of torsion can be expressed in another form, as follows.

Let l, m, n be the direction cosines of the binormal, and $L = y'z'' - z'y''$, etc., as in § 267. Then

$$\frac{l}{L} = \frac{m}{M} = \frac{n}{N} = \rho. \quad (1)$$

Whence $$L^2 + M^2 + N^2 = 1/\rho^2. \quad (2)$$

Since the binormal is perpendicular to the tangent and principal normal,

$$lx' + my' + nz' = 0, \quad (3)$$

$$lx'' + my'' + nz'' = 0. \quad (4)$$

Differentiating (3) and using (4),

$$l'x' + m'y' + n'z' = 0. \qquad (5)$$

Differentiating, $\qquad l^2 + m^2 + n^2 = 1. \qquad (6)$

$$\therefore \quad ll' + mm' + nn' = 0. \qquad (7)$$

From (5) and (7) we get

$$\frac{l'}{mz' - ny'} = \frac{m'}{nx' - lz'} = \frac{n'}{l'y - mx'}, \qquad (8)$$

and each of these is equal to

$$\frac{l'x''}{mz'x'' - ny'x''} = \frac{m'y''}{nx'y'' - lz'y''} = \frac{n'z''}{ly'z'' - mx'z''}$$

$$= \frac{l'x'' + m'y'' + n'z''}{lL + mM + nN}. \qquad (9)$$

Differentiating (4),

$$l'x'' + m'y'' + n'z'' + lx''' + my''' + nz''' = 0.$$

Therefore (8) is equal to

$$-\frac{lx''' + my''' + nz'''}{lL + mM + nN} = -\frac{x'''L + y'''M + z'''N}{L^2 + M^2 + N^2}.$$

Remembering that l, m, n; x', y', z' are the direction cosines of two lines at right angles,

$$(mz' - ny')^2 + (nx' - lz')^2 + (ly' - mx')^2 = \sin^2 \tfrac{1}{2}\pi = 1.$$

Therefore, by (2), § 268, and (8),

$$\left(\frac{d\tau}{ds}\right)^2 = \left(\frac{x'''L + y'''M + z'''N}{L^2 + M^2 + N^2}\right)^2,$$

or

$$\frac{1}{\sigma} = \frac{d\tau}{ds} = \rho^2 \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}, \qquad (10)$$

by (2), and the determinant form of

$$x'''L + y'''M + z'''N.$$

270. Spherical Curvature.—Through any four points on a space curve can be passed one determinate sphere. The limit to which converges this sphere when the four points converge to one as a limit is called the osculating sphere, or sphere of curvature.

Differentiating the equation of the sphere,

$$\begin{aligned} &(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = R^2. \\ \therefore \quad &(x - \alpha)x' + (y - \beta)y' + (z - \gamma)z' = 0, \\ &(x - \alpha)x'' + (y - \beta)y'' + (z - \gamma)z'' = -1, \\ &(x - \alpha)x''' + (y - \beta)y''' + (z - \gamma)z''' = 0. \end{aligned}$$

Eliminating between the last three equations,

$$(x-\alpha) = \begin{vmatrix} y' & z' \\ y''' & z''' \end{vmatrix} : \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} = \sigma\rho^2(y'z''' - z'y'');$$

$$y - \beta = \sigma\rho^2(z'x''' - z'''x'); \quad z - \gamma = \sigma\rho^2(x'y''' - x'''y').$$

Squaring and adding,

$$R^2 = \sigma^2\rho^4[(x'y''' - y'x''')^2 + (y'z''' - z'y''')^2 + (z'x''' - x'z''')^2].$$

Clearly the circle of curvature lies on the sphere of curvature. Let P, Q, R, J be four points on a curve and in the same neighborhood, R and ρ the radii of spherical and circular curvature.

Then, C being the center of the circle through P, Q, R, and S that of the sphere through P, Q, R, J, we have directly from the figure,

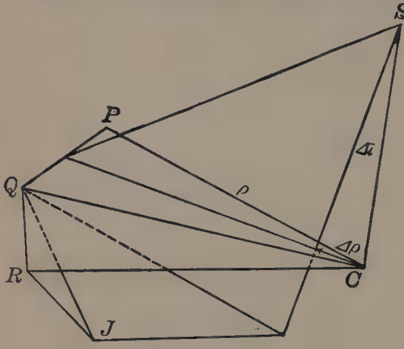

FIG. 147.

$$SC = \frac{\Delta\rho}{\Delta\tau}, \quad R^2 = \rho^2 + \left(\frac{\Delta\rho}{\Delta\tau}\right)^2.$$

$$\therefore \quad R^2 = \rho^2 + \left(\frac{d\rho}{d\tau}\right)^2$$

$$= \rho^2 + \sigma^2\left(\frac{d\rho}{ds}\right)^2.$$

$$SC = \frac{d\rho}{d\tau} = \sigma\frac{d\rho}{ds}.$$

271. The expressions for the value of the radius of curvature and measure of torsion in § 266, and (10), § 269, have been worked out with respect to s, the curve length, as the independent variable. These can be written in differentials, regardless of whatever variable be taken as the independent variable.

Represent by

$$\begin{Vmatrix} dx & dy & dz \\ d^2x & d^2y & d^2z \end{Vmatrix}^2$$

the sum of the squares of the three determinants

$$(dy\,d^2z - dz\,d^2y)^2 + (dz\,d^2x - dx\,d^2z)^2 + (dx\,d^2y - dy\,d^2x)^2.$$

Then, regardless of the independent variable employed,

$$\rho = \frac{(dx^2 + dy^2 + dz^2)^{\frac{3}{2}}}{\sqrt{\begin{Vmatrix} dx & dy & dz \\ d^2x & d^2y & d^2z \end{Vmatrix}^2}}. \tag{1}$$

$$\frac{1}{\sigma} = \frac{\begin{vmatrix} dx & dy & dz \\ d^2x & d^2y & d^2z \\ d^3x & d^3y & d^3z \end{vmatrix}}{\begin{Vmatrix} dx & dy & dz \\ d^2x & d^2y & d^2z \end{Vmatrix}^2}. \tag{2}$$

(1) comes immediately from § 267, and (2) from putting the value of ρ^2 from (1), § 271 in (10), § 269.

EXERCISES.

1. Show that in a plane curve the torsion is 0.

2. The equations of the tangent at x, y, z to the curve whose equations are $ax^2 + by^2 + cz^2 = 1$, $bx^2 + cy^2 + az^2 = 1$, are

$$\frac{x(X - x)}{ab - c^2} = \frac{y(Y - y)}{bc - a^2} = \frac{z(Z - z)}{ac - b^2}.$$

3. The equations of a line are

$$x^2 + y^2 + z^2 = 4a^2 \quad \text{and} \quad x^2 + z^2 = 2ax.$$

Show that the equations of the tangent line and normal plane are

$$\left.\begin{aligned} (x - a)X + zZ &= ax, \\ yY + aX &= a(4a - x). \end{aligned}\right\}; \quad \frac{X}{x} - \frac{Y}{y} = \left(1 - \frac{a}{x}\right)\left(\frac{Z}{z} - \frac{Y}{y}\right).$$

4. The equation of the normal plane to the intersection of

$$x^2/a + y^2/b + z^2/c = 1 \quad \text{and} \quad x^2 + y^2 + z^2 = d^2$$

is

$$\frac{X}{x}a(b - c) + \frac{Y}{y}b(c - a) + \frac{Z}{z}c(a - b) = 0.$$

5. Show that the curve $z(x + z)(x - a) = a^3$, $z(y + z)(y - a) = a^3$, is a plane curve.

6. If the osculating plane at every point of a curve pass through a fixed point, prove that the curve will be plane.

7. Prove that the surface $x^4 + y^4 + z^4 = \frac{1}{2}a^4$ cuts the sphere

$$x^2 + y^2 + z^2 = a^2$$

in great circles.

8. Show that the equations of the tangent to the curve

$$y^2 = ax - x^2, \quad z^2 = a^2 - ax,$$

are

$$X - x = \frac{2y}{a - 2x}(Y - y) = -\frac{2z}{a}(Z - z).$$

9. Find the osculating plane at any point of the curve

$$x = a \cos t, \quad y = b \sin t, \quad z = ct.$$

Ans. $c(Xy - Yx) + ab(Z - z) = 0.$

10. Find the radius of circular curvature at any point of

$$x/h + y/k = 1, \quad x^2 + z^2 = a^2.$$

Ans. $\dfrac{(h^2a^2 + k^2z^2)^{\frac{3}{2}}}{a^2h^2\sqrt{h^2 + k^2}}.$

11. Show that the curves of greatest slope to xOy on the surfaces $xyz = a^3$ and $cz = xy$ are the lines in which these surfaces are cut by the cylinder $x^2 - y^2 = \text{const.}$

12. Find the osculating plane at any point of the curve

$$x = a \cos\theta + b \sin\theta, \quad y = a \sin\theta + b \cos\theta, \quad z = c \sin 2\theta.$$

13. Find the principal normal at any point of

$$x^2 + y^2 = a^2, \quad az = x^2 - y^2.$$

Hint. Express x, y in terms of z as the independent variable.

14. Given the *helix* $x = a\cos\theta$, $y = a\sin\theta$, $z = b\theta$, show that

(1). The tangent makes a constant angle with the xy plane.
(2). Find the normal and osculating planes, principal normal.
(3). Locus of principal normals.
(4). Coordinates of center and radius of curvature.
(5). Radius of torsion.

Ans. (2). $aX\sin\theta - aY\cos\theta - b(z - b\theta) = 0,$

$bX\sin\theta - bY\cos\theta + a(z - b\theta) = 0.$

(3). $\dfrac{y}{x} = \tan\dfrac{z}{b}.$

(4). $\rho = a(1 + b^2/a^2).$

(5). $\sigma = (a^2 + b^2)/b.$

15. Show that $F = 0$, $F_z' = 0$ are the equations of the line of contact of the vertical enveloping cylinder of $F = 0$, and that the horizontal projection of this line is the envelope of the horizontal projections of parallel plane sections of $F = 0$.

16. Show that the equations of the level lines and lines of steepest slope on the surface $F = 0$ are

$$F = 0, \quad F_x' dx + F_y' dy = 0 \quad \text{and} \quad F = 0, \quad F_x' dy - F_y' dx = 0$$

respectively, and that they cross each other at right angles.

17. Find the lines of steepest slope on the surfaces

$$ax^2 + by^2 + cz^2 = 1 \quad \text{and} \quad z = ax^2 + by^2.$$

18. A line of constant slope on a surface is called a *Loxodrone.* Find the loxodrone on the cone $x^2 + y^2 = k(z - c)^2$. Show that its horizontal projection is a logarithmic spiral.

19. Find the loxodrone on the sphere $x^2 + y^2 + z^2 = a^2$.

20. *A line of curvature* on a surface is a line at every point of which the tangent to the line lies in a principal plane of the surface. Show that through every ordinary point on a surface pass two lines of curvature at right angles.

21. *A geodesic line* on a surface is a line whose osculating plane at any point contains the normal to the surface at that point. Use Meunier's Theorem to show that between two arbitrarily near points on the surface the geodesic is the shortest line that can be drawn on the surface. Show that at every point on a geodesic on the surface $F = 0$, we have

$$F_x'/d^2x = F_y'/d^2y = F_z'/d^2z.$$

CHAPTER XXXV.

ENVELOPES OF SURFACES.

272. Envelope of a Surface-Family having One Variable Parameter.—When $F(x, y, z) = 0$ is the equation of a surface containing an arbitrary parameter α, we can indicate the presence of this arbitrary parameter α by writing the equation

$$F(x, y, z, \alpha) = 0. \tag{1}$$

The position of the surface (1) depends on the value assigned to α. By assigning a continuous series of values to α we have a singly infinite family of surfaces whose equation is (1).

If we assign to α a particular value α_1, we have another position of the surface (1) whose equation is

$$F(x, y, z, \alpha_1) = 0. \tag{2}$$

The two surfaces (1) and (2) will in general intersect in a curve. When $\alpha_1 (=) \alpha$ the surface (2) converges to coincidence with the surface (1), and their line of intersection may converge to a definite position on (1). At any point on the intersection of (1) and (2) the values of x, y, z are the same in both equations. By the law of the mean,

$$F(x, y, z, \alpha_1) = F(x, y, z, \alpha) + (\alpha_1 - \alpha) F'_{\alpha'}(x, y, z, \alpha'),$$

α' being a number between α and α_1.

At any point of intersection of (1) and (2)

$$F(x, y, z, \alpha) = F(x, y, z, \alpha_1) = 0.$$

Therefore at any such point we have

$$F'_{\alpha'}(x, y, z, \alpha') = 0. \tag{3}$$

If, when $\alpha_1 (=) \alpha$, the line of intersection of (1) and (2) converges to a definite position on (1), then the coordinates of all points on this line must satisfy, by (3), the equation

$$\frac{\partial}{\partial \alpha} F(x, y, z, \alpha) = 0, \tag{4}$$

and the surface (4) passes through the limiting position of (1) and (2).

If from equations (1) and (4), i.e.,

$$F(x, y, z, \alpha) = 0, \quad F'_{\alpha}(x, y, z, \alpha) = 0, \tag{5}$$

α be eliminated, the result is an equation $\phi(x, y, z) = 0$, which is the surface generated by the line whose equations are (5), or $\phi = 0$ is the locus of the ultimate intersections of consecutive surfaces of the family (1). This locus is called the *envelope* of the family (1). The line whose equations are (5) is called the *characteristic* of the envelope.

273. Each Member of a Family of One Parameter is Tangent to the Envelope at all Points of the Characteristic.—The parameter α being assigned any constant value, the tangent plane to

$$F(x, y, z, \alpha) = 0, \text{ at } x, y, z, \text{ is}$$

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = 0. \tag{1}$$

But in $F(x, y, z, \alpha) = 0$, as x, y, z vary along the envelope, α also varies, and the equation to the tangent to the envelope is

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz + \frac{\partial F}{\partial \alpha} d\alpha = 0. \tag{2}$$

Since at any point x, y, z common to the surface $F = 0$ and the envelope, that is all along the characteristic, we have $F'_\alpha = 0$, the planes (1) and (2) coincide.

EXAMPLES.

1. Show that the envelope of a family of planes having a single parameter is a *torse* (developable surface).

Let $$z = x\phi(\alpha) + y\psi(\alpha) + \chi(\alpha).$$

$$\therefore \quad \frac{\partial z}{\partial x} = \phi(\alpha), \quad \frac{\partial z}{\partial y} = \psi(\alpha); \quad \because \quad x\phi'_\alpha + y\psi'_\alpha + \chi'_\alpha = 0.$$

Also,

$$\frac{\partial^2 z}{\partial x^2} = \phi'(\alpha)\frac{\partial \alpha}{\partial x}, \quad \frac{\partial^2 z}{\partial y^2} = \psi'(\alpha)\frac{\partial \alpha}{\partial y}, \quad \frac{\partial^2 z}{\partial x\, \partial y} = \phi'(\alpha)\frac{\partial \alpha}{\partial y} = \psi'(\alpha)\frac{\partial \alpha}{\partial x}.$$

Hence $rt - s^2 = 0$. See Ex. 29, § 256.

2. Envelop $\dfrac{x+y}{\alpha} + z\alpha = 2$.

Ans. Hyperbolic cylinder, $xz + yz = 1$.

3. Envelop $x + y - 2\alpha z = \alpha^2$.

Ans. Parabolic cylinder, $x + y + z^2 = 0$.

4. Generally if ϕ, ψ, χ are linear functions of x, y, z, then the envelope of the plane

$$\phi\alpha^2 + 2\psi\alpha + \chi = 0$$

is $\psi^2 = \phi\chi$, a cone or cylinder having $\phi = 0$, $\chi = 0$ as tangent planes, and $\psi = 0$ is a plane through the lines of contact.

5. Find the envelope of the family of spheres whose centers lie on the parabola $x^2 + 4ay = 0$, $z = 0$, and which pass through the origin.

Ans. $x^2 + y^2 + z^2 = 2ax^2/y$.

6. Find the envelope of a plane which forms with the coordinate planes a tetrahedron of constant volume.

Ans. $xyz = \text{const.}$

7. Find the envelope of a plane such that the sum of the squares of its intercepts on the axes is constant.

Ans. $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = \text{const.}$

274. Envelope of a Surface-Family with Two Variable Parameters.

If

$$F(\alpha, \beta) \equiv F(x, y, z, \alpha, \beta) = 0 \tag{1}$$

is a surface of the family, then

$$F(\alpha_1, \beta_1) \equiv F(x, y, z, \alpha_1, \beta_1) = 0 \tag{2}$$

is a second surface of the family.

At any point x, y, z where (1) and (2) meet,

$$F(\alpha_1, \beta_1) = F(\alpha, \beta) + (\alpha_1 - \alpha)\frac{\partial F}{\partial \alpha'} + (\beta_1 - \beta)\frac{\partial F}{\partial \beta'}, \tag{3}$$

where α' is between α_1 and α, β' between β_1 and β.

In virtue of (1) and (2), (3) gives

$$(\alpha_1 - \alpha)\frac{\partial F}{\partial \alpha'} + (\beta_1 - \beta)\frac{\partial F}{\partial \beta'} = 0. \tag{4}$$

This is the equation of a surface passing through the intersection of (1) and (2). But for any fixed values x, y, z, α, β satisfying (1) and (2) there are an indefinite number of surfaces (4) obtained by varying α_1, β_1, all of which cut (1) in lines passing through x, y, z. Consequently there are of these surfaces (4) two particular surfaces,

$$\frac{\partial F}{\partial \alpha'} = 0, \quad \frac{\partial F}{\partial \beta'} = 0,$$

which cut (1) in lines passing through x, y, z.

If now the point x, y, z has a determinate limit when $\alpha_1 (=) \alpha$, $\beta_1 (=) \beta$, then the three surfaces

$$F(\alpha, \beta) = 0, \quad F'_\alpha(\alpha, \beta) = 0, \quad F'_\beta(\alpha, \beta) = 0,$$

pass through and determine that point.

These surfaces (5) intersect, in general, in a discrete set of points. If, however, we eliminate between them α and β, we obtain the equation to the locus of intersections. This locus is a surface called the envelope of the family (1).

275. The Envelope of the Family $F(x, y, z, \alpha, \beta) = 0$ is Tangent to Each Member of the Family.—The tangent plane to any member of the family is

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0. \tag{1}$$

As the point x, y, z moves on the envelope, α and β vary. The plane tangent to the envelope is

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz + \frac{\partial F}{\partial \alpha}d\alpha + \frac{\partial F}{\partial \beta}d\beta = 0. \qquad (2)$$

At a point x, y, z common to the envelope and one of the surfaces, we have $F'_\alpha = 0$, $F'_\beta = 0$, and therefore the planes (1) and (2) coincide. Since this point is the intersection of the line whose equations are $F'_\alpha = 0$, $F'_\beta = 0$ with the surface $F = 0$, the envelope is tangent to the surface at a point, and not along a line.

276. Use of Arbitrary Multipliers.—If $F(x, y, z, \alpha, \beta) = 0$, where α, β are two arbitrary parameters connected by the relation $\phi(\alpha, \beta) = 0$, then $F = 0$ is a family of surfaces depending on a single variable parameter. The equation of the envelope is found by the elimination of α, β, $d\alpha$, $d\beta$ between

$$F = 0, \quad \phi = 0, \quad F'_\alpha\, d\alpha + F'_\beta\, d\beta = 0, \quad \phi'_\alpha\, d\alpha + \phi'_\beta\, d\beta = 0.$$

This is best effected, as in the corresponding problems of maximum and minimum, by the use of arbitrary multipliers. Thus the equation of the envelope is the result obtained by eliminating α, β, λ from

$$F = 0, \quad \phi = 0, \quad F'_\alpha + \lambda\phi'_\alpha = 0, \quad F'_\beta + \lambda\phi'_\beta = 0.$$

The family of surfaces, represented by $F = 0$, containing n parameters which are connected by $n - 1$ or $n - 2$ equations is equivalent to a family containing one or two independent parameters respectively. Such a family, in general, has an envelope. The problem of finding the envelope is generally best solved by introducing arbitrary multipliers to assist the eliminations.

If more than two independent variable parameters are involved, there can be no envelope. For in this case we obtain more than three equations for determining the limiting position of the intersection of one surface with a neighboring surface. From these three equations x, y, z could be eliminated, and a relation between the parameters obtained, which is contrary to the hypothesis that they are independent.

In general, if $F = 0$ contains n arbitrary parameters $\alpha_1, \ldots, \alpha_n$ connected by the [illegible] $- 1$ equations of condition $\phi_1 = 0, \ldots,$ $\phi_{n-1} = 0$, the equation of the envelope is found by eliminating the $2n - 1$ numbers $\alpha_1, \ldots, \alpha_n, \lambda_1, \ldots, \lambda_{n-1}$, between the $2n$ equations

$$F = 0, \quad \phi_1 = 0, \; \ldots, \; \phi_{n-1} = 0,$$

$$F'(\alpha_1) + \lambda_1\phi_1'(\alpha_1) + \ldots + \lambda_{n-1}\phi'_{n-1}(\alpha_1) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$F'(\alpha_n) + \lambda_1\phi_1'(\alpha_n) + \ldots + \lambda_{n-1}\phi'_{n-1}(\alpha_n) = 0.$$

EXERCISES.

1. Find the envelope of (θ being the variable parameter)

$$x \sin\theta - y \cos\theta = a\theta - cz.$$

Ans. $x \sin \dfrac{cz + \sqrt{x^2+y^2-a^2}}{a} - y \cos \dfrac{cz + \sqrt{x^2+y^2-a^2}}{a} = \sqrt{x^2+y^2-a^2}.$

2. Find the envelope of a sphere of constant radius whose center lies on a circle in the xy-plane.

Ans. If $x^2 + y^2 = c^2$ is the circle, and the sphere has radius a, the envelope is the torus

$$x^2 + y^2 = [c + (a^2 - z^2)^{\frac{1}{2}}]^2.$$

3. Find the envelope of the ellipsoids $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, where $a + b + c = k$.

Ans. $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = k^{\frac{2}{3}}.$

4. Find the envelope of the ellipsoids in Ex. 3 when they have a constant volume.

5. Find the envelope of the spheres whose centers are on the x-axis and whose radii are proportional to the distance of the center from the origin.

Ans. $y^2 + z^2 = m^2(x^2 + y^2 + z^2).$

6. Find the envelope of the plane $\alpha x + \beta y + \gamma z = 1$ when the rectangle under the perpendiculars from the points $(a, 0, 0)$ and $(-a, 0, 0)$ on the plane is equal to k^2.

Ans. α, β positive. $\dfrac{x^2}{a^2 + k^2} + \dfrac{y^2 + z^2}{k^2} = 1.$

7. Find the envelope of the planes $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ when $a^n + b^n + c^n = k^n$.

Ans. $x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} + z^{\frac{n}{n+1}} = k^{\frac{n}{n+1}}.$

8. Spheres are described having their centers on $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$, and their radii proportional to the square root of the distance of the centers from the origin; show that the equation of the envelope is (l, m, n being direction cosines)

$$x^2 + y^2 + z^2 = (lx + my + nz + c)^2. \qquad c = \text{const.}$$

9. Envelop the family of spheres having for diameters a series of parallel chords of an ellipsoid.

10. If $F(\alpha) = 0$ is the equation of a family of surfaces containing a single arbitrary parameter α, then the equations of the characteristic line on the envelope are $F(\alpha) = 0$, $F'(\alpha) = 0$. As α varies this line moves on the envelope; it will in general have an enveloping line on that surface. The envelope of the characteristic is called the *edge* of the envelope. Show that the equations of the edge of the envelope are obtained by eliminating α between

$$F(\alpha) = 0, \quad F'(\alpha) = 0, \quad F''(\alpha) = 0.$$

11. Find the equations of the *edge* of the envelope of the plane

$$x \sin\theta - y \cos\theta = a\theta - cz.$$

Ans. $x^2 + y^2 = a^2, \quad y = x \tan \dfrac{cz}{a}.$

12. Envelop a series of planes passing through the center of an ellipsoid and cutting it in sections of constant area.

Let $lx + my + nz = 0$ be the plane; the parameters are connected by

$$l^2 + m^2 + n^2 = 1, \quad l^2a^2 + m^2b^2 + n^2c^2 = d^2.$$

Ans. $$\frac{x^2}{a^2 - d^2} + \frac{y^2}{b^2 - d^2} + \frac{z^2}{c^2 - d^2} = 0.$$

13. Spheres are described on chords of the circle $x^2 + y^2 = 2ax$, $z = 0$ which pass through the origin, as diameters, show that they are enveloped by

$$(x^2 + y^2 + z^2 - ax)^2 = a^2(x^2 + y^2).$$

14. Show that the envelope of planes cutting off a constant volume from the cone $ax^2 + by^2 + cz^2 = 0$ is a hyperboloid of which the cone is the asymptote.

15. Find the envelope of the plane $lx + my + nz = d$, when $l^2 + m^2 + n^2 = 1$,

$$\frac{l^2}{d^2 - a^2} + \frac{m^2}{d^2 - b^2} + \frac{n^2}{d^2 - c^2} = 0.$$

Ans. Fresnel's Wave-surface,

$$\frac{a^2x^2}{x^2 + y^2 + z^2 - a^2} + \frac{b^2y^2}{x^2 + y^2 + z^2 - b^2} + \frac{c^2z^2}{x^2 + y^2 + z^2 - c^2} = 0.$$

16. Find the envelope of a plane passing through the origin, having its direction cosines proportional to the coordinates of a point on the line in which intersect the sphere and cone

$$x^2 + y^2 + z^2 = r^2, \quad x^2/a^2 + y^2/b^2 + z^2/c^2 = 0.$$

17. Find the envelope of a plane which moves in such a manner that the sum of the squares of its distances from the corners of a tetrahedron is constant.

18. Show that the envelope of a plane, the sum of whose distances from n fixed points in space is equal to the constant k, is a sphere whose center is the centroid of the fixed points and whose radius is one nth of k.

19. Show that the envelope of a plane, the sum of the squares of whose distances from n fixed points in space is constant, is a conicoid. Find the equation of the envelope.

20. If right lines radiating from a point be reflected from a given surface, the envelope of the reflected rays is called the *caustic by reflexion.*

Show that the caustic by reflexion of the sphere $x^2 + y^2 + z^2 = r^2$, the radiant point being h, 0, 0, is

$$[4h^2\rho^2 - r^2(\rho^2 + 2hx + h^2)]^3 = 27h^2(y^2 + z^2)(\rho^2 - h^2)^2,$$

in which $\rho^2 \equiv x^2 + y^2 + z^2$.

PART VII.

INTEGRATION FOR MORE THAN ONE VARIABLE. MULTIPLE INTEGRALS.

CHAPTER XXXVI.

DIFFERENTIATION AND INTEGRATION OF INTEGRALS.

277. Differentiation under the Integral Sign. Indefinite Integral.—Let $f(x, y)$ be a function of two independent variables x, y. Let

$$F(x, y) = \int f(x, y) dx,$$

the integration being performed for y constant. This integral is a function of y as well as of x. On differentiating with respect to x,

$$\frac{\partial F}{\partial x} = f(x, y).$$

Again, differentiating this with respect to y,

$$\frac{\partial^2 F}{\partial y\, \partial x} = \frac{\partial f(x, y)}{\partial y}.$$

But

$$\frac{\partial^2 F}{\partial x\, \partial y} = \frac{\partial^2 F}{\partial y\, \partial x} = \frac{\partial f}{\partial y}.$$

$$\therefore \quad d\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} dx.$$

Consequently

$$\frac{\partial F}{\partial y} = \int \frac{\partial f}{\partial y} dx,$$

or

$$\frac{\partial}{\partial y} \int f(x, y) dx = \int \frac{\partial f(x, y)}{\partial y} dx.$$

Therefore, to differentiate with respect to y the integral taken with respect to x of a function of two independent variables x, y, differentiate the function under the integral sign.

In like manner we have

$$\frac{\partial^n}{\partial y^n}\int f(x, y)dx = \int \frac{\partial^n f}{\partial y^n}dx.$$

This process is useful in finding new integrals, from a known integral, of a function containing an arbitrary parameter.

EXAMPLES.

1. We have the known integral

$$\int e^{ax}dx = \frac{e^{ax}}{a}.$$

Differentiating with respect to a, we find

$$\int xe^{ax}dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right).$$

And generally, differentiating n times,

$$\int x^n e^{ax}dx = e^{ax}\left(x + \frac{d}{da}\right)^n\left(\frac{1}{a}\right).$$

2. Since $\displaystyle\int \sin ax\, dx = -\frac{\cos ax}{a}$,

$$\therefore\quad \int x \cos ax\, dx = \frac{x \sin ax}{a} + \frac{\cos ax}{a^2}.$$

3. From $\displaystyle\int (a + bx)^n dx = \frac{(a + bx)^{n+1}}{(n + 1)b}$ show that

$$\int x(a + bx)^{n-1}dx = \frac{(nbx - a)(a + bx)^n}{n(n + 1)b^2}.$$

4. From $\displaystyle\int x^a dx = \frac{x^{a+1}}{a + 1}$ show that

$$\int x^a \log x\, dx = \frac{x^{a+1}}{a + 1}\left(\log x - \frac{1}{a + 1}\right).$$

278. Differentiation of Definite Integrals when the Limits are Constants.

Let

$$u = \int_a^b f(x, y)dx,$$

where a and b are independent of x. Then the result of § 277 holds as before, and

$$\frac{\partial u}{\partial y} = \int_a^b \frac{\partial f}{\partial y}dx.$$

On account of the importance of this an independent proof is added. Let Δu denote the change in u due to the change Δy in y. Then, the limits remaining the same,

$$\Delta u = \int_a^b [f(x, y + \Delta y) - f(x, y)]\,dx.$$

$$\therefore\quad \frac{\Delta u}{\Delta y} = \int_a^b \frac{f(x, y + \Delta y) - (fx, y)}{\Delta y}dx.$$

Hence, when $\Delta y (=) 0$, we have

$$\frac{\partial u}{\partial y} = \int_a^b \frac{\partial f}{\partial y} dx,$$

and, generally,

$$\frac{\partial^n u}{\partial y^n} = \int_a^b \frac{\partial^n f}{\partial y^n} dx.$$

EXAMPLES.

1. If $$\int_0^\infty e^{-ax} dx = \frac{1}{a}$$

be differentiated n times with respect to a, we get

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}.$$

2. From $$\int_0^\infty \frac{dx}{(x^2 + a)} = \frac{\pi}{2a^{\frac{1}{2}}},$$

$$\int_0^\infty \frac{dx}{(x^2 + a)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \ldots (2n - 1)}{2 \cdot 4 \cdot 6 \ldots 2n} \frac{\pi}{2a^{n+\frac{1}{2}}}.$$

The value of a definite integral can frequently be found by this method. Thus:

3. Let $$u = \int_0^1 \frac{(x^a - 1)}{\log x} dx.$$

Then $$\frac{du}{da} = \int_0^1 \frac{x^a \log x}{\log x} dx = \int_0^1 x^a dx = \frac{1}{a + 1}.$$

$$\therefore \quad u = \int \frac{da}{a + 1} = \log (a + 1),$$

no constant being added since $u = 0$ when $a = 0$.

4. Find $$\int_0^\pi \log (1 + a \cos \theta) d\theta.$$

Ans. $\pi \log \left(1 + \sqrt{1 - a^2}\right)$.

279. Integration under the Integral Sign.

I. Indefinite Integral.

Let $$F(x, y) = \int f(x, y) dx.$$

Then will

$$\int \left[\int f(x, y) dx \right] dy = \int \left[\int f(x, y) dy \right] dx.$$

Let $$v = \int f(x, y) dy.$$

Then $$\frac{\partial v}{\partial y} = f(x, y).$$

Also, $$\frac{\partial}{\partial y} \int v \, dx = \int \frac{\partial v}{\partial y} dx = \int f(x, y) dx = F(x, y).$$

$$\therefore \quad \int v \, dx = \int F(x, y) dy,$$

or $$\int \left\{ \int f(x, y) dy \right\} dx = \int \left\{ \int f(x, y) dx \right\} dy.$$

Hence the order in which the integration is performed is indifferent. This shows that in indefinite integration when we integrate a function of two independent variables, first with respect to one variable and then with respect to the other, the result is the same when the order of integration is reversed. This being the case, we can represent the result of the two integrations by either of the compact symbols

$$\int\int f\,dx\,dy = \int\int f\,dy\,dx.$$

As in differentiation, the operation is to be performed first with respect to the variable whose differential is written nearest the function, or integral sign.

II. The same theorem is true for the definite double integral of a function of two independent variables *when the limits are constants.*

Let

$$\int f(x, y)dx = X(x, y), \quad \int f(x, y)dy = Y(x, y);$$

$$\int\int f(x, y)dx\,dy = \int\int f(x, y)dy\,dx = F(x, y).$$

Then

$$\int_{x_1}^{x_2} f\,dx = X(x_2, y) - X(x_1, y),$$

$$\int_{y_1}^{y_2} f\,dy = Y(x, y_2) - Y(x, y_1);$$

$$\int_{y_1}^{y_2}\left\{\int_{x_1}^{x_2} f\,dx\right\}dy = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1),$$

$$\int_{x_1}^{x_2}\left\{\int_{y_1}^{y_2} f\,dy\right\}dx = F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1).$$

The last two values are the same. Hence

$$\int_{y_1}^{y_2}\left\{\int_{x_1}^{x_2} f\,dx\right\}dy = \int_{x_1}^{x_2}\left\{\int_{y_1}^{y_2} f\,dy\right\}dx,$$

or

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y)dx\,dy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)dy\,dx.$$

The integral sign with its appropriate limits and the corresponding differential are written in the same relative position with respect to the function.

EXAMPLES.

1. $\int_0^1 x^{a-1}dx = \frac{1}{a}$. Hence

$$\int_0^1\int_{a_0}^{a_1} x^{a-1}da\,dx = \int_{a_0}^{a_1}\frac{da}{a} = \log\frac{a_1}{a_0}.$$

$$\therefore \quad \int_0^1 \frac{x^{a_1-1} - x^{a_2-1}}{\log x}dx = \log\frac{a_1}{a_0}.$$

Put $x = e^{-z}$.

$$\therefore \quad \int_0^\infty \frac{e^{-a_1 z} - e^{-a_0 z}}{z}\, dz = \log \frac{a_1}{a_0}.$$

2. $\displaystyle\int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2 + b^2}.$

$$\therefore \quad \int_0^\infty \int_{a_0}^{a_1} e^{-ax} \sin bx\, da\, dx = \int_{a_0}^{a_1} \frac{b\, da}{a^2 + b^2},$$

or

$$\int_0^\infty \frac{e^{-a_0 x} - e^{-a_1 x}}{x} \sin bx\, dx = \tan^{-1} \frac{a_1}{b} - \tan^{-1} \frac{a_0}{b}.$$

If $a_0 = 0, \quad a_1 = \infty,$ then

$$\int_0^\infty \frac{\sin bx}{x}\, dx = \tfrac{1}{2}\pi.$$

3. **Evaluate**

$$\int_0^\infty e^{-a^2 x^2}\, dx.$$

Put

$$k = \int_0^\infty e^{-x^2} dx.$$

$$\therefore \quad \int_0^\infty e^{-a^2 x^2} a\, dx = k.$$

and

$$\int_0^\infty e^{-a^2(1+x^2)} a\, dx = k e^{-a^2}.$$

$$\therefore \quad \int_0^\infty \int_0^\infty e^{-a^2(1+x^2)} a\, da\, dx = k \int_0^\infty e^{-a^2} da = k^2.$$

Also,

$$\int_0^\infty e^{-a^2(1+x^2)} a\, da = \frac{1}{2} \frac{1}{1 + x^2}$$

and

$$\tfrac{1}{2} \int_0^\infty \frac{dx}{1 + x^2} = \tfrac{1}{2} \tan^{-1} x \Big]_0^\infty = \tfrac{1}{4}\pi = k^2.$$

$$\therefore \quad \int_0^\infty e^{-x^2} dx = \tfrac{1}{2} \sqrt{\pi},$$

and

$$\int_0^\infty e^{-a^2 x^2} dx = \frac{1}{2a} \sqrt{\pi}.$$

This gives the area of the probability curve.

280. If $F(x, y, z)$ is a function of three independent variables, the same rules as for a function of two independent variables govern the triple integral

$$\int \int \int F\, dx\, dy\, dz.$$

Examples of double and triple integrals will be given in the next chapter.

CHAPTER XXXVII.

APPLICATIONS OF DOUBLE AND TRIPLE INTEGRALS.

PLANE AREAS. DOUBLE INTEGRATION.

281. Rectangular Formulæ.—If x, y are the rectangular coordinates of a point in a plane xOy, then

$$d\omega = \Delta x\ \Delta y = dx\ dy$$

is an element of area, being the area of the rectangle whose sides are Δx and Δy.

Let the entire plane xOy be divided into rectangular spaces by parallels to Ox and Oy, of which $\Delta x\ \Delta y$ is a type. The area of any closed boundary drawn in the plane is the limit of the sum of all the *entire* rectangular elements of type $\Delta y\ \Delta x$ included in the boundary, when for each rectangle $\Delta x(=)0$, $\Delta y(=)0$. For the area within the closed boundary A is equal to

$$A = \Sigma\ \Delta y\ \Delta x$$

plus the sum of the fractional rectangles which are cut by the boundary. This latter sum can be shown to be less than the length of the boundary multiplied by the diagonal of the greatest elementary rectangle, and therefore has the limit zero. Hence

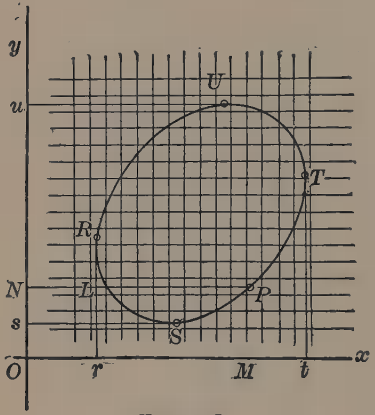

FIG. 148.

$$A = \pounds\Sigma\ \Delta y\ \Delta x,$$

taken throughout the enclosed region, when $\Delta x(=)\Delta y(=)0$.

The summation is effected by summing first the rectangles in a vertical strip PQ and then summing all the vertical strips from R to T; or, first sum the elements in a horizontal strip PL, then sum all the horizontal strips in the boundary from S to U. These summations are clearly represented by the double integrals

$$\int_{x=r}^{x=t}\int_{y=\phi(x)}^{y=\psi(x)} dy\ dx, \quad \int_{y=s}^{y=u}\int_{x=\lambda(y)}^{x=\mu(y)} dx\ dy.$$

In the first integration in either case the limits of the integration are, in general, functions of the other variable which are to be determined from the given boundary.

EXAMPLES.

1. Required the area between the parabola $y^2 = ax$ and the circle $y^2 = 2ax - x^2$, in the first quadrant.

The curves meet at the origin and at $x = a$.

(1).
$$A = \int_0^a \int_{y=\sqrt{ax}}^{y=\sqrt{2ax-x^2}} dy\, dx = \int_0^a \left[\sqrt{2ax-x^2} - \sqrt{2ax}\right] dx$$
$$= \frac{\pi a^2}{4} - \frac{2a^2}{3}.$$

(2).
$$A = \int_{y=0}^{y=a} \int_{x=a-\sqrt{a^2-x^2}}^{x=y^2/a} dx\, dy$$
$$= \int_0^a \left\{ \frac{y^2}{a} - a + \sqrt{a^2-y^2} \right\} dy$$
$$= \frac{\pi a^2}{4} - \frac{2a^2}{3}.$$

2. Find the area outside the parabola $y^2 = 4a(a - x)$ and inside the circle $y^2 = 4a^2 - x^2$.

3. Find the area common to the parabola $3y^2 = 25x$ and $5x^2 = 9y$. *Ans.* 5.

282. Polar Coordinates.—The surface of the plane is divided into checks bounded by rays drawn from the pole and concentric circles drawn with center at the pole.

The exact area of any check PQ bounded by arcs with radii ρ, $\rho + \Delta\rho$, and these radii including the angle $\Delta\theta$, is

$$\tfrac{1}{2}\{(\rho + \Delta\rho)^2 - \rho^2\}\Delta\theta$$
$$= \rho\,\Delta\rho\,\Delta\theta + \tfrac{1}{2}\Delta\rho^2\,\Delta\theta.$$

The entire area in any closed boundary is the limit of the sum of the entire checks in the boundary. The sum of the partial checks on the boundary being 0 when $\Delta\rho(=)\Delta\theta(=)0$, as in § 281.

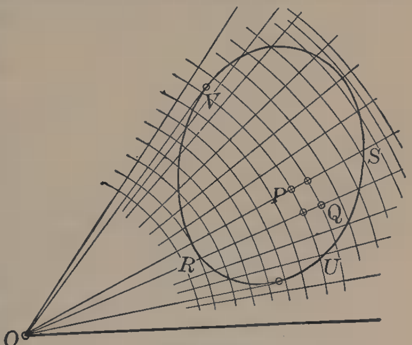

FIG. 149.

But, since

$$\pounds\frac{\rho\Delta\rho\,\Delta\theta + \frac{1}{2}\Delta\rho^2\Delta\theta}{\rho\Delta\rho\,\Delta\theta} = 1 + \tfrac{1}{2}\pounds\frac{\Delta\rho}{\rho},$$
$$= 1,$$

when $\Delta\rho(=)\Delta\theta(=)0$, the area within any closed boundary is equal to

$$A = \pounds\Sigma\rho\,\Delta\rho\,\Delta\theta$$

when $\Delta\rho(=)\Delta\theta(=)0$.

This summation can be effected in two ways:

(1). We can sum the checks along a radius vector RS, keeping $\Delta\theta$ constant, then sum the tier of checks thus obtained from one value of θ to another.

(2). We can sum the checks along the ring UV, keeping ρ and $\Delta\rho$ constant, then sum the rings from one value of ρ to another.

These operations clearly give the double integrals

$$\int_{\theta_1}^{\theta_2}\int_{\rho=\phi(\theta)}^{\rho=\psi(\theta)} \rho\, d\rho\, d\theta, \quad \int_{\rho_1}^{\rho_2}\int_{\theta=\lambda(\rho)}^{\theta=\mu(\rho)} d\theta\, \rho\, d\rho.$$

EXAMPLES.

1. Find the area between the two circles $\rho = a\cos\theta$, $\rho = b\cos\theta$, $b > a$.

$$(1).\ A = \int_0^{\frac{\pi}{2}}\int_{a\cos\theta}^{b\cos\theta} \rho\, d\rho\, d\theta,$$

$$= \int_0^{\frac{\pi}{2}} \tfrac{1}{2}(b^2 - a^2)\cos^2\theta\, d\theta = \frac{\pi}{8}(b^2 - c^2).$$

$$(2).\ A = \int_a^b\int_0^{\cos^{-1}\frac{\rho}{b}} d\theta\, \rho\, d\rho + \int_0^a \int_{\cos^{-1}\frac{\rho}{a}}^{\cos^{-1}\frac{\rho}{b}} d\theta\, \rho\, d\rho,$$

which gives the same result as (1).

The double integration is not necessary for finding the areas of curves; it is given here as an illustration of a process which admits of generalization.

VOLUMES OF SOLIDS. DOUBLE AND TRIPLE INTEGRATION.

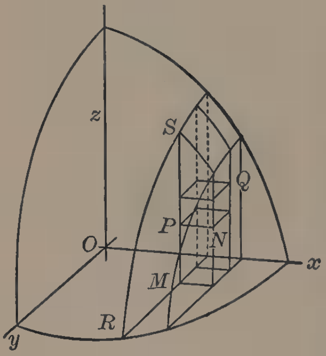

FIG. 150.

283. Rectangular Coordinates.—Let x, y, z be the coordinates of a point in space referred to orthogonal coordinate axes.

Divide space into a system of rectangular parallelopipeds by planes parallel to the coordinate planes. Let $\Delta x, \Delta y, \Delta z$ be the edges of a typical elementary parallelopiped. Then the volume

$$\Delta x\, \Delta y\, \Delta z$$

is the elementary space volume.

The volume of any closed surface is the limit of the sum of the entire elementary parallelopipeds included by the surface when $\Delta x (=) \Delta y (=) \Delta z (=) 0$.

$$V = \pounds\Sigma\, \Delta x\, \Delta y\, \Delta z,$$

taken throughout the enclosed space.

(1). Let $x, y, \Delta y, \Delta x$ be constant. Sum the elementary volumes between the two values of z, obtaining the volume of a column MS of the solid. The result expresses z as a function of x and y given by the equation or equations of the boundary.

(2). Let x, Δx be constant. Sum the columns between two values of y for $\Delta y (=) 0$. The result is the slice of the solid on the cross-section $x =$ constant, having thickness Δx.

(3). Sum the slices between two values of x for $\Delta x (=) 0$. The result is the total sum of the elements, expressed by the integral

$$V = \int_{x_1}^{x_2} \int_{y=\phi(x)}^{y=\psi(x)} \int_{z=\lambda(x, y)}^{z=\mu(x, y)} dz\, dy\, dx,$$

$$= \int_{x_1}^{x_2} \int_{y_1}^{y_2} z\, dy\, dx,$$

$$= \int_{x_1}^{x_2} A_x\, dx.$$

Clearly, if more convenient we may change the order of integration, making the proper changes in the limits ot integration.

EXAMPLES.

1. Find the volume of one eighth the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

$$V = \int_0^a \int_0^{b\sqrt{1 - \frac{x^2}{a^2}}} \int_0^{c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} dz\, dy\, dx,$$

$$= \int_0^a \int_0^{b\sqrt{1 - \frac{x^2}{a^2}}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dy\, dx,$$

$$= \frac{bc}{4} \pi \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = \tfrac{1}{6}\pi abc.$$

2. Find the volume bounded by the hyperbolic paraboloid $xy = az$, the xOy plane, and the four planes $x = x_1$, $x = x_2$, $y = y_1$, $y = y_2$.

$$V = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_0^{\frac{xy}{a}} dz\, dy\, dx,$$

$$= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{xy}{a}\, dy\, dx,$$

$$= \int_{x_1}^{x_2} \frac{y_2^2 - y_1^2}{2a}\, x\, dx,$$

$$= \frac{1}{4a} (y_2^2 - y_1^2)(x_2^2 - x_1^2),$$

$$= \frac{1}{4a} (x_2 - x_1)(y_2 - y_1)(x_1 y_1 + x_2 y_2 + x_1 y_2 + x_2 y_1),$$

$$= \tfrac{1}{4} (x_2 - x_1)(x_2 - y_1)(z_1 + z_2 + z_3 + z_4).$$

The volume is therefore equal to the area of the rectangular base multiplied by the average of the elevations of the corners. This is the engineer's rule for calculating earthwork volumes.

284. Polar Coordinates.—The polar coordinates of a point P in space are ρ, the distance of the point from the origin; θ, the angle which this radius vector OP makes with the vertical Oz; and ϕ, the angle which the vertical plane POz makes with the fixed plane xOz.

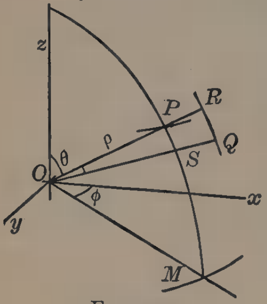

FIG. 151.

Through P draw a vertical circle PM with radius ρ. Prolong OP to R, $PR = \Delta\rho$. Draw the circle RQ in the plane POM with radius $\rho + \Delta\rho$. If ΔA is the area $PRQS$, then

$$\mathcal{L}\frac{\Delta A}{\Delta\rho\,\rho\,\Delta\theta} = 1.$$

We may therefore take $dA = \rho\,d\rho\,d\theta$. This area revolving around Oz generates a ring of volume

$$2\pi\,\rho \sin\theta\,dA.$$

Therefore the volume generated by dA revolving through the arc $ds = \rho\sin\theta\,d\phi$ is in the same proportion to the volume of the ring as is the arc to the whole circumference, or the element of volume is

$$\rho^2 \sin\theta\,d\phi\,d\rho\,d\theta.$$

We divide space into elementary volumes by a series of concentric spheres having the origin as center, and a series of cones of revolution having Oz for axis, and a series of planes through Oz

The volume of any closed surface is the limit of the sum of the entire elementary solids included in the surface when

$$\Delta\rho(=)\Delta\phi(=)\Delta\theta(=)0.$$

Or, the volume is equal to the triple integral

$$V = \int\int\int \rho^2 \sin\theta\,d\phi\,d\rho\,d\theta,$$

taken with the proper limits as determined by the boundaries of the surface.

EXAMPLES.

1. Find the volume of one eighth the sphere $\rho = a$.

$$V = \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\rho=0}^{\rho=a} \rho^2 d\rho \cdot \sin\theta\,d\theta \cdot d\phi,$$

$$= \frac{a^3}{3}\int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin\theta\,d\theta \cdot d\phi$$

$$= \frac{a^3}{3}\int_0^{\frac{\pi}{2}} d\phi = \tfrac{1}{6}\pi a^3$$

The first integration gives a pyramid with vertex O and spherical base $a^2 \sin\theta\, d\theta\, d\phi$. The next integration gives the volume of a wedge-shaped element of a solid between two vertical planes determined by ϕ and $\phi + \Delta\phi$. The last integration sums up these wedges.

2. A right cone has its vertex on the surface of a sphere, and its axis coincident with the diameter of the sphere passing through the vertex; find the volume common to the sphere and cone.

Let a be the radius of the sphere, α the semi-vertical angle of the cone. The polar equation of the sphere with the vertex of the cone as origin is $\rho = 2a\cos\theta$.

$$\therefore \quad V = \int_0^{2\pi}\int_0^{\alpha}\int_0^{2a\cos\theta} \rho^2\, d\rho\cdot\sin\theta\, d\theta\cdot d\phi.$$

3. The curve $\rho = a(1 + \cos\theta)$ revolves about the initial line; find the volume of the solid generated.

$$V = \int_0^{\pi}\int_0^{2\pi}\int_0^{a(1+\cos\theta)} \rho^2\, d\rho\cdot d\phi\cdot\sin\theta\, d\theta,$$

$$= \frac{2\pi a^3}{3}\int_0^{\pi}(1+\cos\theta)^3\sin\theta\, d\theta = \tfrac{8}{3}\pi a^3.$$

285. Mixed Coordinates.—Instead of dividing a solid into columns standing on a rectangular elementary basis, as in the method

$$V = \int\int z\, dx\, dy,$$

it is sometimes advantageous to divide it into columns standing on the polar element of area. Thus the elementary column volume is

$$z\,\rho\, d\theta\, d\rho.$$

Therefore for the volume of a solid we have

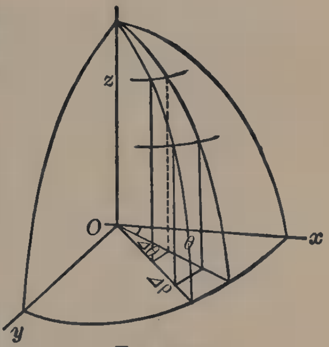

FIG. 152.

$$V = \int\int\int dz\cdot\rho\, d\theta\, d\rho,$$

$$= \int\int z\,\rho\, d\rho\, d\theta,$$

taken between the proper limits.

EXAMPLES.

1. Find the volume bounded by the surfaces $z = 0$,

$$x^2 + y^2 = 4az \quad\text{and}\quad y^2 = 2cx - x^2.$$

Here $z = \rho^2/4a$ and the limits of ρ and θ must be such as to extend the integration over the whole area of the circle $y^2 = 2cx - x^2$. Let $\rho_1 = 2c\cos\theta$; then

$$V = \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{\rho_1}\frac{\rho^3}{4a}\, d\rho\, d\theta = \frac{c^4}{a}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos^4\theta\, d\theta,$$

$$= \frac{2c^4}{a}\int_0^{\frac{1}{2}\pi}\cos^4\theta\, d\theta = \frac{2c^4}{a}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi c^4}{8a}.$$

2. Find the volume of the solid bounded by the plane $z = 0$ and the surface

$$z = ae^{-\frac{x^2+y^2}{c^2}}.$$

Here $\rho^2 = x^2 + y^2$. $\therefore$ $V = a \int\int e^{-\frac{\rho^2}{c^2}} \rho\, d\rho\, d\theta$.

$$\int_0^\infty e^{-\frac{\rho^2}{c^2}} \rho\, d\rho = -\tfrac{1}{2}c^2 e^{-\frac{\rho^2}{c^2}}\Big]_0^\infty = \tfrac{1}{2}c^2.$$

$$\int_0^{2\pi} d\theta = 2\pi.$$

$\therefore$ $V = \pi ac^2$. [See Todhunter, Int. Cal. p. 181.]

SURFACES OF SOLIDS.

286. When the plane through any three points on a surface (the points arbitrarily chosen) converges to a tangent plane as a limit when the three points converge to a fixed point as a limit, then a definite idea of the area of the surface can be had, as follows:

Inscribe in a given bounded portion of the surface a polyhedral surface with triangular plane faces. The area of the given portion of the surface is the limit to which converges the area of the polyhedral surface when the area of each triangular face converges to zero.

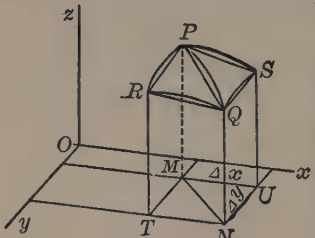

FIG. 153.

To evaluate the limit of the sum of the triangular areas inscribed in the surface we proceed as follows:

Let P be a point x, y, z on a surface, and Q a point $x + \Delta x, y + \Delta y, z + \Delta z$. The prism $MTNU$ on the rectangle whose sides are Δx, Δy cuts the surface in an element of surface $PRQS$. Draw the diagonal MN and the two inscribed triangles PRQ and PSQ. Let perpendiculars to the planes of the triangles PRQ, PSQ at the point P make angles γ_1, γ_2 with Oz respectively. The angles γ_1, γ_2 are then the angles which these planes make with the horizontal plane xOy. Since the area of the orthogonal projection of a plane triangle is equal to the area of the triangle into the cosine of the angle between the plane of the triangle and the plane of projection, we have the areas

$$PRQ = MTN \sec \gamma_1, \quad PSQ = MUN \sec \gamma_2.$$

Also,
$$MTN = MUN.$$

$$\therefore \quad PRQ + PSQ = \frac{\sec \gamma_1 + \sec \gamma_2}{2} \Delta y\, \Delta x.$$

By hypothesis, if $\Delta^2 S$ is the area of the element of surface $PRQS$, then

$$\mathcal{L}\frac{\Delta^2 S}{\frac{\sec\gamma_1+\sec\gamma_2}{2}\,\Delta y\,\Delta x}=1.$$

But when $Q(=)P$ the perpendiculars to the planes PRQ, PSQ have the normal to the surface at P as a limit, since the planes PRQ, PSQ converge to the tangent plane at P as a limit.

If γ is the angle which the tangent plane at P makes with the plane xOy, then

$$\mathcal{L}\frac{\sec\gamma_1+\sec\gamma_2}{2}=\sec\gamma,$$

$$=\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}.$$

$$\therefore\quad \frac{d^2S}{dy\,dx}=\sec\gamma,$$

and

$$S=\iint\sec\gamma\,dy\,dx,$$

$$=\iint\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dy\,dx,$$

taken between the limits determined by the boundary of the portion of the surface whose area is required.

EXAMPLES.

1. Find the area of the sphere-surface $x^2+y^2+z^2=a^2$.

$$\frac{\partial z}{\partial x}=-\frac{x}{z},\quad \frac{\partial z}{\partial y}=-\frac{y}{z}.$$

$$\tfrac{1}{8}S=a\int_{x=0}^{x=a}\int_{y=0}^{y=\sqrt{a^2-x^2}}\frac{dy\,dx}{\sqrt{a^2-x^2-y^2}},$$

$$=a\int_{x=0}^{x=a}\left[\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}}dx,$$

$$=\frac{\pi a}{2}\int_0^a dx=\tfrac{1}{2}\pi a^2.$$

2. The center of a sphere whose radius is a is on the surface of a cylinder of revolution whose radius is $\frac{1}{2}a$. Find the surface of the cylinder intercepted by the sphere.

(1). Let the equations of the sphere and cylinder be

$$x^2+y^2+z^2=a^2,$$
$$x^2+y^2=ax,$$

as in the figure.

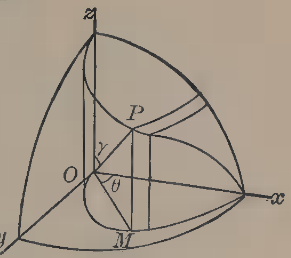

FIG. 154.

$$\therefore \quad S = 4\int_{x=0}^{x=a}\int_{z=0}^{z=\sqrt{a^2-ax}}\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dz\,dx,$$

$$= 4\int_0^a\int_0^{\sqrt{a^2-ax}}\sqrt{1+\left(\frac{a-2x}{2y}\right)^2}\,dz\,dx,$$

$$= 2a\int_0^a\int_0^{\sqrt{a^2-ax}}\frac{dz\,dx}{\sqrt{ax-x^2}},$$

$$= 2a\int_0^a\frac{\sqrt{a^2-ax}}{\sqrt{ax-x^2}}dx = 2a\int_0^a\sqrt{\frac{a}{x}}\,dx = 4a^2.$$

(2). Let s be the length of the arc of the base of the cylinder measured from the origin. Then

$$S = 4\int z\,ds,$$

taken over the semi-circumference. Let γ be the angle which the sphere radius to P makes with Oz, and θ the angle which $OM = \rho$ makes with Ox. Then

$$z = a\cos\gamma = a\sin\theta. \quad s = a\theta, \quad ds = a\,d\theta.$$

$$\therefore \quad S = 4\int_{\theta=0}^{\theta=\frac{1}{2}\pi}a^2\sin\theta\,d\theta = 4a^2.$$

(3). Otherwise, immediately from the geometry of the figure,

$$S = 2a\int_{x=0}^{x=a}\int_{z=0}^{z=\sqrt{a^2-ax}}\frac{dz\,dx}{\sqrt{ax-x^2}} = 2a\int_0^a\sqrt{\frac{a}{x}}\,dx = 4a^2,$$

as in (1).

3. Find the surface of the sphere intercepted by the cylinder in Ex. **2.**
From the figure,

(1). $$\sec\gamma = \frac{a}{z} = \frac{a}{\sqrt{a^2-x^2-y^2}}.$$

$$\tfrac{1}{4}S = a\int_{x=0}^{x=a}\int_{y=0}^{y=\sqrt{ax-x^2}}\frac{dy\,dx}{\sqrt{a^2-x^2-y^2}},$$

$$= a\int_{x=0}^{x=a}\sin^{-1}\sqrt{\frac{x}{a+x}}\,dx.$$

Integrate directly, or put $\sin^2\theta = x/(a+x)$ and integrate
Hence $S = 2a^2(\pi - 2)$.
(2). Again,

$$\tfrac{1}{4}S = \int_0^{\frac{1}{2}\pi}\rho\theta\sec\gamma\,d\rho.$$

$$\rho = a\cos\theta = a\sin\gamma. \quad \therefore \quad \gamma = \tfrac{1}{2}\pi - \theta$$

$$\tfrac{1}{4}S = -a^2\int\theta\cos\theta\,d\theta = -a^2\left[\theta\sin\theta+\cos\theta\right]_0^{\frac{1}{2}\pi} = -a^2(\tfrac{1}{2}\pi - 1)$$

LENGTHS OF CURVES IN SPACE.

287. As in plane curves, the length of a curve in space is defined to be the limit to which converges the sum of the lengths of the sides of a polygonal line inscribed in the curve.

Since $$\left(\frac{\Delta c}{\Delta s}\right)^2 \Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2,$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2},$$

with similar values for the derivatives $\frac{ds}{dy}$, $\frac{ds}{dz}$,

$$\therefore \quad s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx,$$

with corresponding values for s when y or z is taken as the independent variable.

If the coordinates of a point on the curve are given in terms of a variable t, then

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2,$$

and

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

EXAMPLES.

1. Find the length of the helix

$$x = a \cos \frac{z}{b}, \quad y = a \sin \frac{z}{b},$$

measured from $z = 0$.

Take z as the independent variable. Then

$$\frac{dx}{dz} = -\frac{a}{b} \sin \frac{z}{b}, \quad \frac{dy}{dz} = \frac{a}{b} \cos \frac{z}{b};$$

$$\therefore \quad s = \int_{z=0}^{z=z} \sqrt{1 + \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2}\, dz,$$

$$= \int \sqrt{1 + \frac{a^2}{b^2}}\, dz = z\sqrt{1 + \frac{a^2}{b^2}}.$$

2. Find the length, measured from the origin, of the curve

$$2ay = x^2, \quad 6a^2 z = x^3.$$

$$s = \int \left(1 - \frac{x^2}{a^2} + \frac{x^4}{4a^4}\right)^{\frac{1}{2}} dx = \int \left(1 + \frac{x^2}{2a^2}\right) dx = x + \frac{x^3}{6a^2} = x + z.$$

3. Show that the length, measured from the origin, of

$$y = a \sin x, \quad 4z = a^2(x + \cos x \sin x),$$

is $x + z$.

4. Find the length of

$$y = 2\sqrt{ax} - x, \quad z = x - \frac{2}{3}\sqrt{\frac{x^3}{a}},$$

measured from the origin. *Ans.* $s = x + y - z$.

5. Find the length, measured from the horizontal plane, of the curve

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad x = \tfrac{1}{2}a\left(e^{\frac{z}{a}} + e^{-\frac{z}{a}}\right).$$

Ans. $s = \dfrac{(a^2 + b^2)^{\frac{1}{2}}}{a}\sqrt{x^2 - a^2}.$

288. Observations on Multiple Integrals. — The problem of integration always reduces ultimately to the irreducible integral

$$\int dE,$$

dE being the element of the subject to be integrated. Or this may be taken as the starting-point and considered as the simplest elementary statement of the problem for solution. This, in simple cases, may be evaluated directly, otherwise it may be necessary to integrate partially two or more times with respect to the different variables which enter the problem. There may be several different ways in which the elements can be summed. A careful study of the problem in each particular case should be made in order to determine the best way of effecting the partial summations, with respect to the limits at each stage of the process.

One is at perfect liberty to take the elements of integration in geometrical problems in any way and of any shape one chooses, as the limit of the sum is independent of the manner in which the subdivision is made (see Appendix). This should be verified by working the same problem in several different ways.

The applications of multiple integration in mechanics are numerous and extensive. Further application beyond the elementary geometrical ones given here is outside the scope of the present work.

EXERCISES.

In these exercises the results should be obtained by double and triple integration, and also by single integration whenever it is possible.

1. Find the volume bounded by the surfaces

$$x^2 + y^2 = a^2, \quad z = 0, \quad z = x \tan \alpha.$$

Ans. $2\displaystyle\int_0^a \int_0^{\sqrt{a^2 - x^2}} \int_0^{x \tan \alpha} dz\, dy\, dx = \tfrac{2}{3}a^3 \tan \alpha.$

2. Find the volume bounded by the plane $z = 0$, the cylinder

$$(x - a)^2 + (y - b)^2 = R^2,$$

and the hyperbolic paraboloid $xy = cz$. *Ans.* $\pi \dfrac{ab}{c} R^2.$

3. Find the volume bounded by the sphere and cylinders

$$x^2 + y^2 + z^2 = a^2, \quad x^2 + y^2 = b^2, \quad \rho^2 = a^2 \cos^2\theta + b^2 \sin^2\theta.$$

Ans. $\tfrac{4}{9}(16 - 3\pi)(a^2 - b^2)^{\frac{3}{2}}.$

4. A sphere is cut by a right cylinder whose surface passes through the center of the sphere; the radius of the cylinder is one half that of the sphere a. Find the volume common to both surfaces. *Ans.* $\frac{2}{3}(\pi - \frac{4}{3})a^3$.

5. Show that the volume included within the surface

$$F\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) = 0.$$

is abc times the volume of the surface

$$F(x, y, z) = 0.$$

6. Show that the volume of the solid bounded by the surfaces

$$z = 0, \quad x^2 + y^2 = 4az, \quad x^2 + y^2 = 2cx, \quad \text{is} \quad \tfrac{3}{8}\pi c^4/a.$$

7. Find the entire volume bounded by the positive sides of the three coordinate planes and the surface

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} + \left(\frac{z}{c}\right)^{\frac{1}{2}} = 1. \qquad \textit{Ans.} \quad \frac{abc}{90}.$$

8. Find the volume bounded by the surface

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}. \qquad \textit{Ans.} \quad \tfrac{4}{35}\pi a^3.$$

9. Find the volume of the surface

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{z}{c}\right)^{\frac{2}{3}} = 1. \qquad \textit{Ans.} \quad \tfrac{4}{35}\pi abc.$$

10. Show that the volume included between the surface of the hyperboloid of one sheet, its asymptotic cone, and two planes parallel to that of the real axes is proportional to the distance between those planes.

11. Find the whole volume of the solid

$$x^2/a^2 + y^2/b^2 + z^4/c^4 = 1. \qquad \textit{Ans.} \quad \tfrac{8}{5}\pi abc.$$

12. Find the whole volume of the solid bounded by

$$(x^2 + y^2 + z^2)^3 = 27a^3xyz. \qquad \textit{Ans.} \quad \tfrac{9}{2}a^3.$$

13. Use § 285 to show that the volume of the torus

$$(x^2 + y^2 + z^2 + c^2 - a^2)^2 = 4c^2(x^2 + y^2) \quad \text{is} \quad 2\pi^2ca^2.$$

14. Find the volume of the solid bounded by the planes $x = 0$, $y = 0$, the surface $(x + y)^2 = 4az$, and the tangent plane to the surface at any point f, g, h.
Ans. $\frac{1}{8}ah^2$.

15. Show that the surfaces $y^2 + z^2 = 4ax$, and $x - z = a$, include a volume $8\pi a^3$.

16. Show that the volume included between the plane $z = 0$, the cylinder $y^2 = 2cx - x^2$, and a paraboloid $ax^2 + by^2 = 2z$ is $\frac{1}{8}\pi c^4(5a^{-1} + b^{-1})$.

17. Show that the whole volume of the surface whose equation is

$$(x^2 + y^2 + z^2)^2 = cxyz \text{ is equal to } c^3/360.$$

18. Show that the volume included between the planes $y = \pm k$ and the surface

$$a^2x^2 + b^2z^2 = 2(ax + bz)y^2 \quad \text{is} \quad 4\pi k^5/5ab.$$

19. Find the form of the surface whose equation is

$$(x^2/a^2 + y^2/b^2 + z^2/c^2)^2 = x^2/a^2 + y^2/b^2 - z^2/c^2,$$

and show that the volume is $\pi^2abc/4\sqrt{2}$.

20. Find the entire surface of the groin, the solid common to two equal cylinders of revolution whose axes intersect at right angles.

Ans. $16R^2$. R being the radius of the cylinders.

21. Find the area of the surface

$$z^2 + (x \cos \alpha + y \sin \alpha)^2 = a^2$$

in the first octant. *Ans.* $2a^2 \csc 2\alpha$.

22. Find the volume of the solid in the first octant bounded by $xy = az$ and $x + y + z = a$. *Ans.* $(\frac{17}{12} - \log 4)a^3$.

23. Find the surface of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant intercepted between the planes $x = 0$, $y = 0$, $x = b$, $y = b$.

$$\textit{Ans.}\ a\left(2b \sin^{-1}\frac{b}{\sqrt{a^2 - b^2}} - a \sin^{-1}\frac{b^2}{a^2 - b^2}\right).$$

24. A curve is traced on a sphere so that its tangent makes always a constant angle with a fixed plane. Find its length from cusp to cusp.

CHAPTER XXXVIII.

INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS.

289. Classification.—A differential equation is an equation which involves derivatives or differentials.

An ordinary differential equation is one in which the derivatives are taken with respect to *one* independent variable. These are the only kind that we shall consider.

Differential equations are classified according to the *order* and *degree* of the equation. The *order* of a differential equation is the order of the highest derivative contained in the equation. The *degree* of the equation is the highest power of the highest derivative involved.

290. We shall consider in this text only examples of ordinary differential equations of the first and second degree in the first order, and a few particular cases of the first degree in the second order.

291. Examples of Equations of the First Order and First Degree.—The derivative equations of the first order and first degree

$$\frac{dy}{dx} = \cos x, \quad 2x\frac{dy}{dx} = 3y - xy, \quad ax^2y^3\frac{dy}{dx} = 2x\frac{dy}{dx} - y,$$

when multiplied by dx, are equivalent to the differential equations of the first order and first degree

$$dy = \cos x\, dx, \quad 2x\, dy = (3y - xy)dx, \quad ax^2y^3dy = 2x\, dy - y\, dx.$$

In general, any linear function of $\frac{dy}{dx}$,

$$\phi\frac{dy}{dx} + \psi = 0,$$

in which ϕ and ψ are constants, or functions of x or y, or of x and y, is a derivative equation of the first degree and order. When multiplied by dx it becomes the general differential equation of the first degree and order

$$\phi\, dy + \psi\, dx = 0.$$

292. Examples of Equations of the First Order and Second Degree.—The equations

$$\left(\frac{dy}{dx}\right)^2 = ax^3, \quad x\left(\frac{dy}{dx}\right)^2 - 2y\left(\frac{dy}{dx}\right) + ax = 0,$$

are of the second degree and first order. Written differentially,

$$dy^2 = ax^3dx^2, \quad x\, dy^2 - 2y\, dy\, dx + ax\, dx^2 = 0.$$

In genéral, the type of an equation of the first order and second degree is

$$\phi\left(\frac{dy}{dx}\right)^2 + \psi\left(\frac{dy}{dx}\right) + \chi = 0,$$

where ϕ, ψ, χ are functions of x, y or x and y, or constants.

293. Equations of the Second Order and First Degree.—Such equations as

$$\frac{d^2y}{dx^2} - m^2y = 0, \quad \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2e^{2x},$$

$$y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = y^2 \log y, \quad x^2y\frac{d^2y}{dx^2} + \left(x\frac{dy}{dx} - y\right)^2 = 0,$$

are of the second order and first degree.

294. Solution of a Differential Equation.—To solve a given differential equation

$$F(x, y, y') = 0,$$

where $y' \equiv \dfrac{dy}{dx}$, is to find the values x and y which satisfy the equation. Thus, if the values of x and y which satisfy the equation

$$\phi(x, y) = 0$$

satisfy a differential equation $F = 0$, then $\phi = 0$ is a solution of $F = 0$.

The solution of a given differential equation may be a *particular* solution or it may be the *general* solution. The general solution includes all the particular solutions. Or the solution may be a *singular* solution, which is not included in the general solution. The *complete* solution of a differential equation includes the general solution and the singular solution. The meaning of these solutions will be developed in what follows.

The solution of a differential equation is considered as having been effected when it has been reduced to an equation in integrals, whether the actual integrations can be effected in finite terms or not.

EQUATIONS OF THE FIRST DEGREE AND FIRST ORDER.

295. The simplest type of an ordinary differential equation of the first order and degree is

$$dy = f(x)dx. \qquad (1)$$

Integrating, we obtain the solution

$$y = F(x) + c, \qquad (2)$$

where $F(x)$ is a primitive of $f(x)$ and c is an arbitrary constant. For a particular assigned value of c, (2) is a particular solution of (1), and is the equation of a particular curve in a definite position. At each point of the curve (2),

$$\frac{dy}{dx} = f(x)$$

is the slope, or direction of the curve (2). For different values of c we have different curves. The ordinates of any two such curves differ by a constant. Equation (2) is then the equation of a family of curves having the arbitrary parameter c. This singly infinite system of curves, or family of curves with a single parameter, is the general solution of the differential equation (1).

296. Every equation of the first order and first degree can be written

$$M\,dx + N\,dy = 0, \tag{1}$$

where, as has been said before, M and N are either constants, functions of x or y, or functions of x and y.

297. Solution by Separation of the Variables.—This solution consists in arranging the equation

$$M\,dx + N\,dy = 0, \tag{1}$$

so that it takes the form

$$\phi(x)dx + \psi(y)dy = 0. \tag{2}$$

The process by which this is effected is called *separation of the variables*. When the variables have been thus separated the solution is obtained by direct integration. Thus, integrating (2),

$$\int \phi(x)\,dx + \int \psi(y)\,dy = c,$$

where c is an arbitrary constant, and is the parameter of the family of curves representing the solution.

I. *Variables Separated by Inspection.*—A considerable number of simple equations can be solved directly by an obvious separation of the variables. The process is best illustrated by examples which follow.

EXAMPLES.

1. Find the curve whose slope to the x-axis is $-x/y$, and which passes through the point 2, 3.

The geometrical conditions give rise to the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}, \quad \text{or} \quad y\,dy + x\,dx = 0.$$

The solution of which, obtained by integration, is the family of circles

$$x^2 + y^2 = c^2.$$

The particular curve of the family through 2, 3 is

$$x^2 + y^2 = 13.$$

2. Find the line whose slope is constant.

$\frac{dy}{dx} = m$ gives the family of parallel straight lines $y = mx + c$.

3. Find the curves whose differential equation is

$$x\,dy + y\,dx = 0.$$

The variables when separated give

$$\frac{dx}{x} + \frac{dy}{y} = 0.$$

$$\therefore \quad \log x + \log y = c, \quad \text{or} \quad xy = k.$$

Otherwise we may write the solution $xy = e^c$. This is a family of hyperbolæ having for asymptotes the coordinate axes.

If we observe that $x\,dy + y\,dx$ is nothing more than $d(xy)$, the solution $xy = c$ is obvious.

4. Find the curve whose slope at any point is equal to the ordinate at the point.

Here $$\frac{dy}{dx} = y. \quad \therefore \quad \frac{dy}{y} = dx.$$

Hence $$\log y = x + c, \quad \text{or} \quad y = e^{x+c} = e^c e^x = ae^x,$$

which is the exponential family of curves.

5. Find the curve whose slope is proportional to the abscissa.

Ans. The family of parabolæ $y = ax^2 + c$, in which c, the constant of integration, is the parameter.

6. Find the curve whose slope at x, y is equal to xy. *Ans.* $y = ce^{\frac{1}{2}x^2}$.

7. Find the curve whose subtangent is proportional to the abscissa of the point of contact.

Here $$y\frac{dx}{dy} = ax. \quad \therefore \quad \frac{dx}{x} = a\frac{dy}{y} \quad \text{gives}$$

$$\log x = a \log y + c, \quad \text{or} \quad y^a = kx.$$

8. Find the curve whose subnormal is constant.

$$y\frac{dy}{dx} = a \quad \text{gives} \quad y^2 = 2ax + c, \quad \text{the parabola.}$$

9. Find the curve whose subtangent is constant. *Ans.* $y = ce^{\frac{x}{a}}$.

10. Find the curve whose subnormal is proportional to the nth power of the ordinate. What is the curve when n is 2?

11. Find the curve whose normal-length is constant.

Here the geometrical conditions give the differential equation

$$y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = a. \quad \therefore \quad dx = \frac{y\,dy}{\sqrt{a^2 - y^2}}.$$

Integrating, $x - c = -(a^2 - y^2)^{\frac{1}{2}}$, or the family of circles

$$(x - c)^2 + y^2 = a^2,$$

with radius a, having their centers on the x-axis.

12. Find the curve in which the perpendicular on the tangent drawn from the foot of the ordinate of the point of contact is constant and equal to a.

The differential equation of condition is

$$\frac{y}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = a. \quad \therefore \quad \frac{a\,dy}{\sqrt{y^2-a^2}} = dx.$$

The solution is therefore the family of curves

$$c + x = a \log\left(y + \sqrt{y^2-a^2}\right).$$

When $c = 0$ this is the catenary with Oy as axis.

13. Find the curve in which the subtangent is proportional to the subnormal.

14. Determine the curve in which the length of the arc measured from a fixed point to any point P is proportional to (1) the abscissa, (2) the square of the abscissa, (3) the square root of the abscissa of the point P.

(1). A straight line.

(2). The condition is $$s = \frac{x^2}{2a}.$$

$$\therefore \quad ds^2 = dx^2 + dy^2 = \frac{x^2}{a^2}dx^2,$$

or $$a\,dy = \sqrt{x^2-a^2}\,dx.$$

The solution of this is

$$c + ay = \tfrac{1}{2}x\sqrt{x^2-a^2} - \tfrac{1}{2}a^2 \log\left[x + \sqrt{x^2-a^2}\right].$$

(3). The geometrical condition can be written $s = 2\sqrt{ax}$.

$$\therefore \quad ds = \sqrt{\frac{a}{x}}\,dx. \qquad dx^2 + dy^2 = ds^2 = \frac{a}{x}dx \text{ gives}$$

$$dy = \sqrt{\frac{a-x}{x}}\,dx.$$

Put $x = z^2$ and integrate. The result is the cycloid

$$c + y = \sqrt{x(a-x)} + a\sin^{-1}\sqrt{\frac{x}{a}}.$$

Ex. 14, really leads to a differential equation of the first order and second degree, which furnishes two solutions which are the same.

15. Find the curve in which the polar subnormal is proportional to (1) the radius vector, (2) to the sine of the vectorial angle. (1). $\rho = ce^{a\theta}$. (2). $\rho = c - a\cos\theta$.

16. Find the curve in which the polar subtangent is proportional to the length of the radius vector, and also that curve in which the polar subtangent and polar sub-normal are in constant ratio. *Ans.* $\rho = ce^{a\theta}$.

17. Determine the curve in which the angle between the radius vector and the tangent is one half the vectorial angle. *Ans.* $\rho = c(1-\cos\theta)$.

18. Determine the curve such that the area bounded by the axes, the curve, and any ordinate is proportional to that ordinate.

If Ω is the area, $\Omega = ay. \;\therefore\; d\Omega = y\,dx = a\,dy. \;\therefore\; y = ce^{\frac{x}{a}}.$

19. Determine the curve such that the area bounded by the x axis, the curve, and two ordinates is proportional to the arc between two ordinates.

$$\Omega = as. \quad \therefore \quad y\,dx = a\,ds, \quad dx = a\frac{dy}{\sqrt{y^2-a^2}}.$$

This gives, on integration, the catenary

$$c + x = a \log\left(y + \sqrt{y^2 - a^2}\right).$$

20. Find the curve in which the square of the slope of the tangent is equal to the slope of the radius vector to the point of contact.

The parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = c^{\frac{1}{2}}$, or $(x-y)^2 - 2c(x+y) + c^2 = 0$.

21. Solve $M\,dx + N\,dy$, when $Mx \pm Ny = 0$.

(1). $Mx + Ny = 0$ gives $M/N = -y/x$.

Substituting in the equation, $\dfrac{dx}{x} = \dfrac{dy}{y}$. $\therefore$ $x = cy$.

(2). $Mx - Ny = 0$ gives $M/N = y/x$.

Substituting in the equation, $\dfrac{dx}{x} + \dfrac{dy}{y} = 0$. $\therefore$ $xy = c$.

II. *Solution when the Equation is homogeneous in x and y.*—When the equation

$$M\,dx + N\,dy = 0$$

is such that $M \equiv \phi(x, y)$, $N \equiv \psi(x, y)$ are homogeneous functions of x and y and of the same degree, the solution can be obtained by the substitution $y = zx$.

We have

$$\frac{M}{N} = \frac{\phi(x,y)}{\psi(x,y)} = F(z).$$

Divide the numerator and denominator by x^n, n being the degree of ϕ or ψ.

$$\therefore \quad \frac{dy}{dx} = z + x\frac{dz}{dx} = -F(z).$$

Hence

$$\frac{dx}{x} + \frac{dz}{z + F(z)} = 0,$$

and the variables are separated. The integration of this gives an equation in x and z. On substituting y/x for z the solution of the original equation is obtained.

EXAMPLES.

1. Solve the equation $(2x^2 - y^2)dy - 2xy\,dx = 0$.

Put $y = zx$. $\therefore$ $z + x\dfrac{dz}{dx} = \dfrac{2z}{2 - z^2}$, or

$$\frac{dx}{x} = \frac{2 - z^2}{z^3}dz.$$

Integrating,

$$\log x = c - \frac{1}{z^2} - \log z.$$

Replacing z by y/x, we have

$$x^2 = y^2(c - \log y).$$

2. Determine the curve in which the perpendicular from the origin on the tangent is equal to the abscissa of the point of contact.

Ans. The circles $x^2 + y^2 = 2cx$.

3. Find the curve in which the intercept of the normal on the x-axis is proportional to the ordinate of the point of contact.

$$x + y\frac{dy}{dx} = my. \quad \therefore \quad (x - my)dx + y\,dy = 0, \quad \text{etc.}$$

4. Find the curve in which the subnormal is equal to the sum of the abscissa and radius vector.

5. Find the curve whose slope at any point is equal to the ratio of the arithmetic to the geometric mean of the coordinates of the point.

6. Solve $y^2dx + (xy + x^2)dy = 0$. *Ans.* $xy^2 = c^2(x + 2y)$.

7. Solve $x^2y\,dx = (x^3 + y^3)dy$. *Ans.* $\log\frac{y}{c} = \frac{x^3}{3y^3}$.

298. Solution when *M* and *N* are of the First Degree.—The equation

$$(a_1x + b_1y + c_1)dx = (a_2x + b_2y + c_2)dy \tag{1}$$

can always be solved as follows :

Put $x = x' + h$, $y = y' + k$, where h and k are arbitrary constants. Then (1) becomes

$$\frac{dy'}{dx'} = \frac{a_1x' + b_1y' + a_1h + b_1k + c_1}{a_2x' + b_2y' + a_2h + b_2k + c_2}. \tag{2}$$

I. If $a_1b_2 \neq a_2b_1$, assign to h, k the values which satisfy

$$\left.\begin{aligned} a_1h + b_1k + c_1 &= 0, \\ a_2h + b_2k + c_2 &= 0. \end{aligned}\right\} \tag{3}$$

Then (2) becomes

$$\frac{dy'}{dx'} = \frac{a_1x' + b_1y'}{a_2x' + b_2y'}. \tag{4}$$

This is homogeneous and can be solved by § 297.

If $f(x', y') = 0$ is the solution of (4), then $f(x - h, y - k) = 0$ is the solution of (1).

II. If $a_1b_2 = a_2b_1$, let $\frac{a_2}{a_1} = \frac{b_2}{b_1} = m$.

Then (1) becomes

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{m(a_1x + b_1y) + c_2}. \tag{5}$$

Put $z = a_1x + b_1y$. Then (5) becomes

$$\frac{dz}{dx} = a_1 + b_1\frac{z + c_1}{mz + c_2},$$

in which the variables can be readily separated.

[illegible]

EXAMPLES.

1. Solve $(3y - 7x + 7)dx + (7y - 3x + 3)dy = 0$.

Ans. $(y - x + 1)^2(y + x - 1)^5 = c$.

2. Solve $(2x + y + 1)dx + (4x + 2y - 1)dy = 0$.

Ans. $x + 2y + \log(2x + y - 1) = c$.

3. Solve. $(7y + x + 2)dx - (3x + 5y + 6)dy = 0$.

Ans. $x + 5y + 2 = c(x - y + 2)^4$.

299. The Exact Differential Equation.—The differential equation

$$M\,dx + N\,dy = 0$$

is said to be an *exact differential equation* when it is the *immediate* result of differentiating an implicit function $f(x, y) = 0$.

In fact, if

$$u = f(x, y) = 0,$$

then

$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

gives an exact differential equation.

300. Condition that $M\,dx + N\,dy = 0$ be Exact.—Since M must be the first partial derivative with respect to x, and N the first partial derivative with respect to y of some function $f(x, y)$, then

$$M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y}.$$

But since

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y},$$

we must have the relation

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{1}$$

existing between M and N in order that $Mdx + Ndy = 0$ shall be exact. This condition is also sufficient, and when (1) is satisfied $Mdx + Ndy$ is an exact differential.

For,* let $V = \int Mdx$.

$$\therefore \quad \frac{\partial V}{\partial x} = M, \quad \frac{\partial^2 V}{\partial y\,\partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x};$$

$$\therefore \quad \frac{\partial N}{\partial x} = \frac{\partial^2 V}{\partial x\,\partial y}.$$

Hence

$$N = \int \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right)dx = \frac{\partial V}{\partial y} + \phi'(y),$$

---

* This is due to Professor James McMahon.

where the constant of integration $\phi'(y)$ is some function of y or a constant independent of x. Therefore

$$Mdx + Ndy = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \phi'(y)dy,$$
$$= d[V + \phi(y)],$$

an exact differential.

301. Solution of the Exact Equation.

If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, there exists a function u, of x and y, such that

$$du = M\,dx + N\,dy. \tag{1}$$

Since $M = \dfrac{\partial u}{\partial x}$, M contains the derivatives of only those terms in u which contain x. Integrating (1) with respect to x (y being constant), we have

$$u = \int M\,dx + \phi(y), \tag{2}$$

where $\phi(y)$ represents the terms in u which do not contain x.

To find $\phi(y)$, differentiate (2) with respect to y.

$$\therefore \quad \frac{\partial u}{\partial y} = N = \frac{\partial}{\partial y}\int M\,dx + \frac{\partial \phi}{\partial y}.$$

Hence

$$\frac{\partial \phi}{\partial y} = N - \frac{\partial}{\partial y}\int M\,dx. \tag{3}$$

As was said, ϕ is independent of x and so also is $\dfrac{\partial \phi}{\partial y}$, as is verified by differentiating (3) with respect to x;

$$\frac{\partial}{\partial x}\left\{N - \frac{\partial}{\partial y}\int M\,dx\right\} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

Integrate (3) with respect to y.

$$\therefore \quad \phi(y) = \int\left\{N - \frac{\partial}{\partial y}\int M\,dx\right\}dy + c.$$

Therefore the solution of (1) is

$$u = \int M\,dx + \int\left\{N - \frac{\partial}{\partial y}\int M\,dx\right\}dy + c = 0. \tag{4}$$

In like manner, working first with N instead of M,

$$\int N\,dy + \int\left\{M - \frac{\partial}{\partial x}\int N\,dy\right\}dx + c = 0 \tag{5}$$

is also a solution of (1).

302. Rule for Solving the Exact Equation.

$\int M\,dx$ contains all the terms of the primitive containing x. Also, since $N - \dfrac{\partial}{\partial x}\displaystyle\int M\,dx$ is independent of x, $\dfrac{\partial}{\partial x}\displaystyle\int M\,dx$ must contain those terms in N containing x. Therefore to obtain

$$\int \left\{ N - \frac{\partial}{\partial y}\int M\,dx \right\} dy,$$

integrate only those terms in N which do not contain x. Hence the rule. Integrate $M\,dx$ as if y were constant; integrate those terms in $N\,dy$ which do not contain x; equate the sum of these integrals to a constant.

A like rule follows for effecting (5), § 301.

EXAMPLES.

1. Solve $\quad (3x^2 - 4xy - 2y^2)dx + (3y^2 - 4xy - 2x^2)dy = 0.$

Here
$$\frac{\partial M}{\partial y} = -4x - 4y = \frac{\partial N}{\partial x}.$$

$$\int M\,dx = x^3 - 2x^2y - 2xy^2; \quad \int 3y^2\,dy = y^3.$$

Therefore the solution is

$$x^3 - 2x^2y - 2xy^2 + y^3 = c.$$

2. Solve $\quad (x^2 + y^2)(x\,dx + y\,dy) + x\,dy - y\,dx = 0.$

Ans. $\dfrac{x^2 + y^2}{2} + \tan^{-1}\dfrac{y}{x} = c.$

3. Solve $\quad (a^2 + 8xy - 2y^2)dx + (2x - y)^2dy = 0.$

Ans. $a^2x + \frac{1}{3}y^3 - 2xy^2 + 4x^2y = c.$

4. Solve $\quad (2ax + by + g)dx + (2cy + bx + e)dy = 0.$

Ans. $ax^2 + bxy + cy^2 + gx + ey = k.$

5. Solve $\quad (m\,dx + n\,dy)\sin(mx + ny) = (n\,dx + m\,dy)\cos(nx + my).$

Ans. $\cos(mx + ny) + \sin(nx + my) = c.$

6. Solve $\quad 2x(x + 2y)dx + (2x^2 - y^2)dy = 0.$

Ans. $x^3 + 3x^2y - 2y^3 = c.$

303. Non-Exact Equations of the First Order and Degree.—We have seen that when a primitive equation $f(x, y) = 0$ is differentiated there results the exact differential equation $\phi(x, y, y') = 0$, writing y' for the derivative of y with respect to x.

If now between $f = 0$ and $\phi = 0$ we eliminate any constant occurring in f and ϕ, we get another equation, $\psi(x, y, y') = 0$, which is a differential equation satisfied at every point on $f = 0$. Therefore $f = 0$ is a primitive of $\psi = 0$. But $\psi = 0$ will not be an exact differential of the primitive $f = 0$, although $f = 0$ is a solution of the differential equation $\psi = 0$.

To fix the ideas, consider the equation

$$ax + by + cxy + k = 0. \tag{1}$$

The exact differential equation of (1) is

$$(a + cy)dx + (b + cx)dy = 0. \tag{2}$$

When (2) is integrated the constant of integration restores the parameter k of the family (1) and (1) is the solution of (2). That is to say, the family of curves (1) obtained by varying the parameter k gives the solution of the exact differential equation (2).

The constant k was eliminated from (2) by the operation of differentiation and restored by the process of integration.

Eliminate a between (1) and (2) by substituting

$$a + cy = -\frac{by + k}{x}$$

from (1) in (2). There results the differential equation

$$\frac{dy}{by + k} = \frac{dx}{x(b + cx)}, \tag{3}$$

$$= \frac{1}{b}\frac{dx}{x} - \frac{c}{b}\frac{dx}{b + cx}$$

or

$$\frac{b\,dy}{by + k} = \frac{dx}{x} - \frac{c\,dx}{cx + b}.$$

Integrating and adding the arbitrary constant $-\log c'$,

$$\log(by + k) + \log(cx + b) - \log x - \log c' = 0.$$

$$\therefore \quad (by + k)(cx + b) = c'x,$$

or

$$(kc - c')x + b^2y + bc\,xy + kb = 0.$$

Putting the arbitrary parameter in the form $kc - c' = ab$, this equation becomes the original primitive

$$ax + by + cxy + k = 0.$$

This equation with the variable parameter a is the solution of the differential equation (3).

The differential equation (3), or

$$(by + k)dx - x(b + cx)dy = 0 \tag{4}$$

is not an exact equation, for

$$\frac{\partial}{\partial x}(by + k) = b, \quad \frac{\partial}{\partial x}(-bx - cx^2) = -b - 2cx.$$

But (1) is the primitive of (3) as well as of (2).

Again, if we eliminate first b and then c between (1) and (2), we shall get two other differential equations, neither of which is exact, but each of which has (1) for solution with variable parameters b and c respectively.

Observe particularly that if (4) be multiplied by $1/x^2$, it becomes an exact differential,

$$\frac{by + k}{x^2} dx - \frac{b + cx}{x} dy = 0, \tag{5}$$

since

$$\frac{\partial}{\partial y}\left(\frac{by + k}{x^2}\right) = \frac{\partial}{\partial x}\left(- \frac{b + cx}{x}\right) = \frac{b}{x^2}.$$

Integrating this exact equation (5) under the rule § 302, the solution is

$$qx + by + cxy + k = 0,$$

the same equation as (1) with q for parameter.

304. Integrating Factors.—In the preceding article we have seen that the same group of primitives can have a number of different differential equations of the first order and degree. The form of any particular differential equation depending on the manner in which an arbitrary constant has been eliminated between the primitive and its exact differential equation. In the example above, when the differential equation was not exact, it was made exact by multiplying by $1/x^2$. Such a factor is called an *integrating factor* of the differential equation which it renders exact.

The number of integrating factors for any equation

$$M\,dx + N\,dy = 0 \tag{1}$$

is infinite. For, let μ be an integrating factor of (1). Then $\mu(M\,dx + N\,dy)$ is an exact differential, say du, and

$$\mu(M\,dx + N\,dy) = du.$$

Multiply both sides of this equation by any integrable function of u, say $f(u)$,

$$\mu f(u)(M\,dx + N\,dy) = f(u)du. \tag{2}$$

The second member of (2) is an exact differential, and therefore also is the first. Hence, when μ is an integrating factor of (1), so also is $\mu f(u)$, where $f(u)$ is any arbitrary integrable function of u.

In illustration consider the equation

$$y\,dx - x\,dy = 0.$$

This is not exact, but when multiplied by either $\frac{1}{y^2}$, $\frac{1}{xy}$, or $\frac{1}{x^2}$ it becomes exact and has for solution

$$\frac{x}{y} = \text{constant}.$$

The general solution of the differential equation

$$M\,dx + N\,dy = 0$$

consists in finding an integrating factor μ such that

$$\mu(M\,dx + N\,dy) = 0$$

is an exact differential, then integrating by the method given as the solution of the exact equation.

The integrating factor always exists, but there is no known method by which it can be determined generally. The rules for determining an integrating factor for a few important equations will now be given.

305. Rules for Integrating Factors.

I. *By Inspection.*—While the process of finding an integrating factor by inspection does not, strictly speaking, constitute a rule, in the absence of a general law for finding the integrating factor it is an important method of procedure. An equation should always be examined first with the view of being able to recognize a factor of integration. The process is best illustrated by examples.

EXAMPLES.

1. Solve $\qquad y\,dx - x\,dy + f(x)dx = 0.$

The last term is exact; its product by any function of x is exact. Therefore any function of x that will make $y\,dx - x\,dy$ exact is an integrating factor. Such a factor is obviously $1/x^2$.

$$\therefore \quad \frac{y\,dx - x\,dy}{x^2} + \frac{f(x)}{x^2}dx = 0,$$

or

$$d\left(\frac{y}{x}\right) + \frac{f(x)}{x^2}dx = 0,$$

gives the solution

$$\frac{y}{x} + \int \frac{f(x)}{x^2}\,dx = c.$$

2. Solve $\quad y\,dx + \log x\,dx = x\,dy.$

Ans. $\quad cx + y + \log x + 1 = 0.$

3. Solve $\quad (1 + xy)y\,dx + (1 - xy)x\,dy = 0. \quad$ (Factor $1/x^2y^2$).

Ans. $\quad cx = ye^{\frac{1}{xy}}.$

4. Integrate $\quad x^{\alpha}y^{\beta}(ay\,dx + bx\,dy) = 0.$

Obviously $\quad x^{ka-1-\alpha}y^{kb-1-\beta}$ is an integrating factor, where k is any number. On multiplying by the factor we get

$$ax^{ka-1}y^{kb}\,dx + bx^{ka}y^{kb-1}\,dy = \frac{1}{k}d(x^{ka}y^{kb}) = 0,$$

the solution of which is evident.

5. Integrate

$$x^{\alpha}y^{\beta}(ay\,dx + bx\,dy) + x^{\alpha_1}y^{\beta_1}(a_1y\,dx + b_1x\,dy) = 0.$$

The factors $x^{ka-1-\alpha}y^{kb-1-\beta}$, $x^{k_1a_1-1-\alpha_1}y^{k_1b_1-1-\beta_1}$ make the expressions

$$x^{\alpha}y^{\beta}(ay\,dx + bx\,dy) \quad\text{and}\quad x^{\alpha_1}y^{\beta_1}(a_1y\,dx + b_1x\,dy)$$

exact differentials respectively, whatever be the values of the arbitrary numbers k and k_1.

Therefore, if k and k_1 be determined so as to satisfy

$$ka - 1 - \alpha = k_1a_1 - 1 - \alpha_1,$$
$$kb - 1 - \beta = k_1b_1 - 1 - \beta_1,$$

the factors are identical and these values of k and k_1 furnish the integrating factor of the equation proposed.

6. Solve $(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0.$

Ans. $x^2y^2(y^2 - x^2) = c.$

7. Solve the equation

$$[y + xf(x^2 + y^2)]dx = [x - yf(x^2 + y^2)]dy. \qquad (1)$$

This is the differential equation of the group or family of rotations. Put

$$x^2 + y^2 = r^2.$$

Rearranging (1),

$$y\,dx - x\,dy + f(r^2)\cdot(x\,dx + y\,dy) = 0,$$
$$2(y\,dx - x\,dy) + f(r^2)dr^2 = 0.$$

This can be written

$$(y\,dx - x\,dy) - (x\,dy - y\,dx) + f(r^2)dr^2 = 0,$$

or

$$y^2\,d\left(\frac{x}{y}\right) - x^2\,d\left(\frac{y}{x}\right) + f(r^2)dr^2 = 0.$$

An integrating factor is obviously $\dfrac{1}{x^2 + y^2}$. Whence

$$\frac{d\dfrac{x}{y}}{1 + \dfrac{x^2}{y^2}} - \frac{d\dfrac{y}{x}}{1 + \dfrac{y^2}{x^2}} + \frac{f(r^2)}{r^2}\,dr^2 = 0.$$

Integrating,

$$\tan^{-1}\frac{x}{y} - \tan^{-1}\frac{y}{x} + \int\frac{f(r^2)}{r^2}dr^2 = c.$$

II. *Whenever an integrating factor exists which is a function of x only or of y only, it can be found.*

Making use of the fact that e^z is always a factor of its derivative:

(a). Let z be a function of x.

In
$$e^z(M\,dx + N\,dy) = 0,$$
put
$$M' = e^zM, \quad N' = e^zN.$$

Then
$$\frac{\partial M'}{\partial y} = e^z\frac{\partial M}{\partial y}; \quad \frac{\partial N'}{\partial x} = e^zN\frac{dz}{dx} + e^z\frac{\partial N}{\partial x}.$$

The condition that e^z shall be an integrating factor is

$$e^z\frac{\partial M}{\partial y} = e^zN\frac{dz}{dx} + e^z\frac{\partial N}{\partial x},$$

or

$$dz = \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}dx.$$

If, therefore,

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \phi(x)$$

is a function of x only, then

$$z = \int \phi(x) dx.$$

Hence $e^{\int \phi(x) dx}$ is an integrating factor of $M\ dx + N\ dy = 0$ whenever

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} \tag{1}$$

is a function, $\phi(x)$, of x only.

(*b*). In like manner, letting e^z be a function of y only, we find that $e^{\int \psi(y) dy}$ is an integrating factor of

$$M\ dx + N\ dy = 0$$

when

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} \tag{2}$$

is a function, $\psi(y)$, of y only.

(*c*). Whenever the expression (1), (2), or $\phi(x)$, $\psi(y)$ is constant, then e^x or e^y, respectively, is the integrating factor.

EXAMPLES.

1. One of the most important equations under this head is **Leibnitz's linear equation,**

$$\frac{dy}{dx} + Py = Q, \tag{1}$$

where P and Q are functions of x or are constants.

This equation, $(Py - Q)dx + dy = 0$, is such that

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = P.$$

Therefore it has the integrating factor $e^{\int P dx}$.

$$e^{\int P dx}(dy + Py\ dx) = e^{\int P dx} Q\ dx. \tag{2}$$

Since
$$e^{\int P dx} dy + e^{\int P dx} Py\ dx = d\left(y e^{\int P dx}\right),$$
on integrating (2),

$$y = e^{-\int P dx} \left\{ \int e^{\int P dx} Q\ dx + c \right\}. \tag{3}$$

This is the solution of the linear equation (1).

2. Bernoulli's Equation.—The equation known as Bernoulli's

$$\frac{dy}{dx} + Py = Qy^n,$$

in which P, Q are functions of x or are constants, reduces to Leibnitz's linear equation. For, multiply by $(-n + 1)/y^n$, and put $v = y^{-n+1}$. Equation (1) becomes

$$\frac{dv}{dx} + (1 - n)Pv = (1 - n)Q,$$

which is linear in v.

3. Solve $f'(y)\dfrac{dy}{dx} + Pf(y) = Q$, where P, Q are functions of x.

Put $v = f(y)$. The equation becomes

$$\frac{dv}{dx} + Pv = Q,$$

which is linear in v.

III. *When $Mx \pm Ny \neq 0$, there are two cases in which the integrating factor of $M\,dx + N\,dy = 0$ can be assigned.*

(1). When M, N are homogeneous and of the same degree, then $\dfrac{1}{Mx + Ny}$ is an integrating factor.

(2). When M, N are such functions as

$$M = y\phi(x \times y), \quad N = x\psi(x \times y),$$

then $\dfrac{1}{Mx - Ny}$ is an integrating factor.

Proof: We have the identity

$M\,dx + N\,dy$

$$= \tfrac{1}{2}\left\{(Mx + Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right\},$$

$$= \tfrac{1}{2}\{(Mx + Ny)d\log(xy) + (Mx - Ny)d\log(x/y)\}.$$

(1). Divide by $Mx + Ny$.

$$\frac{M\,dx + N\,dy}{Mx + Ny} = \tfrac{1}{2}d\log(xy) + \tfrac{1}{2}\frac{Mx - Ny}{Mx + Ny}d\log\left(\frac{x}{y}\right),$$

$$= \tfrac{1}{2}d\log(xy) + \tfrac{1}{2}f\left(\frac{x}{y}\right)d\log\left(\frac{x}{y}\right),$$

if M, N are homogeneous functions in x, y and of the same degree.

Since $x/y = e^{\log\frac{x}{y}}$, this can be written

$$\frac{M\,dx + N\,dy}{Mx + Ny} = \tfrac{1}{2}d\log(xy) + \tfrac{1}{2}F\left(\log\frac{x}{y}\right)d\log\left(\frac{x}{y}\right),$$

$$= \tfrac{1}{2}du + \tfrac{1}{2}F(v)dv,$$

where $u = \log(xy), \quad v = \log(x/y)$.

This case is otherwise solved by the substitution $y = zx$, see § 297, II.

(2). Divide by $Mx - Ny$

$$\frac{M\,dx + N\,dy}{Mx - Ny} = \frac{1}{2}\,\frac{Mx + Ny}{Mx - Ny}\,d\log(xy) + \tfrac{1}{2}d\log(x/y).$$

If $M \equiv y\phi(xy)$, $N \equiv x\psi(xy)$, then

$$\frac{Mx + Ny}{Mx - Ny} = \frac{\phi(xy) + \psi(xy)}{\phi(xy) - \psi(xy)}.$$

$$\begin{aligned} \therefore \quad \frac{M\,dx + N\,dy}{Mx - Ny} &= \tfrac{1}{2}f(xy)d\log(xy) + \tfrac{1}{2}d\log(x/y), \\ &= \tfrac{1}{2}F(\log xy)d\log xy + \tfrac{1}{2}d\log(x/y), \\ &= \tfrac{1}{2}F(u)du + \tfrac{1}{2}dv. \end{aligned}$$

Writing as before, $xy = e^{\log xy}$, $u = \log xy$, $v = \log x/y$.

(3). The cases in which $Mx \pm Ny = 0$ were solved in § 297, I, Ex. 21.

EXAMPLES.

Solve by integrating factors the following equations:

1. $y\,dx - x\,dy + \log x\,dx = 0.$ (I, Ex. 1.) *Ans.* $cx + y + \log x + 1 = 0.$

2. $a(x\,dy + 2y\,dx) = xy\,dy.$ *Ans.* $a\log x^2y = y + c.$

3. $(x^2 + 2xy - y^2)dx = (x^2 - 2xy - y^2)dy.$ *Ans.* $x^2 + y^2 = c(x + y).$

4. $\dfrac{dx}{x} + \dfrac{dy}{y} = 2\left(\dfrac{dy}{x} - \dfrac{dx}{y}\right).$ *Ans.* $x^2 - y^2 + xy = c.$

5. $(x^2y^2 + xy)y\,dx + (x^2y^2 - 1)\,x\,dy = 0.$ *Ans.* $y = ce^{xy}.$

6. $(x^3y^3 + 1)(x\,dy + y\,dx) + (x^2y^2 + xy)(y\,dx - x\,dy) = 0.$ *Ans.* $xy - \dfrac{1}{xy} = \log cy^2.$

7. $x^3\,dx + (3x^2y + 2y^3)dy = 0.$ *Ans.* $x^2 + 2y^2 = c\sqrt{x^2 + y^2}.$

8. $(y + y\sqrt{xy})dx - (x + x\sqrt{xy})dy = 0.$ *Ans.* $y = cx.$

9. $(x^2 + y^2 + 2x)dx + 2y\,dy = 0.$ *Ans.* $x^2 + y^2 = ce^{-x}.$

10. $(3x^2 - y^2)dy = 2xy\,dx.$ *Ans.* $x^2 - y^2 = cy^3.$

11. $2xy\,dy = (x^2 + y^2)dx.$ *Ans.* $x^2 - y^2 = cx.$

12. $(x^2y - 2xy^2)dx = (x^3 - 3x^2y)dy.$ *Ans.* $\dfrac{x}{y} + \log\dfrac{y^3}{x^2} = c.$

13. $(3x^2y^4 + 2xy)dx = (x^2 - 2x^3y^3)dy.$ *Ans.* $x^3y^3 + x^2 = cy.$

14. $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$ *Ans.* $xy + y^2 + 2x/y^2 = c.$

15. $(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0.$ *Ans.* $5x^{-\frac{36}{13}}y^{-\frac{24}{13}} - 12x^{-\frac{10}{13}}y^{-\frac{15}{13}} = c.$

16. $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0.$ *Ans.* $6\sqrt{xy} = x^{-\frac{3}{2}}y^{\frac{3}{2}} + c.$

17. $x\dfrac{dy}{dx} - ay = x + 1.$ *Ans.* $y = \dfrac{x}{1-a} - \dfrac{1}{a} + cx^a.$

18. $(1 + x^2)dy = (m + xy)dx.$ *Ans.* $y = mx + c\sqrt{1 + x^2}.$

19. $\dfrac{dy}{dx} + \dfrac{\tan y}{x + 1} = (x - 1)\sec y.$ *Ans.* $\sin y = \dfrac{x^3 - 3x + c}{3(x + 1)}.$

20. $\dfrac{dy}{dx} = x - y.$ *Ans.* $y = x - 1 + ce^{-x}.$

21. $\dfrac{dy}{dx} + y = xy^3.$ *Ans.* $\dfrac{1}{y^2} = x + \frac{1}{2} + ce^{2x}.$

22. $\dfrac{dy}{dx} = n\dfrac{y}{x} + e^x x^n.$ *Ans.* $y = x^n(e^x + c).$

23. $\dfrac{dy}{dx} + \dfrac{1 - 2x}{x^2}y = 1.$ *Ans.* $\dfrac{y}{x^2} = 1 + ce^{\frac{1}{x}}.$

306. Solution by Differentiation.—A number of equations can be solved, by means of differentiation as equations of the first order and degree.

EXAMPLES.

1. Let $p = \dfrac{dy}{dx}$. Let the differential equation be

$$x = f(p). \tag{1}$$

Differentiating with respect to p,

$$dx = f'(p)\,dp.$$

Since $dy = p\,dx$, this gives the equation

$$dy = f'(p)\,p\,dp.$$

$$\therefore\quad y = \int f'(p)\,p\,dp + c. \tag{2}$$

The elimination of p between (1) and (2) gives the solution.

2. In like manner, if the differential equation is

$$y = f(p), \tag{1}$$

on differentiation we have

$$dy = f'(p)\,dp.$$

$$\therefore\quad p\,dx = f'(p)\,dp,$$

or

$$dx = \frac{f'(p)}{p}\,dp.$$

$$\therefore\quad x = \int\frac{f'(p)}{p}\,dp + c. \tag{2}$$

The elimination of p between (1) and (2) is the general solution of (1).

3. $x = p + \log p.$

Ans. $x + 1 = \pm\sqrt{2y + c} + \log\left(-1 \pm \sqrt{2y + c}\right).$

4. $x^2p^2 = 1 + p^2.$

Ans. $e^{2y} + 2cxe^y + c^2 = 0.$

5. $y = ap + bp^2.$

Ans. $x \pm \sqrt{a^2 + 4by} = \blacksquare \log\left(a \pm \sqrt{a^2 + 4by}\right) + c.$

EXERCISES.

1. $3e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0.$ *Ans.* $\tan y = c(1 - e^x)^3.$

2. $(4y + 3x)dy + (y - 2x)dx = 0.$

Ans. $c(2y^2 + 2xy - x^2)^{2\sqrt{3}} = \dfrac{2y + (1 + \sqrt{3})x}{2y - (1 - \sqrt{3})x}.$

3. $(2x - y + 1)dx + (x + y - 2)dy = 0.$

Ans. $\log\{2(3x - 1)^2 + (3y - 5)^2\} = \sqrt{2} \tan^{-1} \dfrac{\sqrt{2}(3x - 1)}{3y - 5} + c.$

4. $(x^3e_x - 2mxy^2)dx + 2mx^2y\, dy = 0.$ *Ans.* $x^2e^x + my^2 = cx^2.$

5. $y(2xy + e^x)dx - e^x dy = 0.$ *Ans.* $x^2y + e^x = cy.$

6. $dy + (y - e^{-x})dx = 0.$ *Ans.* $ye^x = x + c.$

7. $\cos^2 x\, dy + (y - \tan x)dx = 0.$ *Ans.* $y - ce^{-\tan x} = \tan x - 1.$

8. $(x + 1)dy = ny\, dx + e^x(x + 1)^{n+1}dx.$ *Ans.* $y = (e^x + c)(x + 1)^n.$

9. $dy = (by + a \sin x)dx.$ *Ans.* $y = ce^{bx} - a\dfrac{b \sin x + \cos x}{1 + b^2}.$

10. $\dfrac{dy}{dx} - \dfrac{2y}{x + 1} = (x + 1)^3.$ *Ans.* $2y = (x + 1)^4 + c(x + 1)^2.$

11. $x\, dy = ny\, dx + e^x x^{n+1} dx.$ *Ans.* $y = x^n(e^x + c).$

12. $dy = (y + 1)x\, dx.$ *Ans.* $y = ce^{\frac{1}{2}x^2} - 1.$

13. $\cos x\, dy + y \sin x\, dx = dx.$ *Ans.* $y = \sin x + c \cos x.$

14. $x(1 - x^2)\dfrac{dy}{dx} + (2x^2 - 1)y = ax^3.$ *Ans.* $y = ax + cx\sqrt{1 - x^2}.$

15. $(x + y)^2 dy = a^2\, dx.$ *Ans.* $\dfrac{x + y}{a} = \tan\dfrac{y + c}{a}.$

16. $(x - y)^2 dy = a^2\, dx.$ *Ans.* $\log\dfrac{x - y + a}{x - y - a} = 2\dfrac{c + y}{a}.$

17. $x^2\, dy + (y - 2xy - x^2)dx = 0.$ *Ans.* $y = x^2\left(1 + ce^{\frac{1}{x}}\right).$

18. $x\dfrac{dy}{dx} + \dfrac{y^2}{x} = y.$ *Ans.* $cx = e^{\frac{x}{y}}.$

19. $(x^2 + y^2 - a^2)x\, dx + (x^2 - y^2 - b^2)y\, dy = 0.$

Ans. $x^4 - y^4 + 2x^2y^2 - 2a^2x^2 - 2b^2y^2 = c.$

20. $dy = (x^2y^2 - 1)xy\, dx.$ *Ans.* $y^2(x^2 + 1 + ce^{x^2}) = 1.$

21. $2xy\, dx + (y^2 - x^2)dy = 0.$ *Ans.* $y^2 + x^2 = cy.$

22. $(x + y)dy + (x - y)dx = 0.$ *Ans.* $\log\sqrt{x^2 + y^2} + \tan^{-1}\dfrac{y}{x} = c.$

23. $(x^3y^3 + x^2y^2 + xy + 1)y\, dx + (x^3y^3 - x^2y^2 - xy + 1)x\, dy = 0.$

Ans. $x^2y^2 - 2xy \log cy = 1.$

24. $\dfrac{dy}{dx} + \dfrac{n}{x}y = \dfrac{a}{x^n}.$ *Ans.* $x^n y = ax + c.$

25. $y - x\dfrac{dy}{dx} = b\left(1 + x^2\dfrac{dy}{dx}\right).$ *Ans.* $c(y - b) = \dfrac{x}{1 + bx}.$

CHAPTER XXXIX.

EXAMPLES OF EQUATIONS OF THE FIRST ORDER AND SECOND DEGREE.

307. The equation of the First order and Second degree is a quadratic equation in $\frac{dy}{dx}$ of the form

$$\left(\frac{dy}{dx}\right)^2 + A\frac{dy}{dx} + B = 0, \tag{1}$$

where A, B are, in general, functions of x and y.

We shall represent $\frac{dy}{dx}$ by p. Equation (1) can be written symbolically

$$f(x, y, p) = 0. \tag{2}$$

308. There are three general methods which should be made use of in solving (1):

(1). *Solve for* y; (2). *Solve for* x; (3). *Solve for* p.

309. Equations Solvable for y.—If (2) can be solved for y, the equation becomes

$$y = F(x, p). \tag{1}$$

Differentiate with respect to x.

$$\therefore \quad p = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial p}\frac{dp}{dx}. \tag{2}$$

This equation (2) is of the first order in $\frac{dp}{dx}$.

$\frac{dp}{dx} = \frac{1}{dz} \cdot p$

The elimination of p between (1) and the solution of (2) furnishes the solution of (1). The elimination of p is frequently inconvenient or impracticable. When this is the case, the expression of x and y in terms of the third variable p is regarded as the solution.

EXAMPLES.

1. Solve $p + 2xy = x^2 + y^2$. (1)

$$\therefore \quad y = x + \sqrt{p}.$$

Differentiating,

$$p = 1 + \frac{1}{2p^{\frac{1}{2}}}\frac{dp}{dx},$$

or
$$dx = \frac{dp}{2p^{\frac{1}{2}}(p-1)}. \tag{2}$$

$$\therefore \quad x = \tfrac{1}{2}\log\frac{p^{\frac{1}{2}}-1}{p^{\frac{1}{2}}+1} + c,$$

or
$$p^{\frac{1}{2}} = \frac{1+e^{2x-2c}}{1-e^{2x-2c}}. \tag{3}$$

Eliminating p, we have for the solution

$$y - x = \frac{k+e^{2x}}{k-e^{2x}}.$$

2. Solve $\quad x - yp = ap^2. \qquad (1)$

Differentiate $\quad y = \dfrac{x-ap^2}{p}, \quad$ with respect to x, and put the result in the form

$$\frac{dx}{dp} - \frac{1}{p(1-p^2)}\,x = \frac{ap}{1-p^2}.$$

Solving this linear equation,

$$x = \frac{p}{\sqrt{1-p^2}}\,(c + a\sin^{-1}p). \tag{2}$$

Substituting in (1),

$$y = -ap + \frac{1}{\sqrt{1-p^2}}\,(c + a\sin^{-1}p). \tag{3}$$

The values of x, y expressed in terms of the third variable p in (2), (3) furnish the solution of (1).

3. Clairaut's Equation.—The important equation, known as Clairaut's,

$$y = px + f(p), \tag{1}$$

can be solved in this manner.

Differentiate with respect to x.

$$\therefore \quad p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx},$$

or,
$$[x + f'(p)]\frac{dp}{dx} = 0. \tag{2}$$

The equation (2) is satisfied by either

$$x + f'(p) = 0, \quad \text{or} \quad \frac{dp}{dx} = 0.$$

The solution of (1) is obtained by eliminating p between either of these equations and (1).

$$\frac{dp}{dx} = 0 \quad \text{gives} \quad p = c, \quad \text{constant.}$$

Therefore one solution is

$$y = cx + f(c), \tag{3}$$

which is the family of straight lines with parameter c.

The second solution is the result of eliminating p between

$$\left.\begin{aligned} y &= px + f(p), \\ \text{and} \qquad 0 &= x + f'(p). \end{aligned}\right\} \tag{4}$$

The second of these equations is the derivative of the first with respect to p; x and y being regarded as constants, p as a variable parameter. This result is clearly the envelope of the family of straight lines representing the first solution (3). This envelope is called the *singular* solution of (1).

Thus the general solution of Clairaut's equation (1) is effected by substituting an arbitrary constant for p in the equation. The singular solution is the envelope of the family of straight lines representing the general solution.

4. Lagrange's Equation.—To integrate

$$y = x f(p) + F(p). \tag{1}$$

Differentiating with respect to x and rearranging,

$$\frac{dx}{dp} + \frac{f'(p)}{f(p) - p} x + \frac{F'(p)}{f(p) - p} = 0. \tag{2}$$

This is a linear equation in x and can be solved by § 305, II, Ex. 1.

Eliminating p between (1) and the solution of (2), the solution of (1) is obtained. Otherwise x and y are obtained in terms of the third variable p.

5. Solve $$y = (1 + p)x + p^2.$$

Differentiating, $$\frac{dx}{dp} + x = -2p.$$

Solving this linear equation,

$$x = 2(1 - p) + ce^{-p};$$

$$\therefore \quad y = 2 - p^2 + (1 + p)ce^{-p}.$$

6. Solve $x^2(y - px) = yp^2$.

Put $x^2 = u$, $y^2 = v$.

$$\therefore \quad v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2,$$

which is Clairaut's form.

$$\therefore \quad v = cu + c^2. \quad \text{Hence} \quad y^2 = cx^2 + c^2.$$

310. Equations Solvable for x.—When this is the case

$$f(x, y, p) = 0$$

becomes

$$x = F(y, p). \tag{1}$$

Differentiate with respect to y.

$$\therefore \quad \frac{1}{p} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial p}\frac{dp}{dy}. \tag{2}$$

This is of the first order in $\frac{dp}{dy}$. The elimination of p between (1) and the integral of (2), or the expression of x and y in terms of p, furnishes the solution of (1).

EXAMPLES.

1. Solve $x = y + p^2$.

$$\frac{1}{p} = 1 + 2p\frac{dp}{dy}, \quad \text{or} \quad dy = -2\frac{p^2 dp}{p - 1}.$$

$$\therefore \quad y = c - [p^2 + 2p + 2\log(p - 1)], \quad x = c - [2p + 2\log(p - 1)].$$

2. $x = y + \log p^a$. *Ans.* $y = c - a\log(p - 1)$, $x = c + a\log\frac{p}{p - 1}$.

3. Solve $p^2 y + 2px = y$. *Ans.* $y^2 = 2cx + c^2$.

311. Equations Solvable for p.—The equation $f(x, y, p) = 0$ is a quadratic in p.

If this can be solved in a suitable form for integration for p, it becomes

$$\{p - \phi(x, y)\}\{p - \psi(x, y)\} = 0.$$

Each of the equations

$$p = \phi(x, y) \quad \text{and} \quad p = \psi(x, y)$$

is of the first order and degree in $\dfrac{dy}{dx}$, and their solutions are solutions of (1).

Such solutions have already been discussed.

EXAMPLES.

1. Solve $p^2 - (x + y)p + xy = 0.$

$$(p - x)(p - y) = 0$$

gives $$dy - x\,dx = 0, \quad \text{and} \quad \frac{dy}{y} - dx = 0.$$

$$\therefore \quad 2y = x^2 + c, \quad \text{and} \quad y = ce^x.$$

2. $p^2 - 5p + 6 = 0.$ *Ans.* $y = 2x + c, \quad y = 3x + c.$

312. In particular, if $f(x, y, p) = 0$ does not contain x or does not contain y, corresponding simplifications of the above processes apply, see § 306.

312. Equations Homogeneous in x and y.—When the equation $f(x, y, p) = 0$ is homogeneous in x and y, it can be written

$$F\left(\frac{dy}{dx}, \frac{y}{x}\right) = 0. \tag{1}$$

(1). Solve, if possible, for p and proceed as in § 297, II.

(2). Solve for y/x. Then the equation becomes

$$y = xf(p). \tag{2}$$

Differentiate (2) with respect to x and rearrange.

$$\therefore \quad \frac{dx}{x} = \frac{f'(p)\,dp}{p-(fp)}.$$

EXAMPLES.

1. Solve $xp^2 - 2yp + ax = 0.$ *Ans.* $2cy = c^2x^2 + a.$

2. Solve $y = yp^2 + 2px.$ *Ans.* $y^2 = 2cx + c^2.$

3. $x^2p^2 - 2xyp - 3y^2 = 0.$ *Ans.* $cy = x^3, \quad xy = c.$

ORTHOGONAL TRAJECTORIES.

313. A curve which cuts a family of curves at a constant angle is called a *trajectory* of the family. We shall be concerned here only with orthogonal trajectories. If each member of a family of curves cuts each member of a second family of curves at right angles, then each family is said to be the *orthogonal trajectories* of the other family.

At any point x, y where two curves cross at right angles, the relation $pp' = -1$ exists between their slopes p, p'.

314. To Find the Orthogonal Trajectories of a given Family of Curves.

Let
$$\phi(x, y, a) = 0 \tag{1}$$
be the equation of a family of curves having for arbitrary parameter a.

Let
$$f(x, y, p) = 0 \tag{2}$$
be the differential equation of the family (1), obtained by the elimination of the parameter a.

The differential equation
$$f\left(x, y, -\frac{1}{p}\right) = 0,$$
or
$$f\left(x, y, -\frac{dx}{dy}\right) = 0, \tag{3}$$
is the differential equation of a family of curves, each member of which cuts each member of (1) at right angles. Therefore the general integral of (3),
$$\psi(x, y, b) = 0, \tag{4}$$
is the equation of the family of orthogonal trajectories of (1).

EXAMPLES.

1. Find the orthogonal trajectories of the family of parabolæ $y^2 = 4ax$.

Differentiating and eliminating a, the differential equation of the family is
$$\frac{dy}{dx} = \frac{y}{2x}.$$

The differential equation of the orthogonal trajectories is
$$-\frac{dx}{dy} = \frac{y}{2x}.$$
The integral of which is $x^2 + \frac{1}{2}y^2 = c^2$, a family of ellipses.

2. Find the orthogonal trajectories of the hyperbolæ $xy = a^2$.

The differential equation is $y + xp = 0$. The differential equation of the orthogonal trajectories is
$$y - x\frac{dx}{dy} = 0,$$
giving the hyperbolæ $x^2 - y^2 = c^2$ for trajectories.

3. Find the orthogonal trajectories of $y = mx$.

4. Show that $x^2 + y^2 - 2cy = 0$ is orthogonal to the family
$$y^2 = 2ax - x^2.$$

5. Find the orthogonal system of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, in which b is the parameter.

Ans. $x^2+y^2=a^2\log x^2+c$.

6. Find the system of curves cutting $x^2+b^2y^2=b^2a^2$ at right angles, a being the parameter of the family.

Ans. $yc=x^{b^2}$.

THE SINGULAR SOLUTION.

314. We have seen in the case of Clairaut's equation, § 309, Ex. 3, that there may exist a solution of a differential equation which is not included in the general solution. Such a solution, called the *singular* solution, we now propose to notice more generally.

315. Singular Solution from the General Solution.

Let

$$\phi(x, y, c) = 0 \tag{1}$$

be the general solution of the differential equation

$$f(x, y, p) = 0. \tag{2}$$

A solution of the differential equation (2) has been defined to be an equation (1) in x, y such that at any point x, y satisfying the equation (1) the x, y, and $p \equiv \frac{dy}{dx}$ derived from this relation satisfies (2).

The general solution (1) being the integral of (2) satisfies the condition for a solution. Also, however, the envelope of the system of curves (1) is a curve such that at any point on it the x, y, p of the envelope is the same as the x, y, p of a point on some one of the system of curves (1), and must therefore satisfy (2). Consequently the envelope of the family (1) is a solution of (2).

This is a *singular* solution. It is not included in the general solution, and cannot be derived from it by assigning a particular value to the parameter c.

We may then find the singular solution of a differential equation (2) by finding the envelope of the family (1) representing the general solution of (2).

Thus the singular solution of (2) is contained in

$$\psi(x, y) = 0,$$

which results from the elimination of c between

$$\phi(x, y, c) = 0 \quad \text{and} \quad \phi'_c(x, y, c) = 0.$$

316. Singular Solution Directly from the Differential Equation.—It is not necessary to obtain the general solution of a differential equation in order to get the singular solution. The singular solution can be obtained directly from the differential equation without any knowledge of the general solution.

Let the differential equation

$$f(x, y, p) = 0 \tag{1}$$

be regarded as a family of curves having the variable parameter p. Find the envelope

$$\chi(x, y) = 0 \tag{2}$$

of (1), as the result of eliminating p between

$$f(x, y, p) = 0 \quad \text{and} \quad f'_p(x, y, p) = 0.$$

Since at any x, y satisfying (2) the x, y, $\frac{dy}{dx}$ of (2) is the same as the x, y, $\frac{dy}{dx}$ of a point on (1), the equation (2) must contain a solution of (1).

EXAMPLES.

1. Find the general and singular solutions of $p^2 + xp = y$.

This is Clairaut's form, and the general solution can be written immediately by putting $p =$ const.

However, independently, we have on differentiation

$$0 = (x + 2p)\frac{dp}{dx}.$$

$\frac{dp}{dx} = 0$ gives $p = c$, and $y = cx + c^2$ for the general solution. Differentiating with respect to c and eliminating c, we find the singular solution $4y + x^2 = 0$.

Integrating the other factor, $x + 2p = 0$, or eliminating p between this and the differential equation, the same singular solution is found.

2. Find the general and singular solutions of the equation $y = px + a\sqrt{1 + p^2}$.

Ans. $x^2 + y^2 = a^2$.

3. Find the singular solution of $x^2p^2 - 3xyp + 2y^2 + x^3 = 0$.

Ans. $x^2(y^2 - 4x^3) = 0$.

317. The Discriminant Equation.—The discriminant of a function $F(x)$ is the simplest equation between the coefficients or constants in $F(x)$ which expresses the condition that F has a double root. If F has two equal roots, equal to a, then

$$F(x) = (x - a)^2\phi(x),$$

where ϕ is some function which does not vanish when $x = a$. Hence, differentiating and putting $x = a$, we have the conditions for a double root at a,

$$F(a) = 0, \quad F'(a) = 0, \quad F''(a) \neq 0.$$

Eliminating a between $F(a) = 0$, $F'(a) = 0$, or, what is the same thing, eliminating x between $F(x) = 0$, $F'(x) = 0$, we obtain the discriminant relation between the coefficients, the condition that $F(x)$ shall have a double root.

318. c-discriminant and p-discriminant.

Let $\phi(x, y, c) = 0$ be the general solution of the differential equation $f(x, y, p) = 0$.

(1). The equation $\psi(x, y) = 0$ which results from the elimination of c between the equations

$$\phi(x, y, c) = 0 \quad \text{and} \quad \phi'_c(x, y, c) = 0$$

is called the c-discriminant, and expresses the condition that the equation $\phi = 0$, in c, shall have equal roots.

(2). The equation $\chi(x, y) = 0$ which results from the elimination of p between the equations

$$f(x, y, p) = 0 \quad \text{and} \quad f'_p(x, y, p) = 0$$

is called the p-discriminant. It expresses the condition that the equation $f = 0$, in p, shall have equal roots.

319. c-discriminant contains Envelope, Node-locus, Cusp-locus.—The c-discriminant is the locus of the ultimate intersections of consecutive curves of the family $\phi(x, y, c) = 0$.

It has been previously shown that the envelope of the family is part of this locus, and also that the envelope is tangent to each member of the family.

Suppose the curves of the family have a double point, node, or cusp. Then, in case of a node, two neighboring curves of the family intersect in two points in the neighborhood of the node, which converge to the node-locus as the curves converge together. In the neighborhood of the envelope two neighboring curves intersect in general in but one point.

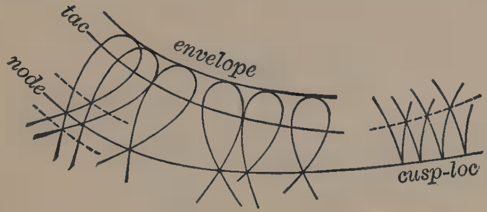

FIG. 155.

In the case of a cusp, two neighboring curves intersect, in general, in three points in the neighborhood of the cusp-locus. Two of these points may be imaginary.

We may expect to find the envelope occurring once, the node-locus twice, the cusp-locus three times as factors in the c-discriminant.

320. p-discriminant contains Envelope, Cusp-Locus, Tac-Locus.—If the curve family $f(x, y, p) = 0$ has a cusp, then for points along the cusp-locus the equation vanishes for two equal values of p, as it does also for points along the envelope. But, in general, the $\frac{dy}{dx}$ of the cusp-locus is not the same as the p of the curve family and therefore does not satisfy the differential equation.

Again, at a point at which non-consecutive members of the curve family $\phi = 0$ are tangent the x, y, p of the point satisfies the equation $f = 0$. The locus of such points is called the *tac-locus*. The $\frac{dy}{dx}$ of the tac-locus is not the same as that of the curve family $\phi = 0$, and the tac-locus therefore is not a solution of $f = 0$.

321. It has been shown by Professor Hill (Proc. Lond. Math. Soc., Vol. XIX, pp. 561) that, in general, the

$$c\text{-discriminant contains} \begin{cases} \text{the envelope} & \textit{once}, \\ \text{the node-locus} & \textit{twice}, \\ \text{the cusp-locus} & \textit{three times}, \end{cases}$$

$$p\text{-discriminant contains} \begin{cases} \text{the envelope} & \textit{once}, \\ \text{the cusp-locus} & \textit{once}, \\ \text{the tac-locus} & \textit{twice}, \end{cases}$$

as a factor. This serves to distinguish these loci. Of these, in general, the envelope alone is a solution of the differential equation. It may be that the node- or cusp-locus coincides with the envelope, and thus appears as a singular solution.* The subject is altogether too abstruse for analytical treatment here.

EXAMPLES.

1. $xp^2 - (x - a)^2 = 0$ has the general solution

$$y + c = \tfrac{2}{3}x^{\frac{3}{2}} - 2ax^{\frac{1}{2}},$$

or

$$9(y + c)^2 = 4x(x - 3a)^2.$$

The p-discriminant condition is $x(x - a)^2 = 0$, the c-discriminant condition is $x(x - 3a)^2 = 0$. $x = 0$ occurs once in each, it also satisfies the differential equation and is the singular solution or envelope. $x = a$ occurs twice in the p-discriminant and does not occur in the c discriminant. $x = a$ is therefore the tac-locus. $x = 3a$ occurs twice in the c- and does not occur in the p-discriminant. $x = 3a$ is therefore a node locus.

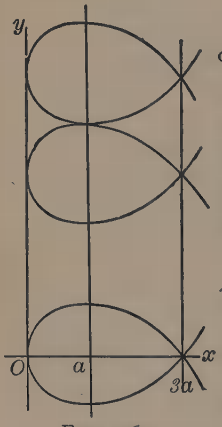

FIG. 156.

2. Show that $(y + c)^2 = x^3$ is the general solution of $4p^2 = 9x$, and $x^3 = 0$ is a cusp-locus. There is no singular solution.

3. Solve and investigate the discriminants in

$$p^2 + 2xp = y.$$

General solution $(2x^3 + 3xy + c)^2 = 4(x^2 + y)^3$. No singular solution. Cusp-locus $x^2 + y = 0$.

4. In $8ap^3 = 27y$, show that the general solution is $ay^2 = (x - c)^3$, singular solution $y = 0$, cusp-locus $y^3 = 0$.

5. Find the general and singular solution of $y = xp - p^2$.

Ans. $y = cx - c^2$, $x^2 = 4y$.

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* Proc. Lond. Math. Soc., Vol. XXII, p. 216. Prof. M. J. M. Hill, "*On node- and cusp-loci which are also envelopes.*"

EXERCISES.

Find the general solutions of the following equations.

1. $p^2 = ax^3$. *Ans.* $25(y + c)^2 = 4ax^5$.

2. $p^3 = ax^4$. *Ans.* $343(y + c)^3 = 27ax^7$.

3. $p^3(x + 2y) + 3p^2(x + y) + p(y + 2x) = 0$. Factor and solve.
Ans. $y = c, \quad x + y = c, \quad xy + x^2 + y^2 = c$.

4. $p^2 - 7p + 12 = 0$. *Ans.* $y = 4x + c. \quad y = 3x + c$.

5. $xp^2 - 2yp + ax = 0$. *Ans.* $2cy = c^2x^2 + a$.

6. $yp^2 + 2xp = y$. *Ans.* $y^2 = 2cx + c^2$.

7. $x^2p^2 - 2xyp + 2y^2 - x^2 = 0$. *Ans.* $\sin^{-1}\frac{y}{x} = \log cx$.

8. $y = p(x - b) + \frac{a}{p}$. *Ans.* $y = c(x - b) + \frac{a}{c}$.

9. $xy^2(p^2 + 2) = 2py^3 + x^3$. *Ans.* $(x^2 - y^2 + c)(x^2 - y^2 + cx^4) = 0$.

10. $y + px = x^4p^2$. *Ans.* $xy = c + c^2x$.

11. $ayp^2 + (2x - b)p - y = 0$. *Ans.* $ac^2 + c(2x - b) - y^2 = 0$.

12. $y - px = \sqrt{1 + p^2}\, f(x^2 + y^2)$. Change to polar coordinates.

$$Ans.\ \theta + c = \int \frac{f(\rho^2)d\rho}{\rho\sqrt{\rho^2 - \{f(\rho^2)\}^2}}.$$

13. $(xp - y)^2 = a(1 + p^2)(x^2 + y^2)^{\frac{3}{2}}$.
Ans. $\tan^{-1}\frac{y}{x} + c = \text{vers}^{-1} 2a\sqrt{x^2 + y^2}$.

14. $(xp - y)^2 = p^2 - 2\frac{y}{x}p + 1$. *Ans.* $\sin^{-1}\frac{y}{x} = \sec^{-1}x + c$.

15. $3p^2y^2 - 2xyp + 4y^2 - x^2 = 0$. Put $x^2 - 3y^2 = v^2$.
Ans. $3(x^2 + y^2) \pm 4cx + c^2 = 0$.

16. $(x^2 + y^2)(1 + p)^2 - 2(x + y)(1 + p)(x + yp) + (x + yp)^2 = 0$.
Ans. $x^2 + y^2 - 2c(x + y) + \frac{1}{4}c^2 = 0$.

17. $x + \frac{p}{\sqrt{1 + p^2}} = a$. *Ans.* $(y + c)^2 + (x - a)^2 = 1$.

18. $y = px + p - p^3$. *Ans.* $y = cx + c - c^3$.

19. $y^2 - 2pxy - 1 = p^2(1 - x^2)$. *Ans.* $(y - cx)^2 = 1 + c^2$.

20. $y = 2px + y^2p^3$. Put $y^2 = z$. *Ans.* $y^2 = cx + \frac{1}{8}c^3$.

21. $x^2(y - px) = yp^2$. *Ans.* $y^2 = cx^2 + c^2$.

22. $(px - y)(py + x) = h^2p$. *Ans.* $y^2 - cx^2 = -\frac{ch^2}{c + 1}$.

23. $y = xp + \sqrt{b^2 + a^2p^2}$. *Ans.* $y = cx + \sqrt{b^2 + a^2c^2}$,
singular solution $x^2/a^2 + y^2/b^2 = 1$.

24. $y = p(x - b) + a/p$, singular solution, $y^2 = 4a(x - b)$.

25. $(y - xp)(mp - n) = mnp$. *Ans.* $(y - cx)(mc - n) = mnc$,
singular solution, $(x/m)^{\frac{1}{2}} \pm (y/n)^{\frac{1}{2}} = 1$.

26. $y^2 - 2xyp + (1 + x^2)p^2 = 1$. *Ans.* $(y - cx)^2 = 1 - c^2$,
singular solution, $y^2 - x^2 = 1$.

27. $p^3 - 4xyp + 8y^2 = 0.$ *Ans.* $y = c(x - c)^2,$
singular solution, $27y = 4x^3.$

28. Find the orthogonal trajectories, λ being the variable parameter, of the following curve families:

(1). $\dfrac{x^2}{a^2} + \dfrac{y^2}{\lambda^2} = 1.$ *Ans.* $x^2 + y^2 = a^2 \log x^2 + c.$

(2). $x^2 + m^2y^2 = m^2\lambda^2.$ *Ans.* $y = cx^{m^2}.$

(3). $\dfrac{x^2}{a^2 + \lambda^2} + \dfrac{y^2}{\lambda^2} = 1.$ *Ans.* $\dfrac{x^2}{a^2 - c^2} - \dfrac{y^2}{c^2} = 1.$

29. Find the orthogonal trajectories of the circles which pass through two fixed points. *Ans.* A system of circles.

30. Find the orthogonal trajectories of the parabolæ of the nth degree
$a^{n-1}y = x^n.$ *Ans.* $ny^2 + x^2 = c^2.$

31. Find the orthogonal trajectories of the confocal and coaxial parabolæ
$y^2 = 4\lambda(x + \lambda).$ *Ans.* Self-orthogonal.

32. Find the ortho-trajectories of the ellipses $x^2/a^2 + y^2/b^2 = \lambda^2.$
Ans. $y^{b^2} = cx^{a^2}.$

33. Show that if

$$f\left(\rho,\ \theta,\ \frac{d\rho}{d\theta}\right) = 0$$

is the differential equation of the family of polar curves $\phi(\rho, \theta, c) = 0$, then

$$f\left(\rho,\ \theta,\ -\rho^2\frac{d\theta}{d\rho}\right) = 0$$

is the differential equation of the orthogonal system.

34. Find the orthogonal trajectories of $\rho = a(1 - \cos\theta).$
Ans. $\rho = c(1 + \cos\theta).$

35. Also the ortho-trajectories of—

(1). $\rho^n \sin n\theta = a^n.$ *Ans.* $\rho^n \cos n\theta = c^n.$
(2). $\rho = \log\tan\theta + a.$ *Ans.* $2/\rho = \sin^2\theta + c.$

CHAPTER XL.

EXAMPLES OF EQUATIONS OF THE SECOND ORDER AND FIRST DEGREE.

322. The differential equation of the second order and first degree is an equation in x, y, p, q,

$$f(x, y, p, q) = 0,$$

where $p \equiv \frac{dy}{dx}$, $q \equiv \frac{dp}{dx} = \frac{d^2y}{dx^2}$, and in the equation q occurs only in the first degree.

We shall attempt the solution of the equation for only a few of the simplest cases.

We have seen that the general solution of the equation of the first order and degree gave rise to a singly infinite number of solutions, represented by a family of curves having a single arbitrary parameter, this parameter being the constant of integration.

In like manner, the general solution of the equation of the second order and first degree, involving two successive integrations, requires at each integration the introduction of an arbitrary constant. The general solution, therefore, contains two arbitrary parameters, and is correspondingly represented by a doubly infinite system of curves, or two families, each having its variable parameter.

The process by which a differential equation of the second order is derived from its primitive is as follows.

Let $$\phi(x, y, c_1, c_2) = 0 \tag{1}$$

be an equation in x, y and two arbitrary constants c_1, c_2. Differentiating (1) twice with respect to x, there results

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0, \tag{2}$$

$$\frac{\partial^2 \phi}{\partial x^2} + 2\frac{\partial^2 f}{\partial x\, \partial y}\frac{dy}{dx} + \frac{\partial^2 f}{\partial x^2}\left(\frac{dy}{dx}\right)^2 + \frac{\partial f}{\partial y}\frac{d^2y}{dx^2} = 0. \tag{3}$$

Between these three equations can be eliminated the two arbitrary parameters c_1, c_2. The result is the differential equation of the second order,

$$f(x, y, p, q) = 0.$$

EXAMPLE.

The simplest equation of the second order is

$$\frac{d^2y}{dx^2} = k.$$

Here the integrations are immediately effected.

$$d\left(\frac{dy}{dx}\right) = k\,dx.$$

$$\therefore \quad \frac{dy}{dx} = kx + c_1,$$

c_1 being the first constant of integration. Integrating again, the general solution is

$$y = \tfrac{1}{2}kx^2 + c_1x + c_2.$$

The two arbitrary parameters c_1, c_2 giving a doubly infinite system of parabolæ.

323. The Five Degenerate Forms.—The ordinary processes of integrating differential equations are of tentative character. We are led to the solution of general forms through the consideration of the simpler cases. Investigation of the general methods of treating this subject is out of place in this text, and we shall consider here only a few interesting and important equations of simple form.

A general method of solution can be proposed for the five degenerate forms of the general equation,

$$1.\ f(x, q) = 0; \qquad 2.\ f(y, q) = 0; \qquad 3.\ f(p, q) = 0;$$
$$4.\ f(x, p, q) = 0; \qquad 5.\ f(y, p, q) = 0.$$

324. Form $f(x, q) = 0$.—This being of the first degree in q,

$$\frac{d^2y}{dx^2} = F(x).$$

The differentials involved are exact, and it is only a question of integrating twice. The solution is

$$\frac{dy}{dx} = \int F(x)dx + c_1.$$

$$\therefore \quad y = \int dx \int F(x)dx + c_1x + c_2.$$

Ex. $q = xe^x$. *Ans.* $y = (x - 2)e^x + c_1x + c_2$.

325. Form $f(y, q) = 0$.—Here

$$\frac{d^2y}{dx^2} = F(y).$$

Put $\frac{dy}{dx} = p$. Then $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dy}{dx}\frac{dp}{dy} = p\frac{dp}{dy}$.

The equation becomes

$$p\,dp = F(y)\,dy.$$

Integrating,

$$\left(\frac{dy}{dx}\right)^2 = 2\int F(y)\,dy + c_1.$$

$$\therefore \quad dx = \frac{dy}{\left\{2\int F(y)\,dy + c_1\right\}^{\frac{1}{2}}}.$$

The integral of this gives the solution.

EXAMPLES.

1. Solve $\dfrac{d^2y}{dx^2} = a^2y.$

$$2\int F(y)dx = a^2y^2. \quad \text{Put} \quad c_1 = a^2c.$$

$$\therefore \quad dx = \frac{dy}{a\sqrt{y^2 + c}}.$$

Hence $$ax = \log(y + \sqrt{y^2 + c}) + c_2.$$

Show that this can be transformed into

$$y = c_1'e^{ax} + c_2'e^{-ax}.$$

Multiply the given differential equation by $2\dfrac{dy}{dx}$.

$$\therefore \quad 2\frac{dy}{dx}\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)^2, \qquad a^2 2y\frac{dy}{dx} = a^2\frac{d}{dx}(y^2).$$

Hence the first integral is, as before,

$$\left(\frac{dy}{dx}\right)^2 = a^2y^2 + k.$$

2. Solve $\dfrac{d^2y}{dx^2} + a^2y = 0.$

Here $$2\int F(y)dy = -a^2y^2. \quad \text{Put} \quad c_1 = a^2c^2.$$

$$\therefore \quad a\,dx = \frac{dy}{\sqrt{c^2 - y^2}}.$$

Hence $$ax + c_2 = \sin^{-1}\frac{y}{c},$$

or $$y = c\sin(ax + c_2),$$

$$= k_1 \sin ax + k_2 \cos ax.$$

Multiply the differential equation by $2p$ and obtain the first integral directly as in Ex. 1.

Examples 1 and 2 are important in Mechanics.

3. Solve $q\sqrt{ay} = 1.$ *Ans.* $3x = 2a^{\frac{1}{2}}(y^{\frac{1}{2}} - 2c_1)(y^{\frac{1}{2}} + c_1)^{\frac{1}{2}} + c_2.$

326. Form $f(p, q) = 0$.

Here $$\frac{d^2y}{dx^2} = F\left(\frac{dy}{dx}\right), \quad \text{or} \quad \frac{dp}{dx} = F(p).$$

$$\therefore \quad x = \int \frac{dp}{F(p)} + c_1.$$

This is an equation of the first order, the solution of which is that of the required equation.

EXAMPLES.

1. Solve $\frac{d^2y}{dx^2} + a\left(\frac{dy}{dx}\right)^2 = 0.$

Integrating $p^{-2}dp + adx,$ we have for the first integral

$$dy = \frac{dx}{ax + c}$$

$$\therefore \quad y = \log(ax + c) + c',$$

or

$$e^y = c_1x + c_2.$$

2. Solve $a\frac{d^2y}{dx^2} = \frac{dy}{dx}.$ *Ans.* $y = c_1e^{\frac{x}{a}} + c_2.$

3. $q = p^2 + 1.$ *Ans.* $e^{-y} = c_2\cos(x + c_1).$

4. $q + p^2 + 1 = 0.$ *Ans.* $y = \log\cos(x - c_1) + c_2.$

327. Form $f(x, p, q) = 0$.—Such equations are reduced to the first order in x and p by the substitution $q = \frac{dp}{dx}.$

$$\therefore \quad f(x, p, q) \equiv f\left(x, p, \frac{dp}{dx}\right) = 0.$$

EXAMPLES.

1. $(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + ax = 0$ is equivalent to

$$\frac{dp}{dx} + \frac{x}{1 + x^2}p + \frac{ax}{1 + x^2} = 0.$$

The first integral is

$$p = -a + \frac{c_1}{\sqrt{1 + x^2}}.$$

The second integration gives

$$y = c_2 - ax + c_1\log\left(x + \sqrt{1 + x^2}\right).$$

2. $(1 + x^2)q + p^2 + 1 = 0.$ *Ans.* $y = c_1x + (c_1^2 + 1)\log(x - c_1) + c_2.$

328. Form $f(y, p, q) = 0.$

$$q = \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dy}{dx}\frac{dp}{dy} = p\frac{dp}{dy}.$$

Substituting for q, the equation is reduced to the first order in y and p.

EXAMPLES.

1. $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0.$ *Ans.* $\frac{a + y}{a - y} = e^{a(x+b)}.$

2. $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1.$ *Ans.* $y^2 = x^2 + c_1x + c_2.$

3. $yq - p^2 = y^2\log y.$ *Ans.* $\log y = c_1e^x + c_2e^{-x}.$

329. Solution of the Linear Equation

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0, \tag{1}$$

in which A, B are constants.

The solution of this equation is suggested by the solution of the corresponding equation of the first order

$$\frac{dy}{dx} + ay = 0,$$

which gives $\frac{dy}{y} = -adx$, the solution of which is $y = ce^{-ax}$.

If we try $y = e^{mx}$ in (1), we have

$$\frac{d^2e^{mx}}{dx^2} + A\frac{de^{mx}}{dx} + Be^{mx} \equiv (m^2 + Am + B)e^{mx}. \tag{2}$$

I. *Roots of the Auxiliary Equation Real and Unequal.*—The function (2) vanishes if m be one of the roots of the *auxiliary* equation

$$m^2 + Am + B \equiv (m - m_1)(m - m_2) = 0. \tag{3}$$

Hence $y = e^{m_1x}$ is a solution. Also, $y = c_1e^{m_1x}$ is a solution for any arbitrary constant c_1. In like manner $y = c_2e^{m_2x}$ is a solution. The sum of these two,

$$y = c_1e^{m_1x} + c_2e^{m_2x}, \tag{4}$$

is also a solution, and is the general solution of (1) since it contains two independent arbitrary constants, c_1 and c_2.

II. *Roots of the Auxiliary Equation Real and Equal.*—If $m_1 = m_2$, the solution (4) fails to give the general solution, since then

$$y = (c_1 + c_2)e^{mx},$$

and $c_1 + c_2 = c'$ is only one arbitrary parameter.

The solution in this case is immediately discovered on differentiating (2) with respect to m. For then

$$\frac{d^2xe^{mx}}{dx^2} + A\frac{dxe^{mx}}{dx} + Bxe^{mx} \equiv (2m + A)e^{mx} + (m^2 + Am + B)xe^{mx}.$$

If $m = \mu$ is the double root of (3), then (3) and its derivative vanish when $m = \mu$. Consequently $y = xe^{\mu x}$ is a solution, and also is $y = cxe^{\mu x}$. Hence the sum of the two solutions $c'e^{\mu x}$ and $cxe^{\mu x}$ is the general solution of (1) when μ is a double root of (3), or

$$y = e^{\mu x}(c' + cx). \tag{5}$$

III. *Roots of the Auxiliary Equation Imaginary.*—When the roots of (3) are imaginary and of the forms

$$m_1 = a + ib, \quad m_2 = a - ib,$$

where $i \equiv \sqrt{-1}$, these roots may be used to find the solution. For (4) becomes

$$y = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x},$$
$$= e^{ax}(c_1 e^{ibx} + c_2 e^{-ibx}).$$

We have by Demoivre's formula

$$e^{ibx} = \cos bx + i \sin bx,$$
$$e^{-ibx} = \cos bx - i \sin bx.$$

Therefore the solution is

$$y = e^{ax}\{(c_1 + c_2) \cos bx + (c_1 - c_2)\, i \sin bx\},$$
$$= e^{ax}(k_1 \cos bx + k_2 \sin bx), \tag{6}$$

where $k_1 = c_1 + c_2$, $k_2 = (c_1 - c_2)i$. If the arbitrary constants c_1 and c_2 be assumed conjugate imaginaries, the constants k_1 and k_2 are real.

By writing $\tan \alpha = k_1/k_2$, or $\cot \beta = k_1/k_2$, the solution (6) may be written respectively

$$y = c' e^{ax} \sin(bx + \alpha),$$
$$= c'' e^{ax} \cos(bx - \beta). \tag{7}$$

EXAMPLES.

1. Solve $q - p = 2y$.

The auxiliary equation is

$$m^2 - m - 2 \equiv (m + 1)(m - 2) = 0.$$

The general solution is therefore $y = c_1 e^{-x} + c_2 e^{2x}$.

2. If $q - 2p + y = 0$, $(m - 1)^2 = 0$, $\therefore\ y = e^x(c_1 + c_2 x)$.

3. Solve $q + 3p = 54y$.

$$m^2 + 3m - 54 \equiv (m - 6)(m + 9).$$
$$\therefore\ y = c_1 e^{6x} + c_2 e^{-9x}.$$

4. Solve $q + 8p + 25y = 0$.

$$m^2 + 8m + 25 = 0 \text{ gives } m = -4 \pm 3\sqrt{-1}.$$
$$\therefore\ y = e^{-4x}(k_1 \cos 3x + k_2 \sin 3x).$$

330. Solution of the Equation

$$x^2 \frac{d^2y}{dx^2} + Ax \frac{dy}{dx} + By = 0, \tag{1}$$

A, B being constants.

Put $x = e^z$, then $z = \log x$. Also,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}; \quad \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right).$$

On substitution, equation (1) becomes

$$\frac{d^2y}{dz} + (A - 1)\frac{dy}{dz} + By = 0,$$

which is the form solved in § 329.

Ex. Solve $x^2q - xp + y = 0$.
The equation transforms into

$$\frac{d^2y}{dz^2} - 2\frac{dy}{dz} + y = 0.$$

$$\therefore \quad y = e^z(c_1 + c_2z) = x(c_1 + c_2 \log x).$$

331. Observations on the Solution of Differential Equations.

—The remarks made on the integration of functions are equally applicable to the integration of differential equations. The process is of tentative character, and skill in solving equations comes through experience and familiarity with the known methods of solving the integrable forms.

When the equation is not readily recognizable as one of the standard forms for solution, it can frequently be transformed into a recognizable form by substitution of a new variable.

Most of the processes given in this chapter for the solution of certain forms of the equation of the second order and first degree are immediately applicable to equations of higher orders. In the exercises will be found certain simple equations of higher order than the second, proposed for solution by the methods exposed in the text.

General methods of solving differential equations must be reserved for monographs on the Theory of Differential Equations.

EXERCISES.

1. $\frac{d^2y}{dx^2} = a^2x + b^2y$. Put $a^2x + b^2y = z$, etc.

Ans. $a^2x + b^2y = c_1e^{bx} + c_2e^{-bx}$.

2. $\frac{d^2y}{dx^2} = a^2x - b^2y$. *Ans.* $a^2x - b^2y = c_1 \sin bx + c_2 \cos bx$.

3. $q = e^y$. *Ans.* $c_2e^{c_1x} = \frac{\sqrt{2e^y + c_1^2} - c_1}{\sqrt{2e^y + c_1^2} + c_1}$, $e^{\frac{1}{2}y} = \frac{\sqrt{2}}{c_2 - x}$,

or $2e^y = c_1^2 \sec^2(\frac{1}{2}c_1x + c^2)$, according as the first constant of integration is

$+c_1^2$, 0, or $-c_1^2$.

4. $xq + p = 0$. *Ans.* $y = c_1 \log x + c_2$.

5. $q = xp$. *Ans.* $y = c_1\int e^{\frac{1}{2}x^2}dx + c_2$.

6. $x^2q = 2y$. Put $z = 2y/x^2$. $\therefore$ $xy = c_2x^3 + c_1$.

7. $q + 12y = 7p$. *Ans.* $y = c_1e^{3x} + c_2e^{4x}$.

8. $3(q + y) = 10p$. *Ans.* $y = c_1e^{3x} + c_2e^{\frac{1}{3}x}$.

9. $q + 4p = y$. *Ans.* $ye^{2x} = c_1e^{x\sqrt{5}} + c_2e^{-x\sqrt{5}}$.

10. $ab(y + q) = (a^2 + b^2)p$. *Ans.* $y = c_1e^{\frac{ax}{b}} + c_2e^{\frac{bx}{a}}$.

11. $\frac{d^3y}{dx^3} = 4\frac{dy}{dx}$. *Ans.* $y = c_1e^{2x} + c_2e^{-2x} + c_3$.

12. $q - 6p + 13y = 0.$ *Ans.* $y = (c_1 \sin 2x + c_2 \cos 2x)e^{3x}.$

13. $q - 2ap + b^2y = 0.$ *Ans.* According as $a >$ or $< b$,

$y = e^{ax}(c_1 e^{x\sqrt{a^2-b^2}} + c_2 e^{-x\sqrt{a^2-b^2}})$, or $e^{ax}(c_1 \sin x\sqrt{b^2 - a^2} + c_2 \cos x\sqrt{b^2 - a^2})$.

14. $q - 4abp + (a^2 + b^2)^2 y = 0.$

Ans. $y = e^{2abx}\{c_1 \sin (a^2 - b^2)x + c_2 \cos(a^2 - b^2)x\}.$

15. $q - p \log a^2 + [1 + (\log a)^2]y = 0.$ *Ans.* $y = a^x(c_1 \sin x + c_2 \cos x).$

16. $q - 2ap + a^2y = 0.$ *Ans.* $y = e^{ax}(c_1 + c_2x).$

17. $q = 0.$ *Ans.* $y = c_1 + c_2x.$

18. $\dfrac{d^3y}{dx^3} = 4q.$ *Ans.* $y = c_1e^{4x} + c_2 + c_3x.$

19. $x^2q - xp = 3y.$ *Ans.* $xy = c_1x^4 + c_2.$

20. $(a + bx)^2q + b(a + bx)p + b^2y = 0.$

Ans. $y = c_1 \sin \log (a + bx) + c_2 \cos \log (a + bx).$

21. $x\dfrac{d^3y}{dx^3} = 2.$ *Ans.* $y = c_1 + c_2x + c_3x^2 + x^2 \log x.$

22. $\dfrac{d^3y}{dx^3} = \sin^3 x.$ *Ans.* $y = c_1 + c_2x + c_3x^2 + \frac{7}{9}\cos x - \frac{1}{27}\cos^3 x.$

23. $qy^3 = a.$ *Ans.* $(c_1x + c_2)^2 = c_1y^2 - a.$

24. $a^2q^2 = 1 + p^2.$ *Ans.* $2y/a = c_1e^{\frac{x}{a}} + c_1^{-1}e^{-\frac{x}{a}} + c_2.$

25. $a^2q^2 = (1 + p^2)^3.$ *Ans.* $(x + c_1)^2 + (y + c_2)^2 = a^2.$

26. $(1 - x^2)q - xp = 2.$ *Ans.* $y = c_1 \sin^{-1}x + (\sin^{-1}x)^2 + c_2.$

27. $yq + p^2 = 1.$ *Ans.* $y^2 = x^2 + c_1x + c_2.$

28. $(1 - \log y)yq + (1 + \log y)p^2 = 0.$ *Ans.* $(c_1x + c_2)(\log y - 1) = 1.$

29. $yq - p^2 = y^2 \log y.$ *Ans.* $\log y = c_1e^x + c_2e^{-x}.$

30. $(p - xq)^2 = 1 + q^2.$ *Ans.* $y = \frac{1}{2}c_1x^2 + x\sqrt{1 + c_1{}^2} + c^2.$

31. Find the curve in which the normal is equal and opposite to the radius of curvature. [Catenary.]

32. Find the curve in which the normal is equal to the radius of curvature and in the same direction.

33. Find the curve in which the radius of curvature is twice the normal and opposite to it. The parabola, $x^2 = 4c(y - c).$

34. Determine the curve in which the normal is one half the radius of curvature, and in the same direction.

The cycloid $x + c \sin^{-1}\dfrac{c - y}{c} + \sqrt{2cy - y^2} = 0.$

35. Find the locus of the focus of the parabola $y^2 = 4ax$ as the parabola rolls on a straight line. [Catenary.]

36. Find the locus of a point on a circle as it rolls on a straight line.

37. Express the locus of the center of an ellipse as it rolls on a straight line in terms of an elliptic integral.

38. The problem of *curves of pursuit* was first presented in the form: To find the path described by a dog which runs to overtake its master.

The point A describes a straight line with uniform velocity; it is required to find the curve described by the point B, the motion of which is always directed toward A and the velocity uniform.

Take the path of A for y-axis. The tangent intercept on the y-axis is $y - xp$. By hypothesis the change of this is proportional to the change of arc-length.

$$\therefore \quad -x\,dp = m\sqrt{1 + p^2}\,dx,$$

$$\log x^m + \log\left(p + \sqrt{1 + p^2}\right) + \log c_1 = 0,$$

$$2p = c_1^{-1}x^{-m} - c_1x^m,$$

$$2y = c_2 - c_1\frac{x^{m+1}}{m + 1} - c_1^{-1}\frac{x^{-m+1}}{m - 1}.$$

The curve is algebraic, except when $m = 1$, then we have to substitute $\log x$ for $-x^{-m+1}/(m - 1)$.

APPENDIX.

SUPPLEMENTARY NOTES.

APPENDIX.

NOTE 1.

Supplementing § 30.

Weierstrass's Example of a Continuous Function which has nowhere a Determinate Derivative.*

The function

$$f(x) \equiv \sum_{0}^{\infty} b^n \cos(a^n \pi x),$$

in which x is real, a an odd positive integer, b a positive constant less than 1, is a continuous function which has for no value of x a determinate derivative, if $ab > 1 + \frac{3}{2}\pi$.

Whatever assigned value x may have, we can always assign an integer μ corresponding to an arbitrarily chosen integer m, for which

$$-\tfrac{1}{2} < a^m x - \mu \leqq +\tfrac{1}{2}.$$

Put $\quad x_{m+1} = a^m x - \mu, \quad$ and let

$$x' = \frac{\mu - 1}{a^m}, \quad x'' = \frac{\mu + 1}{a^m}.$$

$$\therefore \quad x' - x = -\frac{1 + x_{m+1}}{a^m}, \quad x'' - x = \frac{1 - x_{m+1}}{a^m},$$

and

$$x' < x < x''.$$

The integer m can be chosen so great that x' and x'' shall differ from x by as small a number as we choose.

We have

$$\frac{f(x') - f(x)}{x' - x} = \sum_{n=0}^{\infty} b^n \frac{\cos(a^n \pi x') - \cos(a^n \pi x)}{x' - x},$$

$$= \sum_{n=0}^{m-1} (ab)^n \frac{\cos(a^n \pi x') - \cos(a^n \pi x)}{a^n(x' - x)}$$

$$+ \sum_{n=0}^{\infty} b^{m+n} \frac{\cos(a^{m+n} \pi x') - \cos(a^{m+n} \pi x)}{x' - x}. \qquad \text{(i)}$$

---

* Taken from Harkness and Morley, Theory of Functions.

Since

$$\frac{\cos(a^n\pi x') - \cos(a^n\pi x)}{a^n(x'-x)} = -\pi\sin\left(a^n\frac{x'+x}{2}\pi\right)\frac{\sin\left(a^n\dfrac{x'-x}{2}\pi\right)}{a^n\dfrac{x'-x}{2}\pi},$$

and since the absolute value of the last factor on the right is less than 1, then the absolute value of the first part of (i) is less than

$$\pi\sum_{0}^{m-1}(ab)^n = \pi\frac{(ab)^m - 1}{ab-1},$$

and therefore less than $\dfrac{\pi(ab)^m}{ab-1}$, if $ab > 1$.

Also, since a is an odd integer,

$$\cos(a^{m+n}\pi x') = \cos[a^n(\mu - 1)\pi] = -(-1)^\mu,$$

$$\cos(a^{m+n}\pi x) = \cos(a^n\mu\pi + a^n x_{m+1}\pi) = (-1)^\mu\cos(a^n x_{m+1}\pi).$$

Therefore

$$\sum_{n=0}^{\infty} b^{m+n}\frac{\cos(a^{m+n}\pi x') - \cos(a^{m+n}\pi x)}{x'-x}$$

$$= (-1)^\mu(ab)^m\sum_{n=0}^{\infty} b^n\frac{1+\cos(a^n x_{m+1}\pi)}{1+x_{m+1}}.$$

All the terms under the Σ on the right are positive, and the first is not less than $\frac{2}{3}$, since $\cos(x_{m+1}\pi)$ is not negative and $1 + x_{m+1}$ lies between $\frac{1}{2}$ and $\frac{3}{2}$.

Consequently

$$\frac{f(x') - f(x)}{x'-x} = (-1)^\mu(ab)^m\xi\left(\frac{2}{3} + \frac{\pi\eta}{ab-1}\right), \qquad \text{(ii)}$$

where ξ is an absolute number > 1, and η lies between -1 and $+1$.

In like manner

$$\frac{f(x'') - f(x)}{x''-x} = -(-1)^\mu(ab)^m\,\xi'\left(\frac{2}{3} + \frac{\pi\eta'}{ab-1}\right), \qquad \text{(iii)}$$

where ξ' is a positive number > 1, and η' lies between -1 and $+1$.

If ab be so chosen as to make

$$ab > 1 + \tfrac{3}{2}\pi,$$

that is,

$$\frac{2}{3} > \frac{\pi}{ab-1},$$

the two difference-quotients have always opposite signs, and both are infinitely great when m increases without limit. Hence $f(x)$ has neither a determinite finite nor determinate infinite derivative.

Every point on such a line, if line it could be called, is a singular point.

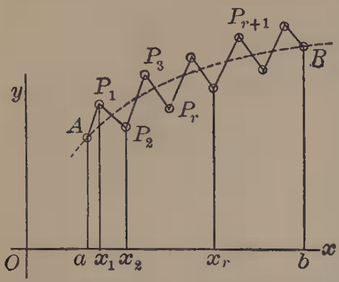

FIG. 157.

Some idea of the character of the geometrical assemblage of points representing such a function can be obtained by selecting two particular fixed points A, B of the assemblage. Between A and B, in progressive order, select points $P_1, P_2, \ldots$ representing the function corresponding to $x_1, x_2, \ldots$ Consider the polygonal line $AP_1P_2 \ldots B$. Increase the number of interpolated points indefinitely, and at the same time let the difference between each consecutive pair converge to 0. Then, since the function $f(x)$ is continuous, each side, P_rP_{r+1}, of the broken line converges to 0. But, instead of each angle between consecutive pairs of sides of this polygonal line converging to two right angles, π, as their lengths diminish indefinitely, as was the case when we defined a curve with definite direction at each point; let now these angles converge alternately to 0 and 2π. The polygonal line folds up in a *zigzag*. The point P converging to the neighborhood of a true curve AB. But the difference-quotient at any point of the zigzag assemblage has no limit, it becomes wholly indeterminate as the two values of the variable converge together. It is also possible that the length representing the sum of the sides of the polygonal between any two points of the assemblage at a finite distance apart (however small) is infinite in the limit.

Such functions are but little understood and have been but little studied. It is possible that they may have in the future far-reaching importance in the study of molecular physics, wherein it becomes necessary to study vibrations of great velocity and small oscillation.

NOTE 2.

Supplementary to § 42.

Geometrical Picture of a Function of a Function.

If $z = f(y)$, where $y = \phi(x)$, we can represent the function z geometrically as follows:

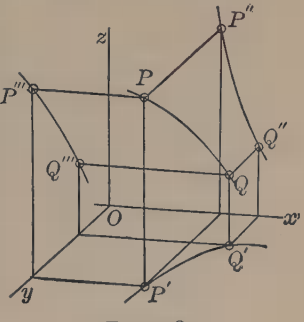

FIG. 158.

Draw through any fixed point O in space three straight lines Ox, Oy, Oz mutually at right angles, so that Ox, Oy are horizontal and Oz is vertical. These lines fix three planes at right angles to each other. xOy is horizontal, xOz and yOz are vertical.

The relation $y = \phi(x)$ can be represented by a curve $P'Q'$ in the plane xOy. At any point P' on this curve we can represent z by drawing $P'P = f(y)$, up if $f(y)$ is positive, down if $f(y)$ is negative. The relation $z = f(y)$ is represented by the curve $P'''Q'''$ in yOz.

$$z = f\{\phi(x)\},$$

as a function of x, is represented by the curve $P''Q''$ in xOz. In other words, z as a function of x and y is represented by a point in space having the corresponding values z, y, x as coordinates with respect to the three planes. The assemblage of points representing z, y, x is a space curve PQ. The orthogonal projections on the three coordinate planes of PQ represent the functional relations

$$(P'Q'),\quad y = \phi(x);\quad (P''Q''),\quad z = f\{\phi(x)\};\quad (P'''Q''),\quad z = f(y).$$

The derivative $D_x y$ is represented by the slope of $P'Q'$ at P' to Ox. The derivative $D_y z$ is represented by the slope of the tangent to $P'''Q'''$ at P''' to Oy; the derivative $D_x z$ by the slope to the axis Ox of the tangent at P'' to $P''Q''$.

The function of a function is represented by a curve in space.

NOTE 3.

Supplementary to § 56.

The *n*th Derivative of the Quotient of Two Functions.

Let $y = u/v$. Then $u = vy$. Applying Leibnitz's formula to this product, we have

$$u = vy,$$

$$\frac{u'}{1!} = \frac{v'}{1!}y + v\frac{y'}{1!},$$

$$\frac{u''}{2!} = \frac{v''}{2!}y + \frac{v'}{1!}\frac{y'}{1!} = v\frac{y''}{2!},$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\frac{u^n}{n!} = \frac{v^n}{n!}y + \frac{v^{n-1}}{(n-1)!}\frac{y'}{1!} + \frac{v^{n-2}}{(n-2)!}\frac{y''}{2!} + \ldots + v\frac{y^n}{n!}.$$

To find y^n, the nth derivative of u/v, in terms of the derivatives of u and v. Eliminate $y, \frac{y'}{1!}, \ldots, \frac{y^{n-1}}{(n-1)!}$ from the $n + 1$ equations. We get

$$\frac{1}{n!}D^n\left(\frac{u}{v}\right) = \frac{(-1)^n}{(v)^{n+1}}\left|\begin{array}{ccccc} u & v & 0 & 0 & \ldots \\ \frac{u'}{1!} & \frac{v'}{1!} & v & 0 & \ldots \\ \frac{u''}{2!} & \frac{v''}{2!} & \frac{v'}{1!} & v & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \multicolumn{5}{c}{(n+1)\text{ rows}} \end{array}\right|$$

Also, in particular, if $u \equiv 1$, we have

$$\frac{1}{n!}D^n\left(\frac{1}{v}\right) = \frac{(-1)^n}{(v)^{n+1}} \begin{vmatrix} \frac{v'}{1!} & v & 0 & 0 & \dots \\ \frac{v''}{2!} & \frac{v'}{1!} & v & 0 & \dots \\ \frac{v'''}{3!} & \frac{v''}{2!} & \frac{v'}{1!} & v & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & n \text{ rows} & & \end{vmatrix}$$

NOTE 4.

Supplementary to § 56.

To Find an Expression for the nth Derivative of a Function of a Function.

Let $z = f(y)$, where $y = \phi(x)$. To find the nth derivative of z with respect to x.

We have, by actual differentiation,

$$f'_x = f'_y y'_x,$$
$$f''_x = f''_y y'^2_x + f'_y y''_x,$$
$$f'''_x = f'''_y y'^3_x - 3 f''_y y'_x y''_x + f'_y y'''_x.$$

The law of formation of these first three derivatives of f with respect to x shows that the nth derivative must be of the form

$$f^n_x = A_1 f'_y + A_2 f''_y + \dots + A_n f^n_y, \tag{1}$$

where the coefficients, A_r, contain only derivatives of y with respect to x and are therefore *independent* of the *form* of the function f. Consequently, if we determine A_r for any particular function f, we have determined the coefficients whatever be the function f. Let then

$$f(y) \equiv \frac{(y-b)^r}{r!}.$$

Then in (1) we have

$$\frac{1}{r!} D^n_x (y-b)^r = \frac{(y-b)^{r-1}}{(r-1)!} A_1 + \dots + \frac{y-b}{1!} A_{r-1} + A_r. \tag{2}$$

Hence, when $b = y$, we have

$$A_r = \frac{1}{r!} \left\{ D^n_x (y-b)^r \right\}_{b=y},$$

which means that $(y-b)^r$ is to be differentiated n times with respect to x and in the result y substituted for b.

$$\therefore \quad \left(\frac{d}{dx}\right)^n f(y) = \sum_{r=1}^{n} \frac{f^r_y(y)}{r!} \left\{ D^n_x (y-b)^r \right\}_{b=y}. \tag{3}$$

This gives the nth derivative of f with respect to x in terms of the derivatives of f with respect to y and those of y with respect to x, and is the generalization of the formula

$$\frac{d}{dx}f(y) = \frac{df(y)}{dy}\,\frac{dy}{dx}.$$

We can give another form to (3), as follows. Let $y = b$ when $x = a$. Then

$$y - b = \phi(x) - \phi(a) = (x - a)v, \qquad (4)$$

where v stands for the difference-quotient

$$\frac{\phi(x) - \phi(a)}{x - a}.$$

Apply Leibnitz's Formula to (4), and we have

$$D_x^n(y - b)^r = D_x^n(x - a)^r v^r,$$

$$= \sum_{p=0}^{n} C_{n,p}\, D_x^{n-p} v^r D_x^p (x - a)^r.$$

But, $D_x^p(x - a)^r = r(r - 1)\ldots(r - p + 1)(x - a)^{r-p}$,

$= 0$ when $p > r$,

$= 0$ when $p < r$ and $x = a$.

$= r!$ when $p = r$ and $x = a$.

Therefore (3) becomes

$$\left(\frac{d}{dx}\right)^n f(y) = \sum_{r=1}^{n} C_{n,r}\, f_y^r(y) \left(\frac{d}{dx}\right)^{n-r} \left(\frac{\phi(x) - \phi(a)}{x - a}\right)^r_{a=x}. \qquad (5)$$

Notes 3 and 4 give some idea of the complicated forms which the higher derivatives of functions assume.

NOTE 5.

§ 64. Footnote.

If a function $f(x)$ and its derivatives are continuous for all values of x in (α, β) except for a particular value a of x at which $f(a) = \infty$, then all the derivatives of $f(x)$ are infinite at a.

Let $x_1 < x_2 < a$. Then

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(\xi),$$

where ξ lies between x_2 and x_1. Let $a - x_1$ be a small but finite number, and let $x_2 (=) a$. Then $f(x_2)$ is infinite, and $f(x_1)$ is finite.

$$\therefore\quad (a - x_1)f'(\xi) = \infty.$$

Since $a - x_1$ is finite, $f'(\xi) = \infty$; and since $f'(\xi)$ is finite if $a - \xi$ is finite, we must have $a - \xi (=) 0$ and

$$f'(a) = \infty.$$

In like manner we show that $f''(a) = \infty$, and so on.

Corollary. If $f(a) = \infty$, then $f'(a) = \infty$, and also

$$\frac{f'(x)}{f(x)}$$

becomes ∞ when $x = a$.

For, considering absolute values, if $f\ (a) = \infty$, then also $\log f(a) = \infty$. By the theorem established above, if $\log f(x)$ is ∞ when $x = a$, then

$$D \log f(x) = \frac{f'(x)}{f(x)}$$

also becomes ∞ when $x = a$.

NOTE 6.

Supplementary to Chapter VI

On the Expansion of Functions by Taylor's Series.

1. This subject cannot be satisfactorily treated except by the Theory of Functions of a Complex Variable. The present note is an effort to present in an elementary manner by the methods of the Differential Calculus a fundamental theorem regarding the elementary functions.

An elementary function may be defined to be one which does not become 0 or ∞ an infinite number of times in any finite interval, however small. Such functions are also called rational.

A function $f(x)$ is said to be unlimitedly differentiable at x when all the derivatives $f^r(x)$ of finite order are finite and determinate at x. We consider only those functions which are such that neither the function nor any of its derivatives become 0 or ∞ an unlimited number of times in the neighborhood of any value x considered.

2. In the same way that a function of the real variable x may be 0 for an imaginary number $p + iq$, such a function may be ∞ for a complex number $p + iq$, where $i \equiv \sqrt{-1}$. For example, the function

$$F(x) \equiv \frac{\phi(x)}{\psi(x)}$$

becomes ∞ at $p + iq$ if $p + iq$ is a root of $\psi(x)$ and not of $\phi(x)$. A value of x at which $F(x)$ is 0 or ∞ is called a *root* or *pole*, respectively, of the function. It being understood that there are not an indefinite number of roots or poles in the same neighborhood.*

---

* A point in whose neighborhood there are an infinite number of poles is called an *essential singularity*. An isolated pole is called a *non-essential singularity*.

The poles of a function, whether imaginary or real, enter into the results which we shall obtain. Wherever we use the word function in this note we mean a uniform function which has only roots and poles, *but no essential singularity*, and which is unlimitedly differentiable everywhere except at a pole.

3. Theorem I.—If $f(x)$ is a one-valued, determinate, and unlimitedly differentiable function at x, then the series

$$S = \sum_{0}^{\infty} \frac{y^r}{r!} f^r(x)$$

is absolutely convergent for all values of y less in absolute value than

$$R = +\sqrt{(x-p)^2 + q^2},$$

where $p + iq$ is the nearest pole of $f(x)$, or any of its derivatives $f^r(x)$, to the number x; and the series S is ∞ for any value of y greater than R.

4. Represent x, y by the coordinates of a point in a plane xOy. Then (see § 15, Ex. 9, 10):

(1). At all points x, y at which

$$\underset{n=\infty}{\mathcal{L}} \frac{y}{n+1} \frac{f^{n+1}(x)}{f^n(x)} \,| < |\, 1,$$

S is absolutely convergent, and also

$$\underset{n=\infty}{\mathcal{L}} \frac{y^n}{n!} f^n(x) = 0.$$

(2). At all points x, y at which

$$\underset{n=\infty}{\mathcal{L}} \frac{y}{n+1} \frac{f^{n+1}(x)}{f^n(x)} \,| > |\, 1,$$

$S = \infty$, and also

$$\underset{n=\infty}{\mathcal{L}} \frac{y^n}{n!} f^n(x) = \infty.$$

5. It follows, therefore, that if

$$\underset{n=\infty}{\mathcal{L}} \frac{y^n}{n!} f^n(x)$$

---

* Remembering that the modulus or absolute value of any number $x + iy$ is

$$|x + iy| = +\sqrt{x^2 + y^2},$$

then of two numbers $p_1 + iq_1$ and $p_2 + iq_2$ that one is nearest x for which we have the difference

$$|p + iq - x|$$

least.

has a finite limit different from 0, it is necessary that

$$\underset{n=\infty}{\mathcal{L}} \frac{y}{n+1} \frac{f^{n+1}(x)}{f^{n}(x)} \,|=|\, 1.$$

6. Since at all points of absolute convergence of S

$$\mathcal{L} \frac{y^n}{n!} f^{n}(x) = 0,$$

and at all points of infinite divergence of S

$$\mathcal{L} \frac{y^n}{n!} f^{n}(x) = \infty,$$

the boundary between absolute convergence and infinite divergence of S is marked by the values of x, y which satisfy

$$\mathcal{L} \frac{y^n}{n!} f^{n}(x) = 1.$$

7. The locus

$$\frac{y^n}{n!} f^{n}(x) = 1,$$

for an arbitrary and great value of n, will be a close approximation to the boundary line we seek. Differentiating, this locus has the differential equation

$$\frac{dy}{dx} = -\frac{y}{n} \frac{f^{n+1}(x)}{f^{n}(x)},$$
$$= -\left(1 + \frac{1}{n}\right) \frac{y}{n+1} \frac{f^{n+1}(x)}{f^{n}(x)}.$$

Which for n arbitrarily great gives, in the limit,

$$\frac{dy}{dx} \,|=|\, 1,$$

in virtue of

$$\mathcal{L} \frac{y}{n+1} \frac{f^{n+1}(x)}{f^{n}(x)} \,|=|\, 1$$

on the boundary.

8. Therefore the absolute value of y is equal to the absolute value of a linear function of x, of the form

$$y^2 = |k - x|^2,$$

for all values of x and y on the boundary.

This is the equation of the family of boundary lines having the parameter k. These lines are fixed by the fact that whenever $f(x)$ or $f^{r}(x)$, for any finite r, is ∞ we have $y = 0$.

9. If, therefore, $f(x) = \infty$ when $x = p$, the corresponding boundary lines for a real pole p are the two straight lines

$$y^2 = (p - x)^2,$$

or

$$y = x - p \quad \text{and} \quad y = -x + p.$$

If $f(x) = \infty$ when $x = p + iq$, then the corresponding boundary lines for a complex pole $p + iq$ are the two branches of the rectangular hyperbola

$$y^2 = |p + iq - x|^2,$$
$$= (p - x)^2 + q^2,$$

or

$$-\frac{(x-p)^2}{q^2} + \frac{y^2}{q^2} = 1,$$

having for asymptotes

$$y = x - p \quad \text{and} \quad y = -x + p.$$

10. Therefore for any function having real and complex poles the boundary lines consist of pairs of straight lines crossing Ox at 45° at the real poles and of right hyperbolæ having as asymptotes similar straight lines crossing Ox at the real part of the complex pole.

The vertices of the hyperbola corresponding to the pole $p + iq$ are $p, \pm q$.

11. The region of absolute convergence of S is that portion of the plane (shaded) such that from any point in it a perpendicular can be drawn to Ox without crossing a boundary line. The nearest boundary lines to Ox make up *the boundary* of the region of convergence of S. It consists of straight lines and hyperbolic arcs.

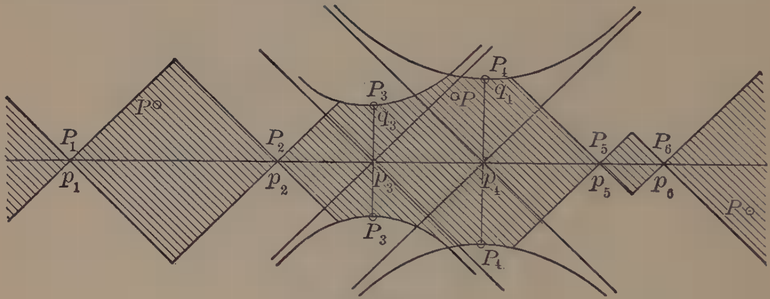

FIG. 159.

The boundary line of the region is symmetrical with respect to Ox. The ordinate at any point of this boundary line of convergence is the radius of convergence for the corresponding abscissa, and is equal to the distance of its foot from the nearest pole point.

For any point on the boundary

$$\underset{n=\infty}{\mathcal{L}} \frac{y}{n+1} \frac{f^{n+1}(x)}{f^{n}(x)} | = | 1,$$

is less than 1 for any point inside, and greater than 1 for any point outside, the region.

12. If a function has two real poles α, β, and no pole between α, β, the region of absolute convergence consists of a square between α and β. If between α and β there is an imaginary pole $p + iq$ such that p lies between α and β, the imaginary pole has no influence on the region of convergence if

$$[p - \tfrac{1}{2}(\alpha + \beta)]^2 + q^2 > \tfrac{1}{4}(\alpha - \beta)^2.$$

If, however,

$$[p - \tfrac{1}{2}(\alpha + \beta)]^2 + q^2 < \tfrac{1}{4}(\alpha - \beta)^2,$$

the hyperbola $y^2 = (p - x)^2 + q^2$ cuts off a portion of the square of convergence.

13. Theorem II. If $f(x)$ is a one-valued, determinate, unlimitedly differentiable function (having only a finite number of roots or poles in any finite interval), then

$$f(x + y) = \sum_0^\infty \frac{y^r}{r!} f^r(x) \qquad (1)$$

for all values of x and y for which the series is absolutely convergent. That is, for all values of y less in absolute value than the radius

$$R = + \sqrt{(x - p)^2 + q^2},$$

where $p + iq$ is the nearest pole of $f(x)$ to x. Equation (1) is not true for any value of y such that $|y| > R$.

Proof: The construction of the region of absolute convergence shows that from any point P *in* this region can be drawn two straight lines making angles of 45° with Ox to meet Ox without crossing or touching the boundary of absolute convergence.

At any point x, y in the region of absolute convergence the series

$$S' = \sum_0^\infty \frac{y^r}{r!} f^{r+1}(x)$$

is absolutely convergent.

But
$$S' = \frac{\partial S}{\partial x} = \frac{\partial S}{\partial y}.$$

Hence

$$\begin{aligned} dS &= \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy, \\ &= \frac{\partial S}{\partial x}(dx + dy) = \frac{\partial S}{\partial y}(dx + dy), \\ &= 0, \text{ if } x + y \text{ is constant.} \end{aligned}$$

Therefore all along the line $x + y = c$, *in* the region of absolute convergence, S must be *constant*. This line passing through any point P in this region meets Ox without touching the boundary. At the point where $x + y = c$ meets Ox we have $y = 0$, $x = c$, and

$$S = f(c) = f(x + y).$$

Consequently all along any such line passing through the region of absolute convergence, and therefore at any point whatever in this region, we have

$$f(x + y) = \sum_0^\infty \frac{y^r}{r!} f^r(x).$$

14. What is the same thing,

$$f(x) = \sum_0^\infty \frac{(x - y)^r}{r!} f^r(y) \tag{1}$$

for all values of x and y which make the series absolutely convergent.*

If we make the investigation in the form (1), the regions consist of parallelograms on the line $y = x$ as diagonal, and having for sides the straight lines

$$x = p, \quad x = 2y - p,$$

corresponding to a real pole p, and hyperbolæ

$$(x - y)^2 = |p + iq - y|^2 = (p - y)^2 + q^2,$$

or

$$x^2 - 2xy + 2py = p^2 + q^2,$$

corresponding to a complex pole $p + iq$.

15. Observations.—In the preceding investigation the object has been to point out as briefly as possible the salient points in the establishment of the theorems proposed. Details have not been entered upon. For example, we might discuss fully the behavior of the approximate boundary line

$$\frac{y^n}{n!} f^n(x) = 1$$

at the zeros of $f^n(x)$. There the curve has vertical asymptotes, but closes up on the asymptote as n increases. Also, $f^n(x)$ cannot have the same zero point for an indefinite number of consecutive integers n unless the function is a polynomial.

Again, if at any assigned point x the derivatives are alternately 0, the radius of convergence is fixed by

$$\mathcal{L}\, n(n + 1) \frac{f^n(x)}{f^{n+2}(x)} | = | R^2,$$

since for absolute convergence we must have

$$\mathcal{L}\, \frac{y^2}{n(n + 1)} \frac{f^{n+2}(x)}{f^n(x)} | < | 1.$$

---

* It being understood that y is at a finite distance from any value of the variable at which the function is ∞.

This simply means that there are two poles that are equidistant from the value x. If the poles of a function are all real, it is impossible for more than alternate derivatives to be zero continually.

If there are more than two poles equidistant from x, then at least one must be complex.

If there be three equidistant poles from x, then one must be real and two imaginary, $p \pm iq$, and conjugate. Then the derivatives at x are 0 alternately in pairs and the radius of convergence there is

$$R^3| = | \mathcal{L}\, n(n+1)(n+2)\frac{f^n(x)}{f^{n+3}(x)},$$

and so on.

Points x, equally distant from several poles, are the singular points on the boundary. Elsewhere, for three poles, we can always write

$$n(n+1)(n+2)\frac{f^n(x)}{f^{n+3}(x)} \equiv n\frac{f^n(x)}{f^{n+1}(x)} \cdot (n+1)\frac{f^{n+1}(x)}{f^{n+2}(x)} \cdot (n+2)\frac{f^{n+2}(x)}{f^{n+3}(x)},$$

the limit of which is K^3, and converges to the value R^3 at the singular point as x converges to the x of such a singular point. The generalization of this is obvious.

EXAMPLES.

1. The region of absolute convergence and of equivalence of the Taylor's series of the functions tan x, cot x, sec x, csc x, consists of the squares whose diagonals are the intervals between the roots of sin x, cos x, respectively.

2. In particular tan x is equivalent to its Maclaurin's series for all values of x in $)-\frac{1}{2}\pi, +\frac{1}{2}\pi($.

Also for sec x in the same interval.

cot x, csc x are equal to their Taylor's series in the interval $)0, \pi($, the base of the expansion being $\frac{1}{2}\pi$.

3. Expand $\dfrac{x}{e^x - 1}$ by Maclaurin's series.

Put y equal to the function. Then

$$ye^x - y = x.$$

Apply Leibnitz's formula, and put $x = 0$ in the result. We have for determining the derivatives of y at 0,

$$ny_0^{(n-1)} + \frac{n(n-1)}{2\,!}y_0^{(n-2)} + \ldots + ny_0' + y_0 = 0.$$

Making $n = 1, 2, 3, \ldots$, we find these derivatives in succession, and therefore

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{B_1}{2\,!}x^2 - \frac{B_2}{4\,!}x^4 + \frac{B_3}{6\,!}x^6 - \ldots,$$

wherein $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$, . . . are called Bernouilli's numbers. They are of importance in connection with the expansion of a number of functions.

Since $e^{\pm i\pi} = -1$, the poles $\pm i\pi$ are the nearest values of x to 0 at which the function becomes ∞. The series is therefore convergent and equal to the function for x in $)-\pi, +\pi($.

4. Show that for x in $)-\tfrac{1}{2}\pi, +\tfrac{1}{2}\pi($,

$$\frac{x}{e^x+1} = \frac{x}{2} - \frac{B_1x^2}{2\,!}(2^2-1) + \frac{B_2x^4}{4\,!}(2^4-1) - \frac{B_3x^6}{6\,!}(2^6-1) + \ldots$$

either directly or from

$$\frac{x}{e^x+1} = \frac{x}{e^x-1} - \frac{2x}{e^{2x}-1}.$$

5. Show directly from 4 that, for the same values of x,

$$\frac{e^x-1}{e^x+1} = B_1x(2^2-1) - \frac{B_2x^3}{4\,!}\,\frac{2^4-1}{2} + \frac{B_3x^5}{6\,!}\,\frac{2^6-1}{2} - \ldots$$

6. Obtain the Maclaurin expansion

$$\sin(m\sin^{-1}x) = \frac{m}{1}x + \frac{m(1^2-m^2)}{3\,!}x^3 + \frac{m(1^2-m^2)(3^2-m^2)}{5\,!}x^5 + \ldots,$$

and find for what values of x the equation is true.

Put $$y = \sin(m\sin^{-1}x).$$

$$\therefore\quad (1-x^2)y'' - xy' + m^2y = 0.$$

Apply Leibnitz's theorem and deduce

$$(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} + (m^2-n^2)y^{(n)} = 0.$$

$$\therefore\quad (1-x^2) - 2x(n+\tfrac{1}{2})\frac{y^{(n+1)}}{y^{(n+2)}} - (n^2-m^2)\frac{y^{(n)}}{y^{(n+2)}} = 0.$$

Since $$\mathop{\pounds}\,(n+\tfrac{1}{2})\frac{y^{(n+1)}}{y^{(n+2)}}\,|=|\,R,$$

$$\mathop{\pounds}\,(n^2-m^2)\frac{y^{(n)}}{y^{(n+2)}} = \mathop{\pounds}\,(n+1)\frac{y^{(n)}}{y^{(n+1)}}\cdot(n+2)\frac{y^{(n+1)}}{y^{(n+2)}}\,|=|\,R^2,$$

we have

$$(1-x^2) - 2x\,R - R^2 = 0,$$

or $$R\,|=|\,x \pm 1.$$

Therefore if x is the base of a Taylor's series for y, the function is equal to the series in $)x-1, x+1($. If $x=0$, the Maclaurin's series is equal to the function in $)-1, +1($.

When $x=0$, the differential equation gives

$$y_0^{(n+2)} = (n^2-m^2)y_0^{(n)},$$

which gives the coefficients in the series.

7. Treat in the same way $\cos(m\sin^{-1}x)$.

8. For what values of x is the Maclaurin's series corresponding to the function y in

$$(1-x^2)y'' - xy' - a^2y = 0$$

equal to the function?

Work as in 6. The function is $e^{a\sin^{-1}x}$.

9. In general, any function y satisfying a differential equation

$$(1+ax^2)y^{(n+2)} + px(n+b)y^{(n+1)} + q(n-c)(n-d)y^{(n)} = 0,$$

where a, b, c, d, p, q are any constants, is equal to its Taylor's series (base x) in the interval $)x-R, x+R($, where R is the radius of absolute convergence, and R is the absolute value of the least root of the quadratic

$$(1+ax^2) + px\,R + qR^2 = 0.$$

A large class of functions can be treated in this way.

10. If u is a function of x having only a finite number of roots in a finite interval, find the region of equivalence of the function $1/u$ with its Taylor's series.

Let $y = 1/u$. Then $yu = 1$. Differentiate n times by Leibnitz's formula. Then

$$y^n u + c_{n,1}\, y^{n-1}u' + c_{n,2}\, y^{n-2}u'' + \dots + u^n = 0.$$

Divide by y^n, and make $n = \infty$. Then, ρ being the radius of convergence, we have, if $u = \phi(x)$,

$$\phi(x) + \frac{\rho}{1}\,\phi'(x) + \frac{\rho^2}{2!}\,\phi''(x) + \dots = 0.$$

But this series is nothing more than $\phi(x + \rho)$.

Therefore $x + \rho$ must be a root of

$$\phi(x + \rho) = 0.$$

Consequently

$$x + \rho = k,$$

or

$$\rho = k - x,$$

where k is the nearest root of $\phi(x)$ to x, the base of the expansion.

NOTE 7.

Supplementary to Note 6.

1. While, in this book, we are not interested in functions of a complex variable $z = x + iy$, it is instructive and interesting to consider the treatment of a function $f(z)$ after the method of Note 6 for a function of the complex variable $z = x + iy$.

We assume that $f(z)$ is one-valued, unlimitedly differentiable with respect to z at all values of z in the finite portion of the plane except at poles of $f(z)$, which are, we assume, the only singularities the function has.

2. Let $z \equiv x + iy$, $\zeta \equiv x' + iy'$. The series

$$S \equiv \sum_{0}^{\infty} \frac{z^r}{r!} f^r(\zeta)$$

is absolutely convergent (when the series of absolute values of its terms is convergent) for all values of z and ζ which satisfy

$$\mathop{\mathcal{L}}_{n=\infty} \left\{ \frac{z^n}{n!} f^n(\zeta) \right\}^{\frac{1}{n}} | < | 1, \quad \text{or} \quad \mathop{\mathcal{L}}_{n=\infty} \frac{z}{n+1}\,\frac{f^{n+1}(\zeta)}{f^n(\zeta)} | < | 1.$$

The series S is ∞ when these limits are greater than 1.

The boundary conditions are

$$\mathcal{L} \left\{ \frac{z^n}{n!} f^n(\zeta) \right\}^{\frac{1}{n}} | = | 1, \qquad \mathcal{L} \frac{z}{n+1}\,\frac{f^{n+1}(\zeta)}{f^n(\zeta)} | = | 1.$$

Therefore for n arbitrarily great the boundary is arbitrarily near

$$\left\{ \frac{z^n}{n!} f^n(\zeta) \right\}^{\frac{1}{n}} = e^{i\alpha}, \quad \frac{z}{n+1}\,\frac{f^{n+1}(\zeta)}{f^n(\zeta)} = e^{i\beta},$$

α and β being arbitrary constant real numbers.

From the first of these equations, we have

$$\frac{dz}{d\zeta} = -\frac{z}{n}\frac{f^{n+1}(\zeta)}{f^{n}(\zeta)},$$

$$= -e^{i\beta}, \qquad \text{when } n = \infty.$$

$$\therefore \quad z = k - e^{i\beta}\zeta,$$

k being an arbitrary constant. But if p is a pole of $f(z)$, then $z = 0$ when $\zeta = p$.

Hence $\qquad 0 = k - e^{i\beta}p, \quad$ or $\quad k = e^{i\beta}p.$

Therefore, corresponding to any assigned ζ, the boundary corresponding to the pole p is fixed by

$$z = e^{i\beta}(p - \zeta),$$

which is a circle about the origin in the z-plane with radius

$$R = |p - \zeta|,$$

since β is arbitrary.

3. If p is the nearest pole of $f(z)$ to ζ, then for all values of z for which

$$|z| < R = |p - \zeta|$$

the series is absolutely convergent, and is infinite for any value of z if $|z| > R$.

4. Put $\xi = z + \zeta$. Then $z = \xi - \zeta$.

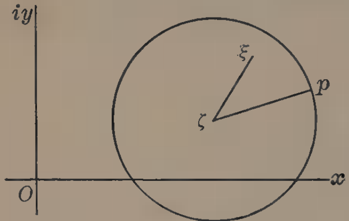

FIG. 160.

The series

$$S = \sum_{0}^{\infty}\frac{(\xi - \zeta)^n}{n\,!}f^n(\zeta)$$

is absolutely convergent at all points ξ inside the circle C described about ζ as a center with radius

$$R = |\zeta - p|,$$

p being the nearest pole of $f(z)$ to ζ.

For any assigned value of ξ in this circle the series S is constant with respect to ζ, since

$$\frac{dS_n}{d\zeta} = \frac{d}{d\zeta}\sum_{0}^{n}\frac{(\xi - \zeta)^r}{r\,!}f^r(\zeta) = \frac{(\xi - \zeta)^n}{n\,!}f^{n+1}(\zeta),$$

and this is 0 when $n = \infty$.

Now we can always move ζ up to ξ along the straight line joining them, the series S remaining constant in value. But when $\zeta = \xi$, we have

$$\sum_{0}^{\infty}\frac{(\xi - \zeta)^n}{n\,!}f^n(\zeta) = f(\xi).$$

Therefore* this equality is true for all values of ξ, ζ which make the series absolutely convergent, i.e., at any point inside the circular boundary corresponding to any assigned ζ and the nearest pole p of $f(z)$, described about ζ as center with radius of absolute convergence,

$$R = |p - \zeta|.$$

NOTE 8.

Supplementary to Note 6.

Pringsheim's Example of a Function for which the Maclaurin's Series is absolutely Convergent and yet the Function and Series are different.

Let

$$f(x) \equiv \sum_{r=0}^{\infty} \frac{\lambda^r}{r\,!} \frac{(-1)^r}{1 + a^{2r}x^2}, \tag{1}$$

λ and a being positive constants, $a > 1$. This function is one-valued, finite, continuous, and unlimitedly differentiable for all finite values of the real variable x. It has, however, infinitely many complex poles

$$\pm \frac{\sqrt{-1}}{a^r}, \qquad r = 1, 2, 3, \ldots$$

an infinite number of which are in the neighborhood of $x = 0$, which is therefore an essentially singular point.

For the nth derivative of $f(x)$ we find ($i \equiv +\sqrt{-1}$)

$$f^n(x) = \frac{1}{2i} n! \sum_{0}^{\infty} (-1)^r \frac{(\lambda a^n)^r}{r\,!} \left\{ \frac{1}{(a^r x - i)^{n+1}} - \frac{1}{(a^r x + i)^{n+1}} \right\}.$$

At $x = 0$,

$$f(0) = e^{-\lambda},$$
$$f^{2m+1}(0) = 0,$$
$$f^{2m}(0) = (-1)^m (2m)!\, e^{-\lambda a^{2m}}.$$

Therefore the Maclaurin's series is

$$S = \sum_{0}^{\infty} (-1)^r \frac{x^{2r}}{e^{\lambda a^{2r}}}, \tag{2}$$
$$= e^{-\lambda} - e^{-\lambda a^2} x^2 + e^{-\lambda a^4} x^4 - \ldots$$

This series is absolutely convergent for all finite real values of x.

---

* This problem was first solved by Cauchy, by means of singular integrals. See any text on the theory of functions of a complex variable.

Now let $\lambda \leqq 1, \quad |x| < 1.$

$$\therefore \quad f(x) > \frac{1}{1+x^2} - \frac{\lambda}{1+a^2x^2} > \frac{1}{1+x^2} - \frac{1}{1+a^2x^2},$$

and $$S < e^{-1}.$$

In particular, let $x = a^{-\frac{1}{2}}$.

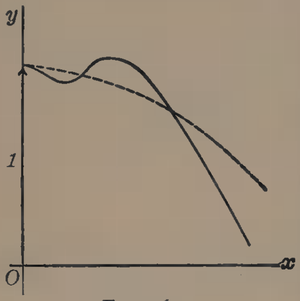

FIG. 161.

$$\therefore \quad f(a^{-\frac{1}{2}}) > \frac{1}{1+1/a} - \frac{1}{1+a} > \frac{a-1}{a+1}.$$

$$\therefore \quad f(a^{-\frac{1}{2}}) > e^{-1} > S \quad \text{when} \quad x = a^{-\frac{1}{2}},$$

when $\dfrac{a-1}{a+1} \geqq e^{-1}$, or $a \geqq \dfrac{e+1}{e-1}$.

The function $f(x)$ and the series S are different.

In the figure the solid line is the curve $y = f(x)$, the dotted line the curve $y = S$, constructed with exaggerated ordinates, for the values $\lambda = \log 2$, $a = 2$.*

NOTE 9.

Supplementary to § 118.

Riemann's Existence Theorem.

Any function $f(x)$ that is one-valued and continuous throughout an interval (a, b) is integrable for that interval.

Let the numbers $x_1, x_2, \ldots, x_{n-1}$ be interpolated in the interval (a, b) taken in order from $a \equiv x_0$ to $b \equiv x_n$.

We have to prove that the sum of the elements

$$S_n \equiv \sum_1^n f(z_r)(x_r - x_{r-1}) \tag{1}$$

converges to a unique determinate limit, when each subinterval converges to zero, whatever be the manner in which the numbers x_r are interpolated in (a, b).

I. The sum S_n must remain finite for all values of n. For $f(x)$ is finite, and if M and m are the greatest and least values of $f(x)$ in (a, b),

$$m(b-a) < S_n < M(b-a).$$

Also, since $f(x)$ is continuous, there exists a value ξ in (a, b) at which

$$S_n = (b-a)f(\xi), \tag{2}$$

$f(\xi)$ being a value of $f(x)$ between m and M.

---

* For further information on this subject, see papers by Pringsheim, *Math. Ann.* Bd. XLII. p. 109. Math. Papers Columbian Exposition, p. 288.

II. Interpolate in the rth subinterval of (1), in any manner, $n'_r - 1$ values $x'_1, \ldots, x_{n'-1}$ of x. Then, as in I,

$$S_{n'_r} \equiv \sum_1^{n'} f(z'_r)(x'_r - x'_{r-1}) = (x_r - x_{r-1})f(\xi'_r), \tag{3}$$

where ξ'_r is some number in the subinterval (x_r, x_{r-1}).

Form similar sums of elements for each of the n subintervals of (1). Let $p = n'_1 + \ldots + n'_n$. Add the n sums of elements such as (3).

Hence

$$\begin{aligned} S_p &= S_{n'_1} + \ldots + S_{n'_n}, \\ &= \sum_1^p f(z_r)(x_r - x_{r-1}), \\ &= \sum_1^n (x_r - x_{r-1})f(\xi'_r). \end{aligned} \tag{4}$$

This is a new element sum containing $p > n$ elements, which is to be regarded as a continuation of (1) by the interpolation of new numbers in each subinterval of (1).

Subtracting (4) from (1), we have

$$S_n - S_p = \sum_1^n [f(z_r) - f(\xi'_r)](x_r - x_{r-1}).$$

Let δ be the greatest absolute value of the difference between the greatest and least values of $f(x)$ in the subinterval $(x_r - x_{r-1})$, $r = 1, \ldots, n$. Then, since $f(z_r)$ and $f(\xi'_r)$ are values of $f(x)$ in (x_r, x_{r-1}),

$$\begin{aligned} S_n - S_p| &< |\delta \sum_1^n (x_r - x_{r-1}), \\ | &< |\delta(b - a), \end{aligned} \tag{5}$$

for all values of the integer p, however great. But when each subinterval converges to 0, then $\delta (=) 0$, since $f(x)$ is continuous, and at the same time $n = \infty$.

Therefore, by the definition of a limit, S_n converges to a limit when $n = \infty$.

III. To show that the limit of S_n is wholly independent of the manner in which the interval (a, b) is subdivided:

Let there be an entirely different and arbitrary interpolation $x_1', \ldots, x'_{m-1}$. Consider the element-sum

$$S'_m \equiv \sum_1^m f(z'_r)(x'_r - x'_{r-1}). \tag{6}$$

Interpolate in (a, b) the numbers

$$x_1, \ldots, x_{n-1}; \quad x'_1, \ldots, x'_{m-1},$$

occurring in (1) and (6), thus dividing (a, b) into $m + n$ intervals.

Interpolate in each of these $m+n$ intervals new numbers, thus dividing (a, b) into $m+n+p$ subintervals. Form the element-sum S_{m+n+p} corresponding to these subintervals.

Then, by II, S_n and S_{m+n+p} converge to the same limit. In like manner S_m and S_{m+n+p} converge to the same limit. Therefore S_n and S_m converge to a common limit. The uniqueness of the limit of (1) under any subdivision whatever of (a, b) is demonstrated.

This theorem gives the means of defining analytically the area and length of a curve, and the volume and surface area of a solid.

NOTE 10.

Supplementary to § 135.

Formulæ for the Reduction of Binomial Differentials of the form

$$x^{\alpha}(a+bx^{n})^{\gamma}dx.$$

Put $y = a + bx^n$. Then

$$\begin{aligned} Dx^{\alpha}y^{\gamma} &= \alpha x^{\alpha-1}y^{\gamma} + n\gamma b x^{\alpha+n-1}y^{\gamma-1}, \\ &= a\alpha x^{\alpha-1}y^{\gamma-1} + (a\alpha + n\gamma)bx^{\alpha+\beta-1}y^{\gamma-1}, \qquad (1) \\ &= (\alpha + n\gamma)x^{\alpha-1}y^{\gamma} - an\gamma x^{\alpha-1}y^{\gamma-1}. \qquad (2) \end{aligned}$$

In (1), put $\alpha = m - n + 1$, $\gamma = p + 1$, then

$$Dx^{m-n+1}y^{p+1} = a(m-n+1)x^{m-n}y^{p} + (np+m+1)bx^{m}y^{p}. \qquad \text{(A)}$$

In (2), put $\alpha = m + 1$, $\gamma = p$, then

$$Dx^{m+1}y^{p} = (np+m+1)x^{m}y^{p} - anpx^{m}y^{p-1}. \qquad \text{(B)}$$

In (1), put $\alpha = m + 1$, $\gamma = p + 1$, then

$$Dx^{m+1}y^{p+1} = a(m+1)x^{m}y^{p} + (np+m+n+1)bx^{m+n}y^{p}. \qquad \text{(C)}$$

In (2), put $\alpha = m + 1$, $\gamma = p + 1$, then

$$Dx^{m+1}y^{p+1} = (np+m+n+1)x^{m}y^{p+1} - an(p+1)x^{m}y^{p}. \qquad \text{(D)}$$

Integrating the formulæ (A), . . . , (D), we have the formulæ of reduction, where $y \equiv a + bx^n$:

$$\int x^{m}y^{p}dx = \frac{x^{m-n+1}y^{p+1}}{(np+m+1)b} - \frac{(m-n+1)a}{(np+m+1)b}\int x^{m-n}y^{p}dx. \qquad \text{(A)}$$

$$\int x^{m}y^{p}dx = \frac{x^{m+1}y^{p}}{np+m+1} + \frac{anp}{np+m+1}\int x^{m}y^{p-1}dx. \qquad \text{(B)}$$

$$\int x^{m}y^{p}dx = \frac{x^{m+1}y^{p+1}}{(m+1)a} - \frac{(np+m+n+1)b}{(m+1)a}\int x^{m+n}y^{p}dx. \qquad \text{(C)}$$

$$\int x^{m}y^{p}dx = -\frac{x^{m+1}y^{p+1}}{an(p+1)} + \frac{np+m+n+1}{an(p+1)}\int x^{m}y^{p+1}dx. \qquad \text{(D)}$$

NOTE 11.

Supplementary to § 165.

If $y = f(x)$ be represented by a curve, and y, Dy, D^2y are uniform and continuous, then we can always take two points P and P_1 on the curve so near together that the curve lies wholly between the chord and the tangents at P and P_1.

Let x, y be the coordinates of P, and X, Y those of P', any point on the curve between P and P_1.

The tangent at P has for its equation

$$Y_t = f(x) + (X - x)f'(x).$$

At any point x, y of ordinary position, not an inflexion, the difference between the ordinate to the curve and the tangent is

$$f(X) - Y_t = \frac{(X-x)^2}{2!}f''(\xi), \quad (1)$$

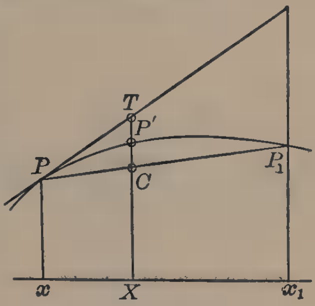

FIG. 162.

where ξ is some number between x and X. We can always take X so near to x that $f''(\xi)$ keeps its sign the same as that of $f''(x)$ for all values of ξ in (x, X). Therefore the difference (1) keeps its sign unchanged in (x, X) or the curve is on one side of the tangent, for this interval.

The equation to the chord PP_1 is

$$Y_c = f(x) + (X - x)f'(\xi_1),$$

where $f'(\xi_1)$ is the slope of the chord PP_1. The difference between the ordinates of the curve and chord is

$$f(X) - Y_c = (X - x)[f'(\xi) - f'(\xi_1)]. \qquad (2)$$

Let x_1 be so near x that $f'(x_1)$, $f'(\xi_1)$ have the same sign as $f'(x)$. Then this difference (2) keeps its sign unchanged for all values of X in (x, x_1). It can now be easily shown that (2) and (1) have opposite signs, and there can always be assigned a number x_1 so near x that the curve PP_1 lies wholly in the triangle formed by the tangents at P, P_1 and the chord PP_1.

NOTE 12.

Supplementary to § 226, IV.

Proof of the Properties of Newton's Analytical Polygon.

1. Let there be any polynomial in x and y, such as

$$f \equiv A_1 x^{\alpha_1} y^{\beta_1} + \ldots + A_m x^{\alpha_m} y^{\beta_m}, \qquad (1)$$

wherein the exponents α, β of each term satisfy the linear relation

$$a\alpha + b\beta = c, \qquad (2)$$

c being taken a positive number.

Let f be arranged according to ascending powers of y, so that $\beta_1 < \beta_2 < \ldots$ Then

$$f = x^{\alpha_1}y^{\beta_1}(A_1 + A_2x^{\alpha_2-\alpha_1}y^{\beta_2-\beta_1} + \ldots),$$

$$= x^{\alpha_1}\,{}^{\beta_1}\Big[A_1 + A_2\Big(yx^{-\frac{b}{a}}\Big)^{\beta_2-\beta_1} + \ldots\Big], \tag{3}$$

$$= x^{\alpha_1}y^{\beta_1}\Big(yx^{-\frac{b}{a}} - k_1\Big) \ldots \Big(yx^{-\frac{b}{a}} - k_{\beta_m-\beta_1}\Big)A_n, \tag{4}$$

where $k_1, \ldots, k_{\beta_m-\beta_1}$ are the roots of the equation in $t \equiv yx^{-\frac{b}{a}}$,

$$A_1 + A_2t^{\beta_2-\beta_1} + \ldots + A_nt^{\beta_m-\beta_1} = 0.$$

Therefore the locus of $f = 0$ consists of $x = 0$, $y = 0$, and the parabolic curves

$$y^a = k_r^a x^b. \qquad (r = 1, \ldots, \beta_m - \beta_1).$$

2. In (3), let $y = kx^{\frac{b}{a}}$, k being constant. Then

$$f = A_1x^{\alpha_1}k^{\beta_1}x^{\beta_1\frac{b}{a}} + A_2x^{\alpha_2}k^{\beta_2}x^{\beta_2\frac{b}{a}} + \ldots,$$

$$= A_1k^{\beta_1}x^{\alpha_1+\beta_1\frac{b}{a}} + A_2k^{\beta_2}x^{\alpha_2+\beta_2\frac{b}{a}} + \ldots,$$

$$= x^{\frac{c}{a}}\Big(A_1k^{\beta_1} + A_2k^{\beta_2} + \ldots\Big)$$

$$= K\,x^{\frac{c}{a}}, \quad K \text{ being constant.}$$

3. Let f' be a function $A'x^{\alpha'}y^{\beta'}$, or the sum of a finite number of such functions, such that the exponents α', β' of each term satisfy the linear equation

$$a\alpha' + b\beta' = c'.$$

Then, as in 2, let $y = kx^{\frac{b}{a}}$, and we have in the same way

$$f' = K'x^{\frac{c'}{a}},$$

K' being a constant.

4. Let a, b and c, c' be positive numbers.

Then

$$\frac{f'}{f} = \frac{K'}{K}x^{\frac{c'-c}{a}},$$

where x and y satisfy $y^a = kx^b$.

(1). If $c' > c$, then

$$\mathcal{L}\frac{f'}{f} = 0, \quad \text{when} \quad x(=)0, \quad y(=)0.$$

(2). If $c' < c$, then

$$\mathcal{L}\frac{f'}{f} = 0, \quad \text{when} \quad x = \infty, \quad y = \infty.$$

5. We are now prepared to prove § 226, IV, (1), (2).

Let $$F(x, y) \equiv \Sigma C_r x^p y^q = 0.$$

(1). Let f represent that part of F which corresponds to a side of the polygon as prescribed in § 226, IV, (1), and F' represent the remainder of F. Then

$$F = f + F',$$

or $$\frac{F}{f} = 1 + \frac{F'}{f}.$$

Through each point corresponding to terms in F' draw a line parallel to the side corresponding to f.

Then by 3, (1), we have

$$\mathcal{L}\frac{F}{f} = 1, \quad \text{since} \quad \mathcal{L}\frac{F'}{f} = 0,$$

when $x (=) 0$, $y (=) 0$.

Therefore in the neighborhood of the origin $F = 0$ and $f = 0$ are the same.

But the form of $f = 0$ in the neighborhood of the origin is that of a parabola

$$y^a = kx^b.$$

Hence $F = 0$ goes through the origin in the same way as does $f = 0$, whose form is that of a parabola of type $y^a = kx^b$.

(2). Let f represent that part of F corresponding to a side of the polygon as prescribed in § 226, IV, (2), and F' the remainder of F.

Then $$\frac{F}{f} = 1 + \frac{F'}{f}.$$

Draw parallels to the side corresponding to f, through all points corresponding to terms in F'.

Then by 3, (2), we have

$$\mathcal{L}\frac{F}{f} = 1, \quad \text{since} \quad \mathcal{L}\frac{F'}{f} = 0,$$

when $x = \infty$, $y = \infty$.

Therefore $F = 0$ and $f = 0$ pass off to ∞ in the same way. Also, $f = 0$ passes off to ∞, as does a parabola of type $y^a = kx^b$.

NOTE.—The same process can be extended to surfaces, using a polyhedron in space. The part of the equation corresponding to a plane face such that there are no points between that face and the origin gives the form of a sheet of the surface at the origin. Likewise the part corresponding to a plane face such that no point lies on the side opposite to the origin gives the form of a sheet at ∞. The plane faces in each case cutting the positive parts of the axes.

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